NUMERICAL SOLITARY WAVE INTERACTION: THE ORDER OF THE INELASTIC EFFECT

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Abstract

Solitary wave interaction is examined using an extended Benjamin-Bona-Mahony (eBBM) equation. This equation includes higher-order nonlinear and dispersive effects and is is asymptotically equivalent to the extended Korteweg-de Vries (eKdV) equation. The eBBM formulation is preferable to the eKdV equation for the numerical modelling of solitary wave collisions, due to the stability of its finite-difference scheme. In particular, it allows the interaction of steep waves to be modelled, which due to numerical instability, is not possible using the eKdV equation.

Numerical simulations of a number of solitary wave collisions of varying nonlinearity are performed for two special cases corresponding to surface water waves. The mass and energy of the dispersive wavetrain generated by the inelastic collision is tabulated and used to show that the change in solitary wave amplitude after interaction is of $O(\alpha^4)$, verifying previously obtained theoretical predictions.

1. Introduction

The KdV equation is the generic model for the study of weakly nonlinear long waves. It arises in physical systems which involve a balance between nonlinearity and dispersion at leading-order. For example, it describes surface waves of long wavelength and small amplitude on shallow water and internal waves in a shallow density-stratified fluid. Many other applications for the KdV equation also exist, such as plasma waves, Rossby waves and magma flow. The KdV solitary wave solution, which consists of a single humped wave, has a number of special properties. Zabusky and Kruskal [11] numerically examined the nonlinear interaction of a large solitary wave overtaking a smaller one. It was found that the collision was elastic. The solitary waves retained their original shapes with the only memory of the collision being a phase shift. Due to

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this special property, amongst others, the solitary wave solution of the KdV equation is termed a soliton. Gardner *et al.* [4] showed that the KdV equation can be solved exactly using the so-called inverse scattering transform. The explicit solution for interacting KdV solitons was developed using inverse scattering by Hirota [5].

The KdV equation arises as an approximate equation governing weakly nonlinear long waves where first-order nonlinear and dispersive terms are retained and are in balance. If second-order terms are retained the eKdV equation

$$
\begin{aligned}
&\eta_t + \eta_x + 6\alpha\eta\eta_x + \alpha\eta_{xxx} + \alpha^2 c_1 \eta^2 \eta_x + \alpha^2 c_2 \eta_x \eta_{xx} \\
&+ \alpha^2 c_3 \eta\eta_{xxx} + \alpha^2 c_4 \eta_{xxxxx} = 0, \quad \alpha \ll 1
\end{aligned} \tag{1}
$$

results, where α is a non-dimensional measure of the (small) wave amplitude and the coefficients of the higher-order terms are c_1, c_2, c_3 and c_4. This equation describes the evolution of steeper waves of shorter wavelength than does the KdV equation. The values of the higher-order coefficients depend on the physical context. In the special case of surface waves on shallow water the coefficients can be obtained in two different ways. First, the perturbation expansion of the Euler water wave equations can be continued to second order, in which case the coefficients are (see Marchant and Smyth [9])

$$
c_1 = -3/2, \quad c_2 = 23/4, \quad c_3 = 5/2, \quad c_4 = 19/40. \tag{2}
$$

Alternatively, Olver [10] considered the perturbation expansion of the water wave Hamiltonian, which incorporated the correct higher-order expansion of the energy functional and Hamiltonian operator. He found the higher-order coefficients to be

$$
c_1 = 15/8, \quad c_2 = 9/4, \quad c_3 = 3/4, \quad c_4 = 0. \tag{3}
$$

As an alternative to the eKdV equation (1), the model retaining second order dispersive and nonlinear effects can be written as an eBBM equation

$$
\begin{aligned}
&\eta_t + \eta_x + 6\alpha\eta\eta_x - \alpha\eta_{xxt} + \alpha^2 b_1 \eta^2 \eta_x + \alpha^2 b_2 \eta_x \eta_{xt} \\
&+ \alpha^2 b_3 \eta\eta_{xxt} + \alpha^2 b_4 \eta_{xxxxt} = 0, \quad \alpha \ll 1,
\end{aligned} \tag{4}
$$

which can be obtained from the eKdV equation by replacing an x derivative in each of the dispersive terms by a t derivative. This is done by the substitution of the appropriately differentiated lower-order equation. The different scaling of time for the eBBM equation means that the higher-order terms appear at $O(\alpha^2)$ rather than at $O(\alpha)$ as in the eKdV equation (1). The solitary wave solution of both equations is the same, of course, except for a scaling of the velocity.

The eKdV and eBBM equations (1) and (4) are asymptotically equivalent, to $O(\alpha^2)$, if the coefficients are related by

$$
b_1 = c_1, \quad b_2 = 18 - c_2, \quad b_3 = 6 - c_3, \quad b_4 = 1 - c_4. \tag{5}
$$

Hence if (5) holds, a simple linear scaling of time transforms a solution of the eBBM equation (4) to a solution of the eKdV equation (1), and vice versa, to the order of approximation of the equations.

The different properties of the eKdV and eBBM equations (1) and (4) can be seen by examining their linearised dispersion relations

$$\omega = k - \alpha k^3 + \alpha^2 c_4 k^5, \quad \omega = k(1 + \alpha k^2 + \alpha^2 b_4 k^4)^{-1}, \tag{6}$$

respectively. The main defect of the eKdV equation is that the wave speed is not bounded for large wavenumber k. Hence when solved numerically, high-frequency waves of small amplitude, generated by the numerical discretisation, are propagated either forwards or backwards at high speeds. This can cause numerical instabilities to develop and limits the numerical methods available for solving the eKdV equation to the case when the wave amplitude α is small, see Marchant and Smyth [9]. As long as the higher-order coefficient b_4 is not negative the eBBM model has no such defects; the dispersion relation remains finite and small amplitude high-frequency waves are not propagated. Moreover waves are only propagated forwards, in keeping with the unidirectional assumption made in the derivation of the model. For the surface water wave models, (5) with coefficients (2) or (3), the coefficient b_4 is positive, hence the eBBM dispersion relation is physically sensible.

Kodama [6] described a method for the asymptotic transformation of the eKdV equation (1) to a higher-order member of the KdV integrable hierarchy. Hence the eKdV equation is nearly integrable and the solitary waves are asymptotic solitons. Marchant and Smyth [9] used a local variant of the transformation of Kodama [6] to construct the asymptotic two-soliton solution of the eKdV equation (1). The $O(\alpha)$ corrections to the phase shifts of the eKdV solitary waves after collision were also found. Numerical solutions, for small wave amplitude α, showed that a dispersive wavetrain of small amplitude is generated by the collision, indicating inelastic behaviour. Also, a good comparison was found between the numerical and theoretical phase shifts.

Zou and Su [12] considered higher-order interactions of solitary waves on shallow water by using a perturbation expansion of the Euler water wave equations. At first order the solution was assumed to be the KdV two-soliton solution. At second and third order, partial differential equations describing the solitary wave collision were found and solved numerically. At second order (at $O(\alpha)$) the solitary wave collision was found to be elastic, while at third order inelastic effects occurred with a dispersive wavetrain generated behind the smaller solitary wave after collision. For the example presented, the dispersive wavetrain had an amplitude about 10% of the correction to the phase shift at $O(\alpha^2)$. The change in solitary wave amplitude was presumed to be an $O(\alpha^3)$ effect.

Kodama [7] considered solitary wave interaction for a high-order KdV equation.

This was transformed to a higher-order member of the KdV integrable hierarchy plus inelastic terms at $O(\alpha^2)$. The inverse scattering perturbation method was then used to determine the inelastic effect of the collision. It was found that a change in solitary wave amplitude and dispersive wavetrain was generated, of magnitude $O(\alpha^4)$.

The aim of this paper is to examine solitary wave interactions of the eBBM equation and numerically verify that the inelastic effect is $O(\alpha^4)$, as shown theoretically by Kodama [7]. In Section 2 numerical results for eBBM solitary wave collisions, corresponding to surface water waves, are presented. The coefficients for the usual second-order model, (5) with (2), and the Hamiltonian model, (5) with (3), are both used. In particular, the mass and energy of the dispersive wavetrain generated by the inelastic collision is tabulated for a range of wave amplitudes. The logarithms of the data are plotted and it is shown that the inelastic effect is indeed of $O(\alpha^4)$, as predicted theoretically. In the Appendix the numerical scheme used to solve the eBBM equation (4) is detailed.

2. Numerical results

The asymptotic solution of the eBBM equation (4) for a single solitary wave is

$$\eta = aS^2 + \alpha b_5 a^2 S^2 + \alpha b_6 a^2 S^4,$$

$$b_5 = \left(\frac{b_2}{6} - \frac{b_1}{6} + \frac{2b_3}{3} + 5b_4 - 12\right), \quad b_6 = \left(\frac{b_1}{12} - \frac{b_2}{4} - \frac{b_3}{2} - \frac{15b_4}{2}\right), \qquad (7)$$

$$V = 1 + 2a\alpha + \alpha^2 4(1 - b_4)a^2, \qquad\qquad S = \operatorname{sech} k(x - Vt).$$

When a eBBM solitary wave (7) propagates alone, it evolves to a numerically exact solitary wave solution of (4). This evolution, with a change in amplitude of magnitude $O(\alpha^2)$, causes mass and energy to be shed, resulting in the production of a dispersive wavetrain behind the solitary wave. Hence if the form (7) is used for the initial condition then any effects due to the interaction will be obscured by the change in amplitude and the dispersive wavetrain generated by the numerical evolution.

To eliminate this effect, exact eBBM solitary waves are generated by the numerical propagation of a single asymptotic eBBM solitary wave. This procedure was also used by Bona and Chen [1] for isolating numerically "clean" solitary waves of a Boussinesq system. The numerically exact eBBM solitary waves, trimmed to a finite width where the truncated tail has a magnitude less than 1×10^{-8}, are copied to a new numerical grid. The initial condition is then composed of the two numerically exact eBBM solitary waves with the larger one located at $x = 270$ and the smaller one at $x = 315$. The scaled wave amplitudes are chosen to be approximately 1 and 1/3 so the waves interact at time $t \approx 34\alpha^{-1}$ with the calculation continued up to time $t \approx 150\alpha^{-1}$ to

allow the eBBM solitary waves to separate and any dispersive wavetrain to completely form behind the smaller wave.

During an inelastic collision the two solitary waves suffer a change in amplitude and shed mass and energy, which forms a dispersive wavetrain behind the waves after collision. Over much of the parameter range considered in the following examples, the changes in solitary wave amplitudes after collision are too small to be measured directly. They can however be measured indirectly from the dispersive wavetrain using mass and energy conservation, see Marchant [8]. Hence the mass and energy of the dispersive wavetrain shall be used in this paper to estimate the order of the inelastic effect of the collision.

We investigate eBBM solitary wave collisions numerically using both the usual second-order model, (5) with (2), and the Hamiltonian model, (5) with (3). Solitary wave collisions are examined ranging from $\alpha = 0.1$ to $\alpha = 0.35$ and the mass and energy of the dispersive wavetrain generated after the collision is calculated. For both models the mass of the dispersive wavetrain is an $O(10^{-5})$ quantity for $\alpha = 0.1$. It grows to $O(10^{-3})$ for $\alpha = 0.35$ indicating the increased inelastic effect for large α. The amplitude of the largest wave in the dispersive wavetrain ranges from a magnitude of $O(10^{-4})$ to $O(10^{-2})$. For example, for the usual second-order model with $\alpha = 0.1$, the amplitude of the largest wave in the dispersive wavetrain is about 6.3×10^{-4}. Even when the amplitude of the dispersive wavetrain is very small, it is not a numerical artifice. Richardson extrapolation indicates that the error in these estimates is negligible. Also note that no discernible dispersive wavetrain is generated by the propagation of single numerically exact eBBM solitary waves. Hence the dispersive wavetrain is taken as evidence that the interaction of eBBM solitary waves are inelastic, even though no change in solitary wave amplitude could be directly detected after the interaction. In the case of steeper waves ($\alpha \geq 0.3$) the change in amplitude is large enough to be measured directly also. Mass and energy conservation indicates that the larger wave increases and the smaller wave decreases in amplitude, after the collision. The Hamiltonian model predicts changes in solitary wave amplitudes which are larger in magnitude than those from the usual second-order model. For example, with $\alpha = 0.2$, the Hamiltonian model predicts an increase in amplitude of 5.1×10^{-5} for the larger wave and a decrease of 2.8×10^{-4} for the smaller wave. These compare with changes in solitary wave amplitude of 4.8×10^{-5} and 1.7×10^{-5} for the corresponding collision using the usual second-order model.

Figure 1 shows the logarithms of the mass and energy versus the logarithm of the wave amplitude α. In all cases the absolute value of the logarithms is used, so positive quantities are obtained. Shown are the mass and energy of the dispersive wavetrain for the usual second-order model, (5) with (2), (diamonds and plus-signs respectively) and for the Hamiltonian model, (5) with (3), (squares and crosses respectively). As the data for the two cases lie close together, the Hamiltonian model data has been
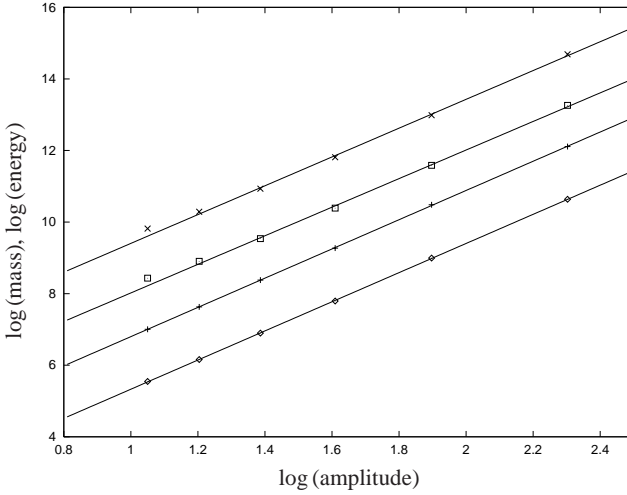
FIGURE 1. The logarithms of mass and energy of the dispersive wavetrain versus wave amplitude

shifted by three units in the vertical direction. The solid lines are fitted to each data set using the method of least squares. For the Hamiltonian model the data from the $\alpha = 0.35$ collision is shown on the figure, but not used for the curve fitting.

It is assumed that the various interaction quantities vary like $k\alpha^n$, hence the slopes of the fitted curves represent numerical estimates for the order of the inelastic effect. For the usual second-order model the slopes of the mass and energy curves are 4.07 and 4.08 respectively. In the case of the Hamiltonian model the corresponding slopes are 3.99 and 4.02. These values are very close to the expected theoretical slope of four, hence these results provide strong numerical confirmation that the inelastic effect of the collisions is $O(\alpha^4)$. For the Hamiltonian model the data from the $\alpha = 0.35$ collision is diverging from the quartic power-law expression. This is because higher-order contributions to the inelastic effect (beyond α^4) can become significant for collisions involving very steep waves.

3. Summary

Numerical simulations have been performed for the collision of solitary waves on shallow water using the eBBM equation. Two choices are used for the higher-order coefficients; those derived from the usual perturbation expansion of the Euler water wave equations and those from a perturbation of the Hamiltonian structure for water waves. Results are presented for a large number of collisions with the wave amplitudes ranging from $\alpha = 0.1$ to 0.35. Numerical evidence is presented that indicates the inelastic effect of the eBBM collision is $O(\alpha^4)$, verifying existing theoretical results.

The eKdV and eBBM models provide asymptotically equivalent descriptions of weakly-nonlinear long wave phenomena. Numerical schemes for the eKdV equation are only stable up to $\alpha \approx 0.1$, limiting their usefulness. Moreover, even at small amplitudes a large space step and small time step must be used to achieve stability. On the other hand, the numerical scheme for the eBBM equation has much more favourable stability properties, allowing the numerical modelling of steeper waves. The eBBM formulation allows both small space and time steps to be used in the numerical method. Hence extremely accurate results can be obtained without the need for excessive computational effort. Also, the high-spatial accuracy delivered by the method is vital to resolve the small dispersive wavetrain generated by the solitary wave collision and hence determine the magnitude of the inelastic effect.

A. The numerical scheme

The numerical solutions of the eBBM equation (4) are obtained by using an implicit, three level, finite-difference scheme with second-order accuracy. Given that the solution at the time t_i

$$\eta_{i,j} = \eta(t_i = i\Delta t, x_j = j\Delta x), \quad j = 1, \ldots, N, \tag{8}$$

and at the previous time step t_{i-1} is known, then the solution at time t_{i+1} is given by

$$
\begin{aligned}
&\alpha^2 \frac{b_4}{\Delta x^4}(\eta_{i+1,j+2} + \eta_{i+1,j-2}) - \alpha\left(\frac{4\alpha b_4}{\Delta x^4} + \frac{1 - \alpha b_3 \eta_{i,j}}{\Delta x^2}\right)(\eta_{i+1,j+1} + \eta_{i+1,j-1}) \\
&+ \alpha^2 \frac{b_2}{4\Delta x^2}(\eta_{i,j+1} - \eta_{i,j-1})(\eta_{i+1,j+1} - \eta_{i+1,j-1}) \\
&+ \left(1 + \alpha^2 \frac{6b_4}{\Delta x^4} + \alpha\frac{2}{\Delta x^2}(1 - \alpha b_3 \eta_{i,j})\right)\eta_{i+1,j} \\
&= \alpha^2 \frac{b_4}{\Delta x^4}(\eta_{i-1,j+2} + \eta_{i-1,j-2}) \\
&\quad - \alpha\left(\frac{4\alpha b_4}{\Delta x^4} + \frac{1 - \alpha b_3 \eta_{i,j}}{\Delta x^2}\right)(\eta_{i-1,j+1} + \eta_{i-1,j-1}) \\
&\quad + \alpha^2 \frac{b_2}{4\Delta x^2}(\eta_{i,j+1} - \eta_{i,j-1})(\eta_{i-1,j+1} - \eta_{i-1,j-1}) \\
&\quad + \left(1 + \alpha^2 \frac{6b_4}{\Delta x^4} + \alpha\frac{2}{\Delta x^2}(1 - \alpha b_3 \eta_{i,j})\right)\eta_{i-1,j} \\
&\quad - \frac{\Delta t}{\Delta x}(\eta_{i,j+1} - \eta_{i,j-1})(1 + 6\alpha\eta_{i,j} + \alpha^2 b_1 \eta_{i,j}^2), \quad j = 1, \ldots, N, \tag{9}
\end{aligned}
$$

where Δt and Δx are the time and space steps. Equation (9) requires the solution of a penta-diagonal matrix at each time step. A fast algorithm for this task is detailed in

(Conte and deBoor [2]). When all the higher-order coefficients of the eBBM equation (4) are zero it reduces to the BBM equation. The finite-difference scheme (9) then becomes that of Eilbeck and McGuire [3] for the BBM equation. In this special case a tri-diagonal matrix needs to be solved at each time step.

Eilbeck and McGuire [3] show that their scheme is stable for all practical choices of Δt and Δx. The scheme (9) for the eBBM equation is similarly stable, if the higher-order coefficient b_4 is non-negative. For b_4 negative the scheme is only stable for very small values of Δx. This is related to the form of the linear dispersion relation (the second equality of (6)).

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