ANZIAM J. 45(2003), 75–89

A NOTION OF LOCAL PROPER EFFICIENCY IN THE BORWEIN SENSE IN VECTOR OPTIMISATION

B. JIMÉNEZ[1] and V. NOVO[2]

(Received 16 March, 2001; revised 1 June, 2001)

Abstract

In this paper we define two types of proper efficient solutions in the Borwein sense for vector optimisation problems and we compare them with the notions of local Borwein, Ishizuka-Tuan, Kuhn-Tucker and strict efficiency. A sufficient condition for a proper solution is also proved.

1. Introduction

In many different fields, such as economics, management science, engineering, industry or operations research, there arise problems in which various functions (objectives) are to be simultaneously optimised. This is why we need tools for nonlinear programming capable of handling several conflicting objectives. In this case, methods of traditional single objective optimisation are not enough, and we need the new concepts and methods of nonlinear multiobjective optimisation. This area is undergoing rapid development and its importance can be seen from the large variety of applications presented in the literature (see Miettinen [18] and the references therein).

One of the main aims of vector optimisation theory is the determination of all the efficient points for a problem. However, this is not always enough, and we can select solutions which are better in some sense. These are the proper efficient solutions. In the finite-dimensional case, the idea of proper Pareto optimal solutions is that unbounded trade-offs between objectives are not allowed. Practically, a proper Pareto optimal solution with very high or very low trade-offs does not essentially differ from a weak Pareto optimal solution for a human decision maker.

[1]Department of Applied Mathematics, UNED, Apartado 60149 (28080) Madrid, Spain; e-mail: bjimen1@encina.pntic.mec.es.
[2]Department of Applied Mathematics, UNED, Apartado 60149 (28080) Madrid, Spain; e-mail: vnovo@ind.uned.es.

The first notion of *proper efficiency* was introduced by Kuhn and Tucker [15] in their well-known work about nonlinear programming and many other notions have been proposed since then. The best-known are those by Hurwicz [9], Geoffrion [6], Borwein [2, 3], Benson [1] and Henig [8]. The reader is referred to [20] for a good presentation of these notions in the finite-dimensional case and to [7, 17] for the infinite-dimensional case and for a comparison between them.

There are two main motivations for introducing proper efficiency. First of all, it makes possible the exclusion of some efficient solutions with undesirable properties, as was observed by Klinger [14] in setting up the starting point of Geoffrion's definition. Secondly, it allows us to set up equivalent scalar problems whose solutions produce most of the optimal solutions, that is, the proper ones.

In this paper, two notions of proper efficiency in the Borwein sense for vector optimisation problems are introduced and they are compared with four others: one was considered by Borwein [3], another two were introduced by Ishizuka and Tuan [10] and the fourth one was introduced in [12]. Section 3 is devoted to the first two definitions, studying implications between them under different assumptions. In Section 4 we introduce two notions of local efficiency in the Borwein sense and the relationships between these definitions and the four aforementioned ones are discussed. A sufficient condition for the existence of proper local efficient solutions for Hadamard directionally differentiable functions is also presented.

2. Preliminaries

Let E_1, E_2 be real normed linear spaces, $S \subset E_1$, $Y \subset E_2$ and $f : E_1 \to E_2$.

The cone generated by S is denoted by $\operatorname{cone} S = \{\lambda x : x \in S,\ \lambda \geq 0\}$ and the convex hull of S, by $\operatorname{co} S$.

Throughout this work, D denotes a cone in E_2 and we will assume its vertex lies at $0 \in D$. We do not suppose that D is convex, consequently the order defined by D in E_2 is not transitive. Recall that D is said to be pointed if $D \cap (-D) = \{0\}$. Cones that are not necessarily pointed have often been considered by authors (see for instance, Borwein [3], Ishizuka and Tuan [10], Khanh [13]), although some authors only consider pointed cones [7, 17]. In the present work, we suppose that the cone D is not pointed, although we need a pointed cone D to obtain the main results.

Let us denote by $T(Y, y_0)$ the tangent cone to Y at $y_0 \in \operatorname{cl} Y$ ($\operatorname{cl} Y$ is the closure of Y), that is, the set of limits of the form

$$v = \lim_{n\to\infty} \lambda_n(y_n - y_0),$$

where (λ_n) is a sequence of positive real numbers and (y_n) is a sequence in Y with limit y_0.

It is said that the subset $\mathscr{B}$ of D is called a base for the cone D if $0 \notin \operatorname{cl} \mathscr{B}$ and every $d \in D \setminus \{0\}$ has an unique representation as $d = \lambda b$, with $\lambda > 0$ and $b \in \mathscr{B}$. Following Luc [16, Definition 1.5] we do not assume that $\mathscr{B}$ is convex.

The existence of a base for a cone has some relevant consequences for the cone itself.

REMARK 2.1. (1) If D has a closed bounded base, then D is closed [16, Proposition 1.7].
(2) If D has a convex base, then D is convex and pointed [11, Lemma 1.14]. Therefore if D has a convex compact base, then D is convex, closed and pointed.
(3) In this paper, we frequently use a cone D with a convex compact base. In this case, we point out that if E_2 is a normed vector lattice with positive cone D, then E_2 is finite dimensional (Dauer and Gallagher, [5, Theorem 3.1]).

For $y_0 \in Y$ and D fixed, various notions of optimality are defined as follows.

The point $y_0 \in Y$ is an *efficient* element of Y (with respect to D), denoted $y_0 \in \operatorname{Min}(Y, D)$, if there exists no $y \in Y$ for which $y_0 - y \in D \setminus (-D)$. Such a point is also called minimal or Pareto optimal. The point y_0 is a *weak efficient* element of Y, written $y_0 \in \operatorname{WMin}(Y, D)$, if there exists no $y \in Y$ satisfying $y_0 - y \in \operatorname{int} D \setminus (-\operatorname{int} D)$. The point y_0 is called a *local efficient* element of Y, denoted $y_0 \in \operatorname{LMin}(Y, D)$, if it has a neighbourhood V such that y_0 is an efficient element of $Y \cap V$. A *local weak efficient* element is defined similarly. It will be denoted $y_0 \in \operatorname{LWMin}(Y, D)$.

With respect to the weak efficient elements, we remark that D is assumed to have a nonempty interior.

Obviously, it is straightforward to verify that

$$\operatorname{Min}(Y, D) \subset \left\{ \begin{matrix} \operatorname{LMin}(Y, D) \\ \operatorname{WMin}(Y, D) \end{matrix} \right\} \subset \operatorname{LWMin}(Y, D).$$

The weak notions are the same as the non weak ones if the cone D satisfies $D = \operatorname{int} D \cup \{0\}$.

Let $f : E_1 \to E_2$ be a function and $S \subset E_1$. The vector optimisation problem considered here is

$$\operatorname{Min}\{ f(x) : x \in S \}, \tag{VP}$$

that is, the problem of determining all $x_0 \in S$ for which $f(x_0) \in \operatorname{Min}(f(S), D)$. Such an x_0 is called an efficient (or minimal) solution for (VP); it will be denoted $x_0 \in \operatorname{Min}(f, S)$. Finally, it is said that x_0 is a local efficient solution for (VP) if $f(x_0) \in \operatorname{Min}(f(S \cap U), D)$, for some neighbourhood U of x_0. Notice that this is not equivalent to $f(x_0) \in \operatorname{LMin}(f(S), D)$. A local weak efficient solution is defined similarly. The Hadamard directional derivative of f at $x_0 \in E_1$ in the direction $v \in E_1$ is defined to be

$$f'(x_0, v) = \lim_{(t,u)\to(0^+,v)} \frac{f(x_0 + tu) - f(x_0)}{t}.$$

The function f is called Hadamard directionally differentiable at a point x_0 if $f'(x_0, v)$ exists and is finite for all $v \in E_1$.

3. Local Borwein proper efficiency

In this section various notions of local efficiency in the Borwein sense are discussed. The following is a well-known definition due to Borwein [2].

DEFINITION 3.1 (*Borwein proper efficiency*). The point $y_0 \in Y$ is called a Borwein proper efficient element, written $y_0 \in \mathrm{Bor}(Y, D)$, if

$$T(Y + D, y_0) \cap (-D) \subset D. \tag{3.1}$$

Borwein's original definition [2] requires that the point y_0 be an efficient element of Y, but Sawaragi *et al*. [20, Proposition 3.1.5] show that if $E_2 = \mathbb{R}^p$ and D is convex and closed, then such a condition is unnecessary. In the next proposition this result is generalised.

PROPOSITION 3.2. *If $y_0 \in Y$ is a Borwein proper efficient element, then y_0 is an efficient element of Y, that is,* $\mathrm{Bor}(Y, D) \subset \mathrm{Min}(Y, D)$.

PROOF. If $y_0 \notin \mathrm{Min}(Y, D)$, then there exists $y \in Y$ such that $y - y_0 \in (-D) \setminus D$. Then the segment $[y, y_0] \subset Y + D$, since for all $\alpha \in [0, 1]$ we have

$$y_\alpha = \alpha y_0 + (1 - \alpha)y = y + \alpha(y_0 - y) \in Y + D.$$

Obviously $y_\alpha = y_0 + (1 - \alpha)(y - y_0)$. Let (y_n) be the sequence defined by taking $\alpha = 1 - 1/n$, that is,

$$y_n = y_0 + \frac{1}{n}(y - y_0) \in Y + D,$$

therefore $y_n \to y_0$, and $\lim_{n\to\infty} n(y_n - y_0) = y - y_0 \in T(Y + D, y_0) \cap [(-D) \setminus D]$ in contradiction to (3.1).

We consider now two local definitions of proper efficiency.

DEFINITION 3.3 (*Local Borwein proper efficiency*, Borwein [3, Definition 2 (a)]). The point $y_0 \in Y$ is said to be a local Borwein proper efficient element, denoted $y_0 \in \mathrm{LBor}(Y, D)$, if it is a local efficient element of Y and

$$T(Y, y_0) \cap (-D) \subset D. \tag{3.2}$$

Guerraggio *et al.* use a slightly different notion [7, Definition 6.5].

It will be proved in Proposition 3.5 that if the cone D is pointed and has a compact base, then it is not necessary to require that y_0 be a local efficient point of Y because this is implied by condition (3.2). To prove this result, we need the following lemma.

LEMMA 3.4. *Let D be a cone with a compact base. If the sequence $\{d_n\} \subset D$ and $\|d_n\| = 1$, then there is a subsequence which converges to some $d \in D$ with $\|d\| = 1$.*

PROOF. Let $\mathscr{B}$ be a compact base of D. We have that $d_n = \lambda_n b_n$ with $\lambda_n > 0$ and $b_n \in \mathscr{B}$. By the compactness of $\mathscr{B}$, there exists a subsequence $\{b_k\}$ convergent to some $b \in \mathscr{B}$, $b \neq 0$. Thus $\|d_k\| = \lambda_k \|b_k\|$, $\lambda_k = \|d_k\|/\|b_k\| = 1/\|b_k\|$ and $\lim_{k\to\infty} \lambda_k = 1/\|b\|$.

Consequently $\lim_{k\to\infty} d_k = \lim_{k\to\infty} b_k/\|b_k\| = b/\|b\|$. Since D is closed, we have $d = b/\|b\| \in D$.

PROPOSITION 3.5. *Let D be a pointed cone with a compact base and $y_0 \in Y$. If $T(Y, y_0) \cap (-D) = \{0\}$, then y_0 is a local efficient element of Y.*

PROOF. Suppose that $y_0 \notin \mathrm{LMin}(Y, D)$. Then $y_0 \notin \mathrm{Min}(Y \cap B(y_0, 1/n), D)$ for all $n \in \mathbb{N}$. Therefore there exists $y_n \in Y \cap B(y_0, 1/n)$ such that $y_n - y_0 \in (-D) \setminus \{0\}$. It follows that $y_n \to y_0$.

Let $d_n = (y_0 - y_n)/\|y_0 - y_n\|$, then $d_n \in D$ and $\|d_n\| = 1$. From Lemma 3.4, taking a subsequence, if necessary, we may assume that

$$\lim_{n\to\infty} \frac{y_n - y_0}{\|y_n - y_0\|} = -d \in -D \quad \text{with} \quad \|d\| = 1.$$

Then, by definition, $-d \in T(Y, y_0)$. But, by the hypotheses, $T(Y, y_0) \cap (-D) = \{0\}$. Thus $d = 0$, which is a contradiction.

Another notion of local proper efficiency was introduced by Ishizuka and Tuan [10, Definition 3.5].

DEFINITION 3.6 (*Local Borwein proper efficiency in the sense of Ishizuka-Tuan*). The point $y_0 \in Y$ is called a local IT-proper efficient element, written $y_0 \in \mathrm{IT}(Y, D)$, if there exists a neighbourhood V of y_0 such that

$$T(Y \cap V + D, y_0) \cap (-D) \subset D. \tag{3.3}$$

Clearly, $\mathrm{Bor}(Y, D) \subset \mathrm{IT}(Y, D)$.

It is proved by Ishizuka and Tuan [10, Proposition 3.1] that if f is continuous at x_0 and $y_0 = f(x_0)$ verifies (3.3) for $Y = f(S)$, then x_0 is a local efficient solution for (VP). We actually prove that from (3.3) it follows that y_0 is a local efficient element of $f(S)$, and then Ishizuka and Tuan's result follows from this.

PROPOSITION 3.7. *If y_0 is a local IT-proper efficient element, then y_0 is a local efficient element. That is,* $\mathrm{IT}(Y, D) \subset \mathrm{LMin}(Y, D)$.

PROOF. By definition, there exists a neighbourhood V of y_0 such that (3.3) holds, that is, $y_0 \in \mathrm{Bor}(Y \cap V, D)$. Then, from Proposition 3.2, $y_0 \in \mathrm{Min}(Y \cap V, D)$, therefore $y_0 \in \mathrm{LMin}(Y, D)$.

The next corollary follows from this proposition and from Result 4.1 in Corley [4].

COROLLARY 3.8. *Let $f : E_1 \to E_2$ be continuous at $x_0 \in S \subset E_1$. If $f(x_0)$ is a local IT-proper efficient element of $f(S)$, then x_0 is a local efficient solution for* (VP).

Now we turn our attention to the relationships between the two notions.

THEOREM 3.9. (a) $\mathrm{IT}(Y, D) \subset \mathrm{LBor}(Y, D)$.

(b) *If D is a pointed convex cone with a compact base, then* $\mathrm{LBor}(Y, D) = \mathrm{IT}(Y, D)$.

PROOF. (a) Let y_0 be a local IT-proper efficient element, then there exists a neighbourhood V of y_0 such that $T(Y \cap V + D, y_0) \cap (-D) \subset D$. By Proposition 3.7, y_0 is a local efficient element.

Since $Y \cap V \subset Y \cap V + D$, it follows that

$$T(Y, y_0) = T(Y \cap V, y_0) \subset T(Y \cap V + D, y_0),$$

and therefore that $T(Y, y_0) \cap (-D) \subset D$, that is, y_0 is a local Borwein proper efficient element.

(b) Let us prove that $\mathrm{LBor}(Y, D) \subset \mathrm{IT}(Y, D)$. Suppose that y_0 is a local Borwein proper efficient element, but that it is not a local IT-proper efficient element. Then for each $B(y_0, 1/n)$ there exists $-d_n \in T(Y \cap B(y_0, 1/n) + D, y_0) \cap (-D)$ and $d_n \neq 0$. We can suppose that $\|d_n\| = 1$ since the sets $T(Y \cap B(y_0, 1/n) + D, y_0)$ and D are cones.

From Lemma 3.4, taking a subsequence if necessary, we deduce that

$$\lim_{n\to\infty} d_n = d \in D \ \text{ with } \ \|d\| = 1 \quad \text{and} \quad -d \in T(Y \cap B(y_0, 1) + D, y_0)$$

since this last set is closed and

$$-d_n \in T(Y \cap B(y_0, 1/n) + D, y_0) \subset T(Y \cap B(y_0, 1) + D, y_0).$$

From the definition of a tangent cone, for each n there exist

$$y_{n,k} \in Y \cap B(y_0, 1/n) \quad \text{and} \quad d_{n,k} \in D, k = 1, 2, \dots, \tag{3.4}$$

such that

$$y_{n,k} + d_{n,k} \to y_0 \quad \text{when } k \to \infty \tag{3.5}$$

and

$$\lim_{k\to\infty} \lambda_{n,k}(y_{n,k} + d_{n,k} - y_0) = -d_n \quad \text{with } \lambda_{n,k} > 0. \tag{3.6}$$

From (3.6), given $\varepsilon = 1/n$, there exists $k_n \in \mathbb{N}$ such that

$$\|\lambda_{n,k_n}(y_{n,k_n} + d_{n,k_n} - y_0) + d_n\| < 1/n.$$

Therefore

$$\lim_{n\to\infty} \lambda_{n,k_n}(y_{n,k_n} + d_{n,k_n} - y_0) = -d. \tag{3.7}$$

Set $y_n = y_{n,k_n}$, $d'_n = d_{n,k_n}$ and $\lambda_n = \lambda_{n,k_n}$. From (3.4) it follows that $\lim_{n\to\infty} y_n = y_0$. From this result and from (3.5) we have $\lim_{n\to\infty} d'_n = 0$.

We rewrite (3.7) in the following form:

$$\lim_{n\to\infty} \big(\lambda_n(y_n - y_0) + \lambda_n d'_n\big) = -d. \tag{3.8}$$

The sequence $\bar{d}_n = \lambda_n d'_n$ which appears in (3.8) may be: (i) bounded or (ii) unbounded.

Case (i). Let $\{\bar{d}_n\}$ be bounded. From Lemma 3.4, we may suppose that

$$\lim_{n\to\infty} \frac{\bar{d}_n}{\|\bar{d}_n\|} = \bar{d} \in D \quad \text{with} \quad \|\bar{d}\| = 1. \tag{3.9}$$

Since $\|\bar{d}_n\|$ is bounded, there exists a subsequence which converges. Then we suppose that $\lim_{n\to\infty} \|\bar{d}_n\| = \lambda$. Hence $\lim_{n\to\infty} \bar{d}_n = \lambda\bar{d}$. From (3.8) it follows that $\lim_{n\to\infty} \lambda_n(y_n - y_0) = v \in T(Y, y_0)$. Then $v + \lambda\bar{d} = -d$, that is, $v = -d - \lambda\bar{d} \in -D$, since D is convex. From the hypotheses, $T(Y, y_0) \cap (-D) = \{0\}$. Hence $v = 0$ and $d \in D \cap (-D) = \{0\}$, that is, a contradiction.

Case (ii). If $\{\bar{d}_n\}$ is unbounded, taking a subsequence, let us assume that

$$\lim_{n\to\infty} \|\bar{d}_n\| = +\infty.$$

From Lemma 3.4, (taking a subsequence, if necessary) we have that (3.9) holds. From (3.8), it follows that $\lim_{n\to\infty}[\lambda_n(y_n - y_0) + \bar{d}_n]/\|\bar{d}_n\| = 0$. Hence

$$\lim_{n\to\infty} \left(\frac{\lambda_n}{\|\bar{d}_n\|}(y_n - y_0) + \frac{\bar{d}_n}{\|\bar{d}_n\|}\right) = 0.$$

Therefore $\lim_{n\to\infty} \lambda_n(y_n - y_0)/\|\bar{d}_n\| = -\bar{d} \in T(Y, y_0) \cap (-D) = \{0\}$, from the hypotheses, and we again have a contradiction and the theorem is proved.

4. Local proper efficiency

In this section two new notions of a local proper efficient solution are proposed and we study the relationships between them, those from the previous section and others. We give a sufficient condition which is very close to a necessary condition for a point to be an efficient element.

In the rest of the paper, the cone D is pointed.

DEFINITION 4.1. (1) The point $x_0 \in S$ is said to be a local proper Borwein efficient solution of type 1 for problem (VP), written $x_0 \in \mathrm{Bor}_1(f, S)$, if there exists a neighbourhood U of x_0 such that

$$T(f(S \cap U) + D,\ f(x_0)) \cap (-D) = \{0\}. \tag{4.1}$$

(2) The point $x_0 \in S$ is said to be a local proper Borwein efficient solution of type 2 for problem (VP), written $x_0 \in \mathrm{Bor}_2(f, S)$, if x_0 is a local efficient solution for (VP) and there exists a neighbourhood U of x_0 such that

$$T(f(S \cap U),\ f(x_0)) \cap (-D) = \{0\}. \tag{4.2}$$

From Proposition 3.5, if D is a cone with a compact base, then (4.2) yields $f(x_0) \in \mathrm{LMin}(f(S \cap U), D)$.

PROPOSITION 4.2. (a) $x_0 \in \mathrm{Bor}_2(f, S)$ *if and only if there exists* U*, a neighbourhood of* x_0*, such that* $f(x_0) \in \mathrm{Min}(f(S \cap U), D)$ *and* $T(f(S \cap U),\ f(x_0)) \cap (-D) = \{0\}$.

(b) *If* $f(x_0) \in \mathrm{IT}(f(S), D)$ *and* f *is continuous at* x_0*, then* $x_0 \in \mathrm{Bor}_1(f, S)$.

(c) *If* $f(x_0) \in \mathrm{LBor}(f(S), D)$*, then* $x_0 \in \mathrm{Bor}_2(f, S)$.

(d) *If* $x_0 \in \mathrm{Bor}_1(f, S)$ *and* f *is continuous at* x_0*, then there exists a neighbourhood* U *of* x_0 *such that* $f(x_0) \in \mathrm{IT}(f(S \cap U), D)$.

(e) *If* $x_0 \in \mathrm{Bor}_2(f, S)$*, then there exists a neighbourhood* U *of* x_0 *such that* $f(x_0) \in \mathrm{LBor}(f(S \cap U), D)$.

(f) *If* $x_0 \in \mathrm{Bor}_1(f, S)$ *and* f *is continuous at* x_0*, then* $x_0 \in \mathrm{Bor}_2(f, S)$.

(g) *If* $x_0 \in \mathrm{Bor}_2(f, S)$*,* f *is continuous at* x_0 *and* D *is a convex cone with a compact base, then* $x_0 \in \mathrm{Bor}_1(f, S)$.

PROOF. Let us suppose that $f(x_0) = y_0$, we have:

(a) If $x_0 \in \mathrm{Bor}_2(f, S)$, then there exist two neighbourhoods U' and U'' of x_0 such that

$$y_0 \in \mathrm{Min}(f(S \cap U'), D) \quad \text{and} \quad T(f(S \cap U''), y_0) \cap (-D) = \{0\}.$$

Taking $U = U' \cap U''$, we obtain this result. The converse is evident.

(b) From the hypotheses, there exists a neighbourhood V of y_0 such that

$$T(f(S)\cap V+D, y_0)\cap(-D)=\{0\}. \tag{4.3}$$

Due to the continuity of f, there exists a neighbourhood U of x_0 such that $f(U)\subset V$. Hence $f(S\cap U)\subset f(S)\cap V$ and therefore $f(S\cap U)+D\subset f(S)\cap V+D$. Then $T(f(S\cap U)+D, y_0)\subset T(f(S)\cap V+D, y_0)$. From (4.3), it follows that $T(f(S\cap U)+D, y_0)\cap(-D)=\{0\}$, that is, $x_0\in \mathrm{Bor}_1(f,S)$.
(c) It is clear.
(d) From the hypotheses, there exists a neighbourhood U' of x_0 such that

$$T(f(S\cap U')+D, y_0)\cap(-D)=\{0\}. \tag{4.4}$$

Given the neighbourhood of y_0, $V=\{y\in E_2 : \|y-y_0\|<1\}$, by continuity we have that $f(U'')\subset V$ for some neighbourhood U'' of x_0. If we define $U=U'\cap U''$, then $f(S\cap U)\subset V$ and therefore $f(S\cap U)\cap V=f(S\cap U)\subset f(S\cap U')$. Hence from (4.4) it follows that $T(f(S\cap U)\cap V+D, y_0)\cap(-D)=\{0\}$.
(e) It follows from (a).
(f) From (d), we have $y_0\in \mathrm{IT}(f(S\cap U), D)$. Besides, from Theorem 3.9 (a) it follows that $y_0\in \mathrm{LBor}(f(S\cap U), D)$, and from (c) that $x_0\in \mathrm{Bor}_2(f,S)$.
(g) To prove this result, we apply successively (e), Theorem 3.9 (b) and (b).

Hence the two notions introduced in Definition 4.1 are equivalent for f continuous at x_0 and D a convex cone with a compact base.

The next example shows that the converses of (b) and (c) in the above proposition are false.

Example 4.3. Let $E_1=E_2=\mathbb{R}^2$, $D=\mathbb{R}^2_+$ the usual cone, the function $f(x,y)=(x-x^2, y-y^2)$, the set $S=\{(x,y) : 0\le x\le 2, 0\le y\le 2\}$ and the point $x_0=(0,0)$. In this case, $T(S,x_0)=\{(x,y) : x\ge 0, y\ge 0\}$ and $f(S)$ contains the sets $[-2,1/4]\times\{0\}$, $\{0\}\times[-2,1/4]$ and $\{(x,x) : -2\le x\le 1/4\}$, which are the images of the subsets of S $\{(x,0) : 0\le x\le 2\}$, $\{(0,y) : 0\le y\le 2\}$ and $\{(x,x) : 0\le x\le 2\}$, respectively. Then $y_0=f(x_0)$ is not an efficient element of $Y=f(S)$, nor is it a local efficient element of Y, that is, $y_0\notin \mathrm{Min}(Y\cap V, D)$ for all neighbourhoods V of y_0.

However, x_0 is a local efficient solution for (VP). It is sufficient to take U, the neighbourhood of x_0, to be $\{(x,y) : -1<x<1, -1<y<1\}$. Then $f(S\cap U)\subset[0,1]\times[0,1]$, and here y_0 is an efficient element: $y_0\in \mathrm{Min}(f(S\cap U), D)$.

Clearly, $y_0\notin \mathrm{LBor}(f(S), D)=\mathrm{IT}(f(S), D)$. However, we have that $x_0\in \mathrm{Bor}_1(f,S)=\mathrm{Bor}_2(f,S)$ since (4.1) holds for the neighbourhood U above.

General optimality conditions for a point to be an efficient element have been collected by Corley [4] considering an arbitrary set Y. In Corley's Theorem 3.1 the following is established:

(a) *Let D be a closed cone in the finite-dimensional space E_2. If $y_0 \in Y$ and $T(Y, y_0) \cap (-D) = \{0\}$ then $y_0 \in \mathrm{LMin}(Y, D)$.*

Corley points out that it is an open question whether the closedness of D or the finite dimensionality of E_2 can be relaxed. We have shown in Proposition 3.5 that (a) holds for E_2 being infinite dimensional but requires that D has a compact base. The next example shows that the closedness and the existence of a compact base cannot be relaxed.

EXAMPLE 4.4. (a) Here D is not closed. Let $E_2 = \mathbb{R}^2$, $D = (\mathrm{int}\, \mathbb{R}^2_+) \cup \{0\}$, $Y = \{(x, y) : y \geq -x^2\}$ and $y_0 = (0, 0)$. It is easy to verify that $T(Y, y_0) \cap (-D) = \{0\}$ and $y_0 \notin \mathrm{LMin}(Y, D)$.

(b) Here D is closed without a compact base. Let $E_2 = E$ be a Hilbert space with an orthonormal base $\mathscr{B} = \{e_n : n \in \mathbb{N}\}$. Let

$$D = \{d = (\alpha_n) \in E : \alpha_n \geq 0\ \forall n \in \mathbb{N}\} = \mathrm{cl}\,\mathrm{co}\,\mathrm{cone}\, \mathscr{B}.$$

Here D is a pointed, closed, convex cone but it has no compact base. Let $Y = D \cup A$, where $A = \{(-1/n)e_n : n \in \mathbb{N}\}$ and $y_0 = 0$.

We have $T(A, y_0) = \{0\}$. In fact, take $v \in T(A, y_0)$. Then, since

$$\lim_{n\to\infty} (-1/n)e_n = 0 = y_0,$$

there exists a subsequence and $\lambda_n > 0$ such that $\lim_{n\to\infty} \lambda_n(-1/k_n)e_{k_n} = v$.

Therefore, for each $i \in \mathbb{N}$, $\lim_{n\to\infty} \lambda_n(-1/k_n)\langle e_{k_n}, e_i\rangle = \langle v, e_i\rangle$. But the sequence within the limit is null (except at the most for a single term). Hence $\langle v, e_i\rangle = 0$ for all $i \in \mathbb{N}$ and it follows that $v = 0$. Then $T(Y, y_0) = T(D, y_0) \cup T(A, y_0) = D$ and $T(Y, y_0) \cap (-D) = \{0\}$, that is, the hypotheses of (a) are verified. However, $y_0 \notin \mathrm{LMin}(Y, y_0)$ because for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$(-1/n)e_n - y_0 \in (Y \cap B(y_0, \varepsilon) - y_0) \cap (-D).$$

Corley [4, Result 4.2] also establishes the following necessary condition for a point to be an efficient solution:

Let $f : E_1 \to E_2$ be Fréchet differentiable at $x_0 \in S \subset E_1$ and D a pointed cone. If x_0 is an efficient solution for (VP), then $\nabla f(x_0)(T(S, x_0)) \cap (-\mathrm{int}\, D) = \emptyset$, where $\nabla f(x_0)$ is the Fréchet differential of f at x_0.

This result can easily be generalised for f Hadamard directionally differentiable at x_0 and x_0 a local weak efficient solution for (VP).

In fact, by the hypothesis there exists a neighbourhood U of x_0 such that

$$f(x_0) = y_0 \in \mathrm{Min}(f(S \cap U), (\mathrm{int}\, D) \cup \{0\}).$$

Hence $T(f(S \cap U), y_0) \cap (-\operatorname{int} D) = \emptyset$ [4, Theorem 3.1(a) and remark of page 75]. Now, since $T(S \cap U, x_0) = T(S, x_0)$, then

$$f'(x_0, \cdot)(T(S, x_0)) = f'(x_0, \cdot)(T(S \cap U, x_0)) \subset T(f(S \cap U), y_0).$$

Therefore $f'(x_0, \cdot)(T(S, x_0)) \cap (-\operatorname{int} D) = \emptyset$. Note that, if $v \in T(S, x_0)$, then $f'(x_0, v) \in T(f(S), f(x_0))$. In fact, we have $v_n = (x_n - x_0)/\lambda_n \to v$, with $\lambda_n \to 0^+$ and $x_n = x_0 + \lambda_n v_n \in S$. Then

$$\lim_{n\to\infty} \frac{f(x_0 + \lambda_n v_n) - f(x_0)}{\lambda_n} = f'(x_0, v) \in T(f(S), f(x_0)).$$

Related to Corley's result, we have the next sufficient condition for a point to be a local proper Borwein efficient solution of type 2 (= type 1 in this case if D also has a convex compact base).

THEOREM 4.5. *Let $x_0 \in S \subset E_1$, E_1 finite dimensional, $D \subset E_2$ a cone with compact base, $f : E_1 \to E_2$ Hadamard directionally differentiable at x_0 and*

$$C(f') = \{v \in E_1 : f'(x_0, v) \in -D\}.$$

If $T(S, x_0) \cap C(f') = \{0\}$, then $x_0 \in \operatorname{Bor}_2(f, S)$.

PROOF. Let $f(x_0) = y_0$. It is sufficient to prove (4.2). In fact, from Proposition 3.5, $y_0 \in \operatorname{LMin}(f(S \cap U), D)$. Hence, by definition, $y_0 \in \operatorname{LBor}(f(S \cap U), D)$. By Proposition 4.2 (c), $x_0 \in \operatorname{Bor}_2(f, S \cap U) = \operatorname{Bor}_2(f, S)$.

Let us prove (4.2). If $T(S, x_0) = \{0\}$, then x_0 is an isolated point. Hence there exists a neighbourhood U of x_0 such that $S \cap U = \{x_0\}$. Therefore $T(f(S \cap U), y_0) = \{0\}$ and the conclusion is true.

Let $T(S, x_0) \neq \{0\}$. Then, for all $\varepsilon > 0$, $T(S \cap B(x_0, \varepsilon), x_0) \neq \{0\}$. Suppose that (4.2) is false. Then for every n we have

$$T(f(S \cap B(x_0, 1/n)), y_0) \cap (-D) \neq \{0\}.$$

Hence there exists

$$-d_n \in T(f(S \cap B(x_0, 1/n)), y_0) \cap (-D) \quad \text{with} \quad d_n \neq 0. \tag{4.5}$$

We can assume that $\|d_n\| = 1$, $d_n \to d$, $\|d\| = 1$, $d \in D$ and $-d \in T\big(f(S \cap B(x_0, 1)), y_0\big)$ since $-d_n \in T(f(S \cap B(x_0, 1)), y_0)$ for every n. From (4.5), there exist

$$x_{n,k} \in B(x_0, 1/n), \quad k = 1, 2, \ldots \tag{4.6}$$

such that

$$\lim_{k\to\infty} \frac{f(x_{n,k}) - f(x_0)}{\lambda_{n,k}} = -d_n \quad \text{with} \quad \lambda_{n,k} = \|f(x_{n,k}) - f(x_0)\| > 0. \tag{4.7}$$

From (4.7), given $\varepsilon = 1/n$, there exists $k_n \in \mathbb{N}$ such that

$$\left\| \frac{f(x_{n,k_n}) - f(x_0)}{\lambda_{n,k_n}} + d_n \right\| < \frac{1}{n}.$$

Therefore

$$\lim_{n\to\infty} \frac{f(x_{n,k_n}) - f(x_0)}{\lambda_{n,k_n}} = -d. \tag{4.8}$$

Set $x_n = x_{n,k_n} \in S \cap B(x_0, 1/n)$ and $\lambda_n = \lambda_{n,k_n} > 0$. Then (4.8) amounts to $\lim_{n\to\infty}(f(x_n) - f(x_0))/\lambda_n = -d$. From (4.6), $\lim_{n\to\infty} x_n = x_0$ and $x_n \neq x_0$ since $f(x_{n,k}) \neq f(x_0)$ by (4.7). Since E_1 is finite dimensional, we may assume that (taking a subsequence if necessary)

$$\lim_{n\to\infty} \frac{x_n - x_0}{\mu_n} = v \in T(S \cap B(x_0, 1), x_0) \quad \text{with} \;\; \mu_n = \|x_n - x_0\| \;\; \text{and} \;\; \|v\| = 1.$$

Then $\lim_{n\to\infty}(f(x_n) - f(x_0))/\mu_n = f'(x_0, v)$. Hence

$$\lim_{n\to\infty} \frac{\|f(x_n) - f(x_0)\|}{\mu_n} = \lim_{n\to\infty} \frac{\lambda_n}{\mu_n} = \|f'(x_0, v)\| = \lambda.$$

Consequently,

$$f'(x_0, v) = \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{\mu_n} = \lim_{n\to\infty} \frac{\lambda_n}{\mu_n} \frac{f(x_n) - f(x_0)}{\lambda_n} = -\lambda d.$$

Hence $f'(x_0, v) \in -D$, but this is a contradiction because $v \in T(S, x_0) \setminus \{0\}$ implies $f'(x_0, v) \notin -D$, from the hypotheses.

The advantage of Theorem 4.5 is that we have to verify a condition in the initial space E_1. This is, in general, easier than verifying the definition in the final space E_2.

The following counterexample shows that the finite dimensionality of E_1 cannot be relaxed.

EXAMPLE 4.6. With the same data as that of Example 4.4 (b), we now take $E_1 = E$, $x_0 = 0$ and $S = D \cup A$. We know that $T(S, x_0) = D$. Let $b_n = (1/n)e_n$.

It is verified that $\sum_{n=1}^{\infty} \|b_n\|^2 = \sum_{n=1}^{\infty} 1/n^2 < \infty$. Hence by [19, Theorem 12.6], for all $x \in E$, the series $\sum_{n=1}^{\infty} \langle b_n, x\rangle$ converges.

Let $f : E \to \mathbb{R}$ be defined by $f(x) = \sum_{n=1}^{\infty} \langle b_n, x\rangle = \sum_{n=1}^{\infty} \alpha_n/n$ with $x = (\alpha_n)$. Then f is a continuous linear functional. Therefore f is Fréchet differentiable with $f'(x_0, v) = \nabla f(x_0)v = f(v)$ and then

$$C(f') = \{v \in E : f'(x_0, v) = f(v) \leq 0\} = \left\{ v = (\beta_n) : \sum_{n=1}^{\infty} \beta_n/n \leq 0 \right\}.$$

We have $T(S,x_0)\cap C(f')=\{0\}$ since if $v\in T(S,x_0)$, then $\beta_n\geq 0\ \forall n$ and $v\in C(f')$ implies $\sum_{n=1}^{\infty}\beta_n/n\leq 0$. Hence $\beta_n=0$ for all n, that is, $v=0$.

However, x_0 is not a local minimum for f over S, because for the points which belong to $A\subset S$, one has $f((-1/n)e_n)=-1/n^2<f(x_0)$ and there are points belonging to A in every neighbourhood of x_0. Therefore we do not even have the guarantee that x_0 be a local minimum for a single-valued function f, and, of course, the equality $T(f(S\cap U),x_0)\cap(-\mathbb{R}_+)=\{0\}$ is false too.

In [12, Definition 3.1], the first author introduces the notion of a strict local efficient minimum for the problem (VP) as follows.

DEFINITION 4.7. Let $m\geq 1$ be an integer. A point $x_0\in S$ is said to be a strict local efficient minimum of order m for $f:E_1\to E_2$ over S if there exist a constant $\alpha>0$ and a neighbourhood U of x_0 such that

$$(f(x)+D)\cap B(f(x_0),\alpha\|x-x_0\|^m)=\emptyset\quad\forall x\in S\cap U\setminus\{x_0\}.$$

If $E_2=\mathbb{R}^p$ and $D=\mathbb{R}^p_+$, such a point is called a strict local Pareto minimum of order m.

Every strict local efficient minimum of order m is a local efficient solution for (VP) [12, Proposition 3.3].

The following result establishes the relationship between this notion and that of a Borwein solution. It follows immediately from [12, Theorem 4.4].

COROLLARY 4.8. *Let $f:\mathbb{R}^n\to\mathbb{R}^p$ be a Hadamard directionally differentiable function at $x_0\in S\subset\mathbb{R}^n$. If x_0 is a strict local Pareto minimum of order* 1 *for f over S, then* $x_0\in\mathrm{Bor}_2(f,S)$.

It is not hard to show that the converse of this corollary is false. Take, for example, $f(x,y)=(x^2+y,x^2-y)$, $S=\mathbb{R}^2$ and $x_0=(0,0)$. Notice that $f(x_0)\in\mathrm{LBor}(f(S),\mathbb{R}^2_+)$, and, by Proposition 4.2(c), $x_0\in\mathrm{Bor}_2(f,S)$.

This concludes our study of proper efficiency in the sense of Kuhn-Tucker.

Ishizuka and Tuan [10, Definition 3.6] consider the following notion of proper efficiency.

DEFINITION 4.9. Let $f:E_1\to E_2$ be a Hadamard directionally differentiable function at $x_0\in S\subset E_1$. It is said that x_0 is a local proper Kuhn-Tucker efficient solution for (VP), written $x_0\in\mathrm{KT}(f,S)$, if $x_0\in\mathrm{LMin}(f,S)$ and $T(S,x_0)\cap C_1(f')=\emptyset$, where $C_1(f')=\{v\in E_1:\ f'(x_0,v)\in -D\setminus\{0\}\}$.

PROPOSITION 4.10. *If* $x_0\in\mathrm{Bor}_2(f,S)$ *then* $x_0\in\mathrm{KT}(f,S)$.

PROOF. By the definition of $\mathrm{Bor}_2(f, S)$, we have $x_0 \in \mathrm{LMin}(f, S)$ and there exists a neighbourhood U of x_0 satisfying (4.2). Suppose that there exists $v \in T(S, x_0) \cap C_1(f')$, and, consequently, $v \in T(S \cap U, x_0)$. Then there exist $x_n \in S \cap U$ and $t_n \to 0^+$ such that $\lim_{n\to\infty}(x_n - x_0)/t_n = v$. As f is Hadamard directionally differentiable at x_0, we deduce that

$$w := \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{t_n} = f'(x_0, v) \in -D \setminus \{0\}, \tag{4.9}$$

because $v \in C_1(f')$. But (4.9) implies $w \in T(f(S \cap U), f(x_0)) \cap (-D)$ with $w \neq 0$, which contradicts (4.2).

The converse of Proposition 4.10 is false in general as the next example shows.

EXAMPLE 4.11. Consider the set $S = \{(x, y, z) \in \mathbb{R}^3 : 2xz \geq x^2 + y^2, z \geq 0\}$, $f(x, y, z) = (x, y)$, $D = \mathbb{R}^2_+$, $x_0 = (0, 0, 0)$ and $y_0 = f(x_0) = (0, 0)$. We have:

(1) $T(S, x_0) \cap C_1(f') = \emptyset$ since $T(S, x_0) = S$ is the convex cone generated by the circle $z = 1$, $(x - 1)^2 + y^2 \leq 1$.
(2) $f(S) = \{(x, y) \in \mathbb{R}^2 : x > 0\} \cup \{(0, 0)\}$.
(3) $f(S \cap U_\varepsilon) = \{(x, y) \in \mathbb{R}^2 : (x - \varepsilon)^2 + y^2 \leq \varepsilon^2\}$, where $U_\varepsilon = \big\{(x, y, z) : \mathrm{Max}\{|x|, |y|, |z|\} \leq 2\varepsilon\big\}$, with $\varepsilon > 0$, is a base of neighbourhoods of x_0. Therefore $T\big(f(S \cap U_\varepsilon), y_0\big) \cap (-\mathbb{R}^2_+) \neq \{(0, 0)\}$, and consequently, $x_0 \notin \mathrm{Bor}_2(f, S)$. However, $x_0 \in \mathrm{KT}(f, S)$.

5. Conclusions

We would like to emphasise that Theorem 4.5 provides a sufficient condition for proper efficient solutions of Bor_2 type. This notion, by Corollary 4.8, is related to strict efficiency, which is in turn a new concept, whose possibilities are still being developed. Using Proposition 4.10, it follows that every solution of Bor_2 type is proper in the Kuhn-Tucker sense, which, in our opinion, is one of the most important notions of proper efficiency. The main advantage of Theorem 4.5 is that it can easily be applied, since if the set S is given by constraints $h(x) = 0$, $g(x) \leq 0$ with $h : \mathbb{R}^n \to \mathbb{R}^r$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ differentiable, and the Abadie constraint qualification

$$T(S, x_0) = \big\{v \in \mathbb{R}^n : \nabla h(x_0)v = 0, \nabla g_j(x_0)v \leq 0\ \forall j \text{ such that } g_j(x_0) = 0\big\}$$

holds at x_0, then all the hypotheses of the theorem can be easily checked. Furthermore, this theorem is very close to the necessary condition for a local efficient solution [4, Result 4.2].

Finally, we find in Theorem 3.9 and Proposition 4.2 that our notions of proper efficiency solutions are equivalent (in very general conditions) and very similar to two previous notions.

Acknowledgement

The authors are indebted to Bruce Craven for his useful comments which led to the present improved version of the paper.

References

[1] H. P. Benson, "An improved definition of proper efficiency for vector minimization with respect to cones", *J. Math. Anal. Appl.* **71** (1979) 232–241.
[2] J. M. Borwein, "Proper efficient points for maximization with respect to cones", *SIAM J. Control Optim.* **15** (1977) 57–63.
[3] J. M. Borwein, "The geometry of Pareto efficiency over cones", *Math. Operationsforsch. Statist. Ser. Optim.* **11** (1980) 235–248.
[4] H. W. Corley, "On optimality conditions for maximizations with respect to cones", *J. Optim. Theory Appl.* **46** (1985) 67–78.
[5] J. P. Dauer and R. J. Gallagher, "Positive proper efficiency and related cone results in vector optimization theory", *SIAM J. Control Optim.* **28** (1990) 158–172.
[6] A. M. Geoffrion, "Proper efficiency and the theory of vector maximization", *J. Math. Anal. Appl.* **22** (1968) 618–630.
[7] A. Guerraggio, E. Molho and A. Zaffaroni, "On the notion of proper efficiency in vector optimization", *J. Optim. Theory Appl.* **82** (1994) 1–21.
[8] M. I. Henig, "Proper efficiency with respect to cones", *J. Optim. Theory Appl.* **36** (1982) 387–407.
[9] L. Hurwicz, "Programming in linear spaces", in *Studies in linear and nonlinear programming* (eds. K. J. Arrow, L. Hurwicz and H. Uzawa), (Standford University Press, Stanford, California, 1958) 38–102.
[10] Y. Ishizuka and H. D. Tuan, "Directionally differentiable multiobjective optimization involving discrete inclusions", *J. Optim. Theory Appl.* **88** (1996) 585–616.
[11] J. Jahn, *Mathematical vector optimization in partially ordered linear spaces* (Verlag Peter Lang, Frankfurt-am-Main, 1986).
[12] B. Jiménez, "Strict efficiency in vector optimization", *J. Math. Anal. Appl.* **265** (2002) 264–284.
[13] P. Q. Khanh, "Proper solutions of vector optimization problems", *J. Optim. Theory Appl.* **74** (1992) 105–130.
[14] A. Klinger, "Improper solutions of the vector maximum problems", *Operations Research* **15** (1967) 570–572.
[15] H. W. Kuhn and A. W. Tucker, "Nonlinear programming", in *Proceedings of the second Berkeley symposium on Mathematics Statistics and Probability*, (Berkeley, California, 1951) 481–492.
[16] D. T. Luc, *Theory of vector optimization* (Springer, Berlin, 1989).
[17] E. K. Makarov and N. N. Rachkovski, "Unified representation of proper efficiency by means of dilating cones", *J. Optim. Theory Appl.* **102** (1999) 141–165.
[18] K. M. Miettinen, *Nonlinear multiobjective optimization* (Kluwer, Boston, 1999).
[19] W. Rudin, *Functional Analysis* (McGraw-Hill, New York, 1973).
[20] Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of multiobjective optimization* (Academic Press, Orlando, 1985).