ANZIAM J. 44(2002), 21–32

WHAT IS THE DISCRETE ANALOGUE OF THE PAINLEVÉ PROPERTY?

A. RAMANI[1] and B. GRAMMATICOS[2]

(Received 29 February, 2000; revised 6 July, 2000)

Abstract

We analyse the various integrability criteria which have been proposed for discrete systems, focusing on the singularity confinement method. We present the exact procedure used for the derivation of discrete Painlevé equations based on the deautonomisation of integrable autonomous mappings. This procedure is then examined in the light of more recent criteria based on the notion of the complexity of the mapping. We show that the low-growth requirements lead, in the case of the discrete Painlevé equations, to exactly the same results as singularity confinement. The analysis of linearisable mappings shows that they have special growth properties which can be used in order to identify them. A working strategy for the study of discrete integrability based on singularity confinement and low-growth considerations is also proposed.

1. In the beginning, Painlevé created . . .

The Painlevé approach [16] is one of the most successful methods for the prediction of integrability of nonlinear differential systems. The main difficulty in integrating the latter, that is, properly defining a function through the solution of the differential equation, was the existence of movable (initial condition dependent) critical singularities. The Painlevé approach consisted of looking for those of the nonlinear differential equations whose solutions were free from movable critical singularities. The Painlevé property has since been used with great success in the detection of integrability [19]. We must stress one important point here. The Painlevé property as introduced by Painlevé is not just a *predictor* of integrability but practically a definition of integrability. As such it becomes a tautology rather than a criterion. It is thus crucial to make the distinction between the Painlevé property and the algorithm for its investigation.

[1]CPT, Ecole Polytechnique, CNRS, UMR 7644, 91128 Palaiseau, France; e-mail: ramani@cpht.polytechnique.fr.

[2]GMPIB, Université Paris VII, Tour 24-14, 5e étage, case 7021, 75251 Paris, France.

The latter can only search for the absence of the Painlevé property *within certain assumptions* [1]. The search can thus lead to a conclusion with questionable validity: if we find that the system passes what is usually referred to as the Painlevé test (in one of its several variants) this does not necessarily mean that the system possesses the Painlevé property. Thus at least as far as its usual practical application is concerned, the Painlevé test may not be sufficient for integrability.

Given the success of the Painlevé approach one can wonder whether these techniques could be transposed *mutatis mutandis* to the study of discrete systems. It is clear that rational mappings have singularities which could play some role in connection with integrability. However, any argument based on singularities in the discrete domain can only bear a superficial resemblance to the situation in the continuous case [14]. One cannot hope to relate directly the singularities of mappings to those of ODEs for the simple reason that there exist discrete systems which do not have any nontrivial continuous limit.

With these arguments in mind we are ready to embark upon a review of the detectors of discrete integrability.

2. The genesis of singularity confinement

While in the introduction we presented a pessimistic view concerning the singularities of discrete systems and their relations to those of continuous systems, we shall here qualify this statement. While a direct relation is elusive the notions of singularity and singlevaluedness can be transposed from the continuum to the discrete setting. In the continuous case, a singularity that introduces multivaluedness is considered incompatible with integrability. The analogous idea in the discrete case was introduced by N. Joshi [13] through the notion of orbits with pole-like behaviour. Starting from finite values of the mapping variable one reaches infinity in a finite number of steps and the orbit can be continued in a single-valued manner beyond the point at infinity to finite values again. Joshi conjectured that for integrability a discrete system should have orbits with pole-like behaviour. However, the criterion of pole-like behaving orbits was not proposed as a discrete integrability detector. This was due to a difficulty which appeared during the first exploratory studies. While studying the mapping

$$x_{n+1} = \frac{x_n^2}{x_n^2 - 1} \tag{1}$$

Joshi observed that it had only orbits with pole-like behaviour while being nonintegrable. However as we have explained in various works of our own, this nonintegrability of (1) is due to some other defect of the mapping. While the forward iteration is well-defined, the backward one is not. Thus a given point has an exponentially increasing number of preimages. This proliferation of preimages is incompatible with integrability.

Still, singularities were to play an important role in the study of the integrability of discrete systems. This was done through the introduction of singularity confinement [10] which is essentially the same as the requirement of the existence of orbits with pole-like behaviour. Singularity confinement was discovered (independently of the results of N. Joshi) in our study of integrable mappings. Let us illustrate this by an example. Consider the mapping

$$x_{n+1} + x_{n-1} = \frac{a}{x_n} + \frac{1}{x_n^2}\,. \tag{2}$$

Obviously, a singularity appears whenever the value of x becomes 0. Iterating this value, one obtains the sequence $\{0, \infty, 0\}$ and then the indeterminate form $\infty - \infty$. As Kruskal points out the real problem lies in the latter, while the occurrence of a simple infinity is something that can easily be dealt with by going to projective space. The way to treat this difficulty is to use an argument of continuity with respect to the initial conditions and introduce a small parameter ϵ. In this case, if we assume that $x_n = \epsilon$ we obtain for the first values of x: $x_{n+1} \approx 1/\epsilon^2$, $x_{n+2} \approx -\epsilon$, and when we carefully compute the next value of x we find that not only is it finite but it also contains the memory of the initial condition x_{n-1}. The singularity has disappeared.

This is the property that we have dubbed singularity confinement and after having analysed a host of discrete systems we concluded that it was characteristic of those which were integrable. Through a bold move, singularity confinement has been elevated to the rank of integrability criterion. In what follows, we shall comment on its necessary and sufficient character.

Several questions had to be answered for singularity confinement to be really operative. The first, that we encountered above, was the one related to the fact that the iteration of a mapping may not be defined uniquely in both directions. Thus we proposed the criterion of preimage non-proliferation [11], which had the advantage of eliminating *en masse* all polynomial nonlinear mappings. The second point is that the notion of 'singularity' had to be refined. Clearly the simple appearance of an infinity in the iteration of a mapping is not really a problem. What is crucial is that a mapping may at some point "lose a degree of freedom". In a mapping of the form $x_{n+1} = f(x_n, x_{n-1})$ this means simply that $\partial x_{n+1}/\partial x_{n-1} = 0$ and the memory of the initial condition x_{n-1} disappears from the iteration. What does "confinement" mean in this case? Clearly, the mapping must recover the lost degree of freedom and the only way to do this is through the appearance of an indeterminate form $0/0$, $\infty - \infty$, *etc.*, in the subsequent iterations.

Once these basic questions were answered, singularity confinement became a most efficient tool for the derivation of new integrable discrete systems. (This is a point that we wish to stress here. In our opinion, it is important to forge a tool and then use it in some useful way despite the fact that it may be far from perfect. Once one has in mind the possible shortcomings of the tool, one can use it perfectly. This we believe is much more constructive than spending all one's energy testing the tool on trivial examples and/or trying to either perfect it or prove its flawed character.)

The most successful application of the singularity confinement method has been the derivation of the discrete analogues of the Painlevé equations [6]. Let us illustrate our approach here. Our starting point is an autonomous integrable mapping and as such we have always used the QRT mapping [18], in its symmetric or asymmetric form. This choice was motivated by the fact that the solutions of the QRT mapping are samplings of elliptic functions. Since the Painlevé equations are nonautonomous extensions of elliptic functions, it makes sense, in the discrete case, to try to construct the discrete Painlevé equations by deautonomising the QRT mapping. The way to apply singularity confinement for deautonomisation is to start from the (confined) singularity pattern of an autonomous (integrable) mapping and ask for the nonautonomous extension with exactly the same singularity pattern. The example (2) above will help us make things clearer. As we have seen, the singularity pattern is $\{0, \infty, 0\}$. Now we assume that a is not a constant anymore but may depend on n. The singularity analysis can be performed in a straightforward way. Assuming that $x_n = \epsilon$, we obtain $x_{n+1} \approx 1/\epsilon^2$, $x_{n+2} \approx -\epsilon$ and requiring x_{n+3} to be finite we obtain the constraint $a_{n+2}-2a_{n+1}+a_n = 0$, that is, a_n is of the form $a_n = \alpha n+\beta$. Thus the nonautonomous form of (2), compatible with the confinement property, is

$$x_{n+1} + x_{n-1} = \frac{\alpha n + \beta}{x_n} + \frac{1}{x_n^2}\,. \tag{3}$$

Mapping (3) is presumably integrable and it turns out that indeed it is. As we have shown in [17], it possesses a Lax pair. Moreover it is the contiguity relation of the solutions of the one-parameter $\mathrm{P_{III}}$ equation [22]. Its continuous limit is $\mathrm{P_I}$ so (3) can be considered as its discrete analogue.

Using the procedure presented above, we have derived discrete analogues for all Painlevé equations ($\mathbb{P}$'s). Along the way we have discovered those genuinely discrete entities, the q-discrete $\mathbb{P}$'s [20]. The domain of discrete $\mathbb{P}$'s has a much more complicated structure than its continuous counterpart. Our findings have made it possible to explore it, chart it and, with the help of the most recent developments based on affine Weyl groups, provide the basis for the classification of the discrete Painlevé equations [23]. But this is a story for grown-ups.

3. Singularity confinement: Examine me and prove me (Psalms 26)

The words of caution we used in the previous section apply as well to the singularity confinement property used as integrability criterion. In this section we shall examine it critically and present its weaknesses. Talking about an integrability criterion one must make clear what one means by integrability. (Let us recall that our approach concerns rational explicit mappings.) Following the analogy with the continuous case, we can introduce various types of integrability:

a) Systems which possess a sufficient number of constants of motion. The QRT family of mappings is a nice example of such a system.

b) Systems which can be reduced to linear mappings.

c) Systems which can be obtained as the compatibility condition for some linear system, that is, systems that possess a Lax pair. (Nice examples of such systems are the discrete Painlevé equations.) Given the Lax pair one can reduce the integration of the nonlinear mapping to the solution of an isomonodromy problem.

It is clear that the integration of a given integrable discrete system may proceed along any of the lines sketched above. One can, for example, perform a first integration using a constant of motion whereupon the system becomes linearisable and so on.

The first question one can ask is whether singularity confinement is necessary for integrability. The answer is no. There exist systems which are integrable while possessing nonconfining singularities. The typical case of nonconfined singularities in an integrable system is that of the discrete derivative of a (discrete) Riccati equation, a case we first encountered in [7]. It is easy to understand the reason for this nonconfined singularity by following the analogy to continuous systems. The ODE which is the derivative of a continuous Riccati corresponds to the limit $n \to \infty$ of the Gambier equation. Since, as we have shown, the analogue of n in the discrete case is the number N of steps needed for confinement, an infinite number of steps is tantamount to a nonconfined singularity. A most simple equation which exhibits a nonconfined singularity while being trivially integrable through linearisation is

$$x_{n+1} = \frac{x_n^2}{x_{n-1}} + a_n x_n. \tag{4}$$

If x_n happens to be zero, which may well occur for nonzero x_{n-1}, then x_{n+1} is also zero and loses the memory of x_{n-1} and this is true of all subsequent iterations, which can *never* recover this lost degree of freedom. The linearisation of (4) is straightforward. Introducing $y_n = x_n/x_{n-1}$ we have $y_{n+1} = y_n + a_n$. Thus at least for a subclass of linearisable systems the singularity confinement condition is violated.

The second question is whether singularity confinement is sufficient for integrability. The answer is again no, as was shown by Hietarinta and Viallet [12]. They examined the mapping

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^2} \tag{5}$$

which has a confined singularity pattern $\{0, \infty, \infty, 0\}$ and showed that it behaves chaotically. Moreover they pointed out that one can construct whole families of nonintegrable mappings which satisfy the confinement criterion.

Thus one wonders what is left from confinement. The answer is clear. We know of a whole domain of mappings for which the confinement property is satisfied: they are those that are integrable through IST methods. The discrete Painlevé equations are most prominent among these. The explanation of this can be sought in the bilinear formalism. Just as in the continuous case, the IST-integrable discrete systems can be described in a bilinear setting, through a dependent variable transformation involving a *finite number* of τ-functions. The latter can only have zeros and an expression in terms of a finite number of τ's leads necessarily to confined singularities. In the case of the mapping (3), the transformation reads

$$x_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} \tag{6}$$

and this explains the singularity pattern $\{0, \infty^2, 0\}$. Of course, a transformation such as (6) does not mean anything if the τ's are not really entire functions. In the case of the discrete Painlevé equations it can be shown, in the framework of our "Grand Scheme" [23], that this is indeed the case, since the τ's are solutions of a system of compatible Hirota-Miwa equations. The same argument applies to the other IST-integrable systems.

Given the shortcomings of the confinement method that we presented above, one may wonder whether this criterion can be used as an integrability detector. We intend to answer (by a qualified affirmative) this question in the final section. For the time being let us adopt the most cautious attitude and wonder whether we can still find a discrete integrability criterion, along lines different from those explored above. The approach we shall follow here is based on the relation of discrete integrability and the complexity of the evolution introduced by Arnold and Veselov. According to Arnold [2] the complexity (in the case of mappings of the plane) is the number of intersection points of a fixed curve with the image of a second curve obtained under the mapping at hand. While the complexity grows exponentially with the iteration for generic mappings, it can be shown [25] to grow only polynomially for a large class of integrable mappings. As Veselov points out, "integrability has an essential correlation with the weak growth of certain characteristics". Veselov himself has used the slow-growth arguments in his study of the integrability of mappings and correspondences. In particular, he has studied the integrability of polynomial mappings and has shown, for example, that the mapping $x_{n+1} - 2x_n + x_{n-1} = f(x_n)$ is integrable only if $f(x_n)$ is linear in x_n.

The notion of complexity was further extended in the works of Viallet and collaborators who focused on rational mappings [3, 5]. They introduced what they called algebraic entropy, which is a global index of the complexity of the mapping. The main idea is that there exists a link between the dynamical complexity of a mapping and the degree of its iterates. If we consider a mapping of degree d (a notion to be made clearer in the examples below) then the n-th iterate will have a degree d^n, unless common factors lead to simplifications. It turns out that when the mapping is integrable such simplifications occur in a massive way leading to a degree growth which is polynomial in n, instead of exponential. Thus while the generic, nonintegrable, mapping has exponential degree growth, a polynomial growth is an indication of integrability.

Let us illustrate this approach by a practical application to a mapping that we have already encountered,

$$x_{n+1} + x_{n-1} = \frac{a}{x_n} + \frac{1}{x_n^2}\,. \tag{7}$$

In order to compute the degree of the iterates, we introduce the homogeneous coordinates by taking $x_0 = p$, $x_1 = q/r$, assigning to p the degree zero, and computing the degree of homogeneity in q and r at every iteration. We could have, of course, introduced a different choice for x_0 but it turns out that the choice of a zero-degree x_0 considerably simplifies the calculations. We obtain thus the degrees: $0, 1, 2, 5, 8, 13, 18, 25, 32, 41, \ldots$. Clearly the degree growth is polynomial. We have $d_{2m} = 2m^2$ and $d_{2m+1} = 2m^2 + 2m + 1$. This is in perfect agreement with the fact that the mapping (7) is integrable (in terms of elliptic functions), being a member of the QRT family of integrable mappings. (A remark is necessary at this point. In order to obtain a closed-form expression for the degrees of the iterates, we start by computing a sufficient number of them. Once the expression of the degree has been heuristically established we compute the next few and check that they agree with the analytical expression predicted.) As a matter of fact, the precise values of the degrees are not important: they are not invariant under coordinate changes. However, the *type of growth* is invariant and can be used as an indication of the integrable or nointegrable character of the mapping.

Let us show what happens in the case of a nonintegrable mapping. We choose one among those examined in [9],

$$x_{n+1} = a + \frac{x_{n-1}}{x_n}\,. \tag{8}$$

Again we take $x_0 = p$, $x_1 = q/r$, and compute the degree of homogeneity in q and r. We find the sequence of degrees $d_n : 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$. This is clearly a Fibonacci sequence obeying the recursion $d_{n+1} = d_n + d_{n-1}$ and thus leading to an exponential growth with asymptotic ratio $\left(1 + \sqrt{5}\right)/2$. As a consequence the mapping (8) is not expected to be integrable, which is in agreement with the findings of [9].

4. Is there life after singularity confinement?

Since there exists a considerable body of results concerning integrable discrete systems derived using the singularity confinement criterion, it is indispensable to reexamine these systems, in particular if their integrability has not been confirmed through an independent approach. The most important result of the application of singularity confinement is the derivation of discrete Painlevé equations. As we have already explained the derivation of the discrete Painlevé equations was based on the deautonomisation of the QRT mapping which is known to be integrable. The deautonomisation procedure consists of finding the dependence of the coefficients of the parameters of the QRT mapping with respect to the independent variable n, which is compatible with the singularity confinement property. Namely, the n-dependence is obtained by asking that the singularities be indeed confined. The reason why this procedure can be justified is the following. Since the autonomous starting point is integrable, it is expected that the growth of the degree of the iterates is polynomial. Now it turns out that the application of the singularity confinement deautonomisation corresponds to the requirement that the nonautonomous mappings lead to the same factorizations and subsequent simplifications and have precisely the same growth properties as the autonomous ones. These considerations will be made more transparent thanks to the examples we present in what follows.

Let us start with the mapping we have studied already,

$$x_{n+1} + x_{n-1} = \frac{a}{x_n} + \frac{1}{x_n^2} \tag{9}$$

where now a depends on n. The singularity confinement result is that a must satisfy the conditions $a_{n+1} - 2a_n + a_{n-1} = 0$, that is, a be linear in n. Assuming now that a is an arbitrary function of n we compute the iterates of (9). We obtain the sequence

$$x_2 = \frac{r^2 + a_1 qr - pq^2}{q^2}, \qquad x_3 = \frac{qQ_4}{r(r^2 + a_1 qr - pq^2)^2},$$
$$x_4 = \frac{(r^2 + a_1 qr - pq^2)Q_7}{qQ_4^2}, \qquad x_5 = \frac{qQ_4 Q_{12}}{r(r^2 + a_1 qr - pq^2)Q_7^2},$$

where the Q_k's are homogeneous polynomials in q, r of degree k. The computation of the degrees of x_n leads to 0, 1, 2, 5, 9, 17, 30, 54, 95, ... The growth is exponential with ratio of the order of 1.76, a clear indication that the mapping is not integrable in general. The simplifications that do occur are insufficient to curb the exponential growth. As a matter of fact, if we follow a particular factor we can check that it keeps appearing either in the numerator or the denominator (where its degree is alternatively 1 and 2). This corresponds to the unconfined singularity pattern

$\{0, \infty^2, 0, \infty, 0, \infty^2, 0, \infty, \ldots\}$. Already at the fourth iteration the degrees differ in the autonomous and nonautonomous cases. Our approach consists of requiring that the degree in the nonautonomous case be *identical* to the one obtained in the autonomous one. If we implement the requirement that d_4 be 8 instead of 9 we find the condition $a_{n+1} - 2a_n + a_{n-1} = 0$, that is, precisely the condition obtained through singularity confinement. Here this condition means that q divides Q_7 exactly. Moreover, once this condition is satisfied, the subsequent degrees of the nonautonomous case coincide with those of the autonomous one. For example both q and $r^2 + a_1 qr - pq^2$ divide Q_{12} exactly, leading to $d_5 = 13$ instead of 17 *etc*. Thus the mapping leads to polynomial growth in agreement with its integrable character.

In what follows, we shall not present all the results on discrete Painlevé equations in detail. They can be found in [15]. The important result is that in our analysis of the growth properties of the d-P's, for all cases the nonautonomous forms obtained through singularity confinement led to the same degrees of the iterates as the autonomous form. Moreover, we have been able to study the degree growth of the generic asymmetric QRT mapping and found $d_n = n^2$. It appears that n^2 is the maximal growth one can obtain for the QRT mapping. As a control, we have also checked the degree of growth of the asymmetric nonautonomous q-$\mathrm{P_{VI}}$ equation and found that it led to exactly the same degree growth $d_n = n^2$. These results are of particular significance since they confirm the finding of the singularity confinement in the case of the discrete Painlevé equations.

As we have seen in Section 3 there exists another class of integrable systems which are integrable in a much simpler way: the linearisable mappings. The degree growth considerations can be (and have been) extended to these systems as well, leading to interesting results [21]. Let us start with the generic second-order projective mapping, in canonical form,

$$x_{n+1} = \alpha + \frac{\beta}{x_n} + \frac{1}{x_n x_{n-1}}, \tag{10}$$

where α, β are free functions of n. The calculation of the degree of the iterates of (10) is straightforward (with $x_0 = p$, $x_1 = q/r$, p being assigned zero degree) and we find $d_n = 1$ for $n > 0$. The constancy of the degree can be explained through the fact that (10) can be obtained as the projective reduction of a system of three linear equations.

Let us now turn to the Riccati derivative we encountered in Section 3,

$$x_{n+1} = \frac{x_n^2}{x_{n-1}} + a_n x_n\,. \tag{11}$$

The degree growth of the iterates of mapping (11) are readily obtained. We find, with $x_0 = p$ (p of homogeneity zero), $x_1 = q/r$, the degrees $d_n = n$. In [21], we have studied in detail the growth properties of linearisable second-order mappings. While for the projective system we have $d_n = 1$, in the other linearisable cases studied we have found a linear degree growth. This has led us to the formulation of the conjecture that linearisable systems are characterised by growth slower than that of the generic integrable mapping of the same order: in the case of second order mappings this means linear growth compared to the quadratic growth of the general integrable mapping. Thus a study of the degree growth can give an indication not only of the integrability but also of the way to integrate a mapping.

At this point we must present a tentative answer to the title of the paper. In the light of the recent findings, what plays the role of the Painlevé property in discrete systems could be the slow growth of some characteristic quantity. On the other hand, the findings on linearisable systems tend to indicate that low complexity is related rather to integrability in general than to the particular kind of integrability associated to the Painlevé property (and all the nice analytical properties the latter entails in the continuous case).

The second question we must answer is that of the title of this section, or, rather in a more operational rephrasing, "how can singularity confinement still be useful in the investigation of integrability?" First, we should remark that the implementation of the degree growth, although straightforward, can lead to prohibitively bulky computations, in particular for systems with more than one degree of freedom. Second, the constraints for the limitation of degree growth, whenever we wish to distinguish the integrable subcases of a more general parametrisation, can appear in a pretty intricate way. In contrast, the singularity confinement approach studies each singularity separately and thus the constraints are usually obtained one by one. These considerations have led us to the proposal of the following strategy for the study of integrability of discrete systems. Given a mapping we start by studying it using the singularity confinement method. This leads to a set of constraints which we consider tentatively as a necessary condition for integrability. We implement these constraints and obtain a system with much less freedom than the initial one. We then study the complexity of this system. If we find an exponential growth, this is an indication that the system is not integrable and we investigate whether some further restriction could curb the exponential growth. If the growth is polynomial, this can be considered as an indication of integrability. However, if the growth is even slower than the maximal polynomial compatible with the degree of the mapping, this can be an indication of linearisability. In this case, the constraints of singularity confinement may be too stringent and we must examine whether some may be released while conserving the same rate of degree growth.

We are convinced that this dual approach will prove a most valuable tool for the study of the integrability of discrete systems.

Acknowledgements

The authors wish to express their deepest gratitude to Martin Kruskal who has, over the years, taken the time to initiate them into the mysteries of integrability, to explain them the meaning of what they have been doing and to provide them with invaluable insights.

Several colleagues have participated in this work through direct collaboration and/or exchange of ideas. The list is becoming so lengthy that the probability of forgetting someone, were we to name them explicitly, is approaching unity. We shall not take risks and just acknowledge here the help of all these friends who have accompanied us in our research over the years (they will recognise themselves in these sentences).

References

[1] M. J. Ablowitz, A. Ramani and H. Segur, "Nonlinear evolution equations and ordinary differential equations of Painlevé type", *Lett. Nuov. Cim.* **23** (1978) 333–338.

[2] V. I. Arnold, "Dynamics of complexity of intersections", *Bol. Soc. Bras. Mat.* **21** (1990) 1–10.

[3] M. P. Bellon, J.-M. Maillard and C.-M. Viallet, "Rational mappings, aborescent iterations, and the symmetries of integrability", *Phys. Rev. Lett.* **67** (1991) 1373–1376.

[4] R. Conte and M. Musette, "A new method to test discrete Painlevé equations", *Phys. Lett. A* **223** (1996) 439–448.

[5] G. Falqui and C.-M. Viallet, "Singularity, complexity, and quasi-integrability of rational mappings", *Comm. Math. Phys.* **154** (1993) 111–125.

[6] B. Grammaticos, F. Nijhoff and A. Ramani, "Discrete Painlevé equations", in *The Painlevé Property: One Century Later* (ed. R. Conte), CRM Series in Mathematical Physics, (Springer, New-York, 1999) 413–516.

[7] B. Grammaticos and A. Ramani, "Discrete Painlevé equations: coalescence, limits and degeneracies", *Physica A* **223** (1996) 160–171.

[8] B. Grammaticos and A. Ramani, "Integrability in a discrete world. Integrability and chaos in discrete systems", *Chaos, Solitons and Fractals* **11** (2000) 7–18.

[9] B. Grammaticos, A. Ramani and S. Lafortune, "The Gambier mapping revisited", *Physica A* **253** (1998) 260–270.

[10] B. Grammaticos, A. Ramani and V. Papageorgiou, "Discrete version of the Painlevé equations", *Phys. Rev. Lett.* **67** (1991) 1825–1832.

[11] B. Grammaticos, A. Ramani and K. M. Tamizhmani, "Nonproliferations of pre-images in integrable mappings", *J. Phys. A* **27** (1994) 559–566.

[12] J. Hietarinta and C. Viallet, "Singularity confinement and chaos in discrete systems", *Phys. Rev. Lett.* **81** (1998) 325–28.

[13] N. Joshi, "Singularity analysis and integrability for discrete dynamical systems", *J. Math. An. and Appl.* **184** (1994) 573–584.

[14] M. D. Kruskal, *private communication*.

[15] Y. Ohta, K. M. Tamizhmani, B. Grammaticos and A. Ramani, "Singularity confinement and algebraic entropy: the case of the discrete Painlevé equations", *Phys. Lett. A.* **262** (1999) 152–157.

[16] P. Painlevé, "Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme", *Acta Math.* **25** (1902) 1–82.

[17] V. G. Papageorgiou, F. W. Nijhoff, B. Grammaticos and A. Ramani, "Isomonodromic deformation problems for discrete analogues of Painlevé equations", *Phys. Lett.* **A164** (1992) 57–64.

[18] G. R. W. Quispel, J. A. G. Roberts and C. J. Thompson, "Integrable mapping and soliton equations. II", *Physica D* **34** (1989) 183–192.

[19] A. Ramani, B. Grammaticos and T. Bountis, "The Painlevé property and singularity analysis of integrable and nonintegrable systems", *Phys. Rep.* **180** (1989) 159–245.

[20] A. Ramani, B. Grammaticos and J. Hietarinta, "Discrete versions of the Painlevé equations", *Phys. Rev. Lett.* **67** (1991) 1829–1832.

[21] A. Ramani, B. Grammaticos, S. Lafortune and Y. Ohta, "Linearisable mappings and the low-growth criterion", *J. Phys. A* **33** (2000) L287–L292.

[22] A. Ramani, B. Grammaticos, T. Tamizhmani and K. M. Tamizhmani, "Special function solutions for asymmetric discrete Painlevé equations", *J. Phys. A* **32** (1999) 4553–4562.

[23] A. Ramani, Y. Ohta, J. Satsuma and B. Grammaticos, "Self-duality and Schlesinger chains for the asymmetric d-P_{II} and q-P_{III} equations", *Comm. Math. Phys.* **192** (1998) 67–76.

[24] J. Satsuma, *private communication*, 1992.

[25] A. P. Veselov, "Growth and integrability in the dynamics of mapping", *Comm. Math. Phys.* **145** (1992) 181–193.