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MINIMAL AND MAXIMAL SOLUTIONS TO SYSTEMS OF DIFFERENTIAL EQUATIONS WITH A SINGULAR MATRIX

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Abstract

The monotone iterative technique is applied to a system of ordinary differential equations with a singular matrix. The existence of extremal solutions is proved.

1. Introduction

Many problems arising in the physical sciences, engineering, biology and applied mathematics lead to mathematical models described by systems of differential equations with initial conditions of the form

$$x'(t) = f_1(t, x(t)), \quad t \in J = [0, T],\ x(0) = x_0 \in \mathbb{R}^p, \tag{1.1}$$

where $f_1 \in C(J \times \mathbb{R}^p, \mathbb{R}^p)$. Conditions on f_1 which guarantee the existence of solutions of problem (1.1) are important analysis theorems. To show that problem (1.1) has a solution, one can employ fixed point theorems (Banach, Schauder), the Leray-Schauder theory of topological degree or the method of successive iterations. Assuming that f_1 satisfies the Lipschitz condition with respect to the last variable one can show that problem (1.1) has a unique solution. If we assume that f_1 satisfies only a one-sided Lipschitz condition, then we can show that problem (1.1) has extremal solutions. Such a result can be obtained when the method of upper and lower solutions is used. This interesting and fruitful technique for proving existence results shows that corresponding monotone sequences converge to the minimal and maximal solutions of our problem (there are some applications of this technique, for example, in [3]). The constructive proofs of existence also provide numerical procedures for the computation

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of solutions. Problem (1.1) may be generalised by adding an algebraic system to obtain the differential-algebraic system

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)), & t \in J,\ x(0) = x_0 \\ y(t) = f_2(t, x(t), y(t)), & t \in J. \end{cases}$$

Note that the last system is a special case of a problem discussed in this paper, namely

$$\begin{cases} Ax'(t) = f(t, x(t), x'(t)), & t \in J = [0, T], \\ x(0) = k_0 \in \mathbb{R}^m, \end{cases} \tag{1.2}$$

where $f \in C(J \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$ and A is a singular square matrix of order m. Note that problem (1.2) is identical to

$$x'(t) = (A + B)^{-1}[f(t, x(t), x'(t)) + Bx'(t)], \quad t \in J,\ x(0) = k_0 \tag{1.3}$$

provided that the matrix B is a square matrix of order m such that $A+B$ is nonsingular.

It is well-known that the method of lower and upper solutions coupled with the monotone iterative technique provides a practical tool to generate monotone sequences that converge to extremal solutions (see [1], see also [3, 4, 2, 5, 6, 7, 8]). The purpose of this paper is to extend this technique to problems of type (1.2). This method is useful since any member of the corresponding linear monotone iterations is an approximate solution of (1.2). In our discussion, we assume that f satisfies a one-sided Lipschitz condition showing that problem (1.2) has extremal solutions. Note that the system of differential-algebraic equations is a special case of (1.2). Some examples are also given.

2. Main results

A function $v \in C^1(J, \mathbb{R}^m)$ is said to be a lower solution of problem (1.2) if

$$\begin{cases} Av'(t) \le f(t, v(t), v'(t)), & t \in J, \\ v(0) \le k_0, \end{cases}$$

and an upper solution of (1.2) if the above inequalities are reversed. In this paper, the vectorial inequalities mean that the same inequalities hold between their corresponding components. Note that if the matrix $(A+B)^{-1}$ exists, $(A+B)^{-1} \ge 0$ and v is a lower solution of problem (1.2), then v satisfies the relations

$$\begin{cases} v'(t) \le (A + B)^{-1}[f(t, v(t), v'(t)) + Bv'(t)], & t \in J, \\ v(0) \le k_0. \end{cases}$$

Here $(A + B)^{-1} \ge 0$ means that some entries of $(A + B)^{-1}$ may be equal to zero.

The next lemma is a special case of [4, Theorem 1.1.4].

LEMMA 2.1. *Assume that $d_{ij}(t) \geq 0$, $t \in J$ for $i \neq j$, where $D = [d_{ij}]$ is a continuous square matrix of order m. Let*

$$\begin{cases} p'(t) \leq D(t)p(t), & t \in J,\ p \in C^1(J, \mathbb{R}^m), \\ p(0) \leq 0 = [\underbrace{0, \dots, 0}_{m}]^T. \end{cases}$$

Then $p(t) \leq 0$ on J.

Let us define the following set:

$$\Omega = \{(t, u, v) : t \in J,\ y_0(t) \leq u \leq z_0(t),\ y_0'(t) \leq v \leq z_0'(t),\ u, v \in \mathbb{R}^m\},$$

where $y_0, z_0 \in C^1(J, \mathbb{R}^m)$.

Now we are in a position to show the following existence result.

THEOREM 2.2. *Assume that $f \in C(\Omega, \mathbb{R}^m)$ and*

(i) *$y_0, z_0 \in C^1(J, \mathbb{R}^m)$ are lower and upper solutions of (1.2), respectively, and such that $y_0(t) \leq z_0(t)$ and $y_0'(t) \leq z_0'(t)$ on J;*

(ii) *there exists a square matrix B of order m such that $(A + B)^{-1}$ exists, $(A + B)^{-1} \geq 0$, and the condition $f(t, u, \alpha) - f(t, u, \bar{\alpha}) \leq B[\bar{\alpha} - \alpha]$ holds for $y_0(t) \leq u \leq z_0(t)$ and $y_0'(t) \leq \alpha \leq \bar{\alpha} \leq z_0'(t)$, $t \in J$;*

(iii) *there exists a square matrix N of order m such that $N \geq 0$, and for $y_0(t) \leq u \leq \bar{u} \leq z_0(t)$, $t \in J$, it holds that $f(t, u, \alpha) - f(t, \bar{u}, \alpha) \leq -N[\bar{u} - u]$.*

Then there exist monotone sequences $\{y_n\}$ and $\{z_n\}$ such that $y_n(t) \to y(t)$ and $z_n(t) \to z(t)$ on J as $n \to \infty$ and this convergence is uniform and monotonic on J. Moreover the functions y and z are minimal and maximal solutions of problem (1.2), respectively.

PROOF. We construct the sequences $\{y_n\}$ and $\{z_n\}$ using the formulas

$$\begin{cases} y_{n+1}'(t) = (A+B)^{-1}\{f(t, y_n, y_n') + By_n'(t) + N[y_{n+1}(t) - y_n(t)]\}, & y_{n+1}(0) = k_0, \\ z_{n+1}'(t) = (A+B)^{-1}\{f(t, z_n, z_n') + Bz_n'(t) + N[z_{n+1}(t) - z_n(t)]\}, & z_{n+1}(0) = k_0. \end{cases}$$

First of all, we are going to show the following relation:

$$\begin{cases} y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \\ y_0'(t) \leq y_1'(t) \leq z_1'(t) \leq z_0'(t), \quad t \in J. \end{cases} \tag{2.1}$$

Put $p = y_0 - y_1$ on J. Then $p(0) \leq 0$. Since $(A + B)^{-1} \geq 0$, by assumption (i) we have

$$\begin{aligned} p'(t) &\leq (A+B)^{-1}\{f(t, y_0, y_0') + By_0'(t) - f(t, y_0, y_0') - By_0'(t) - N[y_1(t) - y_0(t)]\} \\ &= (A+B)^{-1}Np(t). \end{aligned}$$

By Lemma 2.1, we have $p(t) \le 0$ and then $p'(t) \le 0$ on J showing that $y_0(t) \le y_1(t)$, $y_0'(t) \le y_1'(t)$, $t \in J$. Similarly, we can show that $z_1(t) \le z_0(t)$, $z_1'(t) \le z_0'(t)$, $t \in J$.

Put $p = y_1 - z_1$, so $p(0) = 0$. Then, by (ii) and (iii), we have

$$\begin{aligned} p'(t) &= (A+B)^{-1}\{f(t,y_0,y_0') - f(t,z_0,y_0') + f(t,z_0,y_0') - f(t,z_0,z_0') \\ &\quad - B[z_0'(t) - y_0'(t)] + N[y_1(t) - y_0(t) - z_1(t) + z_0(t)]\} \\ &\le (A+B)^{-1}\{-N[z_0(t) - y_0(t)] + B[z_0'(t) - y_0'(t)] \\ &\quad + B[y_0'(t) - z_0'(t)] + N[y_1(t) - y_0(t) - z_1(t) + z_0(t)]\} \\ &= (A+B)^{-1}Np(t), \quad t \in J. \end{aligned}$$

Hence we have $p(t) \le 0$ and then $p'(t) \le 0$ on J showing that $y_1(t) \le z_1(t)$ and $y_1'(t) \le z_1'(t)$, $t \in J$. Thus (2.1) holds.

In the next step we need to show that y_1 and z_1 are lower and upper solutions of problem (1.2), respectively. Then, by assumptions (ii) and (iii), we obtain

$$\begin{aligned} Ay_1'(t) &= f(t,y_0,y_0') + B[y_0'(t) - y_1'(t)] + N[y_1(t) - y_0(t)] \\ &\quad - f(t,y_1,y_0') + f(t,y_1,y_0') - f(t,y_1,y_1') + f(t,y_1,y_1') \\ &\le f(t,y_1,y_1') - N[y_1(t) - y_0(t)] \\ &\quad + B[y_1'(t) - y_0'(t)] + B[y_0'(t) - y_1'(t)] + N[y_1(t) - y_0(t)] \\ &= f(t,y_1,y_1') \end{aligned}$$

and

$$\begin{aligned} Az_1'(t) &= f(t,z_0,z_0') + B[z_0'(t) - z_1'(t)] + N[z_1(t) - z_0(t)] - f(t,z_1,z_0') \\ &\quad + f(t,z_1,z_0') - f(t,z_1,z_1') + f(t,z_1,z_1') \\ &\ge f(t,z_1,z_1') + N[z_0(t) - z_1(t)] - B[z_0'(t) - z_1'(t)] \\ &\quad + B[z_0'(t) - z_1'(t)] + N[z_1(t) - z_0(t)] \\ &= f(t,z_1,z_1'), \end{aligned}$$

showing that y_1 and z_1 are lower and upper solutions of problem (1.2), respectively.

For some $k \ge 1$, let us assume that

$$\begin{cases} y_{k-1}(t) \le y_k(t) \le z_k(t) \le z_{k-1}(t), & t \in J, \\ y_{k-1}'(t) \le y_k'(t) \le z_k'(t) \le z_{k-1}'(t), & t \in J, \end{cases}$$

and let y_k and z_k be lower and upper solutions of problem (1.2), respectively. We shall prove that

$$\begin{cases} y_k(t) \le y_{k+1}(t) \le z_{k+1}(t) \le z_k(t), & t \in J, \\ y_k'(t) \le y_{k+1}'(t) \le z_{k+1}'(t) \le z_k'(t), & t \in J. \end{cases} \tag{2.2}$$

Put $p = y_k - y_{k+1}$. Then

$$\begin{aligned} p'(t) &\le (A+B)^{-1}\{f(t, y_k, y_k') + By_k'(t) - f(t, y_k, y_k') - By_k'(t) - N[y_{k+1}(t) - y_k(t)]\} \\ &= (A+B)^{-1}Np(t) \end{aligned}$$

with $p(0) = 0$. Hence, by Lemma 2.1, $p(t) \le 0$ and $p'(t) \le 0$, $t \in J$, showing that $y_k(t) \le y_{k+1}(t)$ and $y_k'(t) \le y_{k+1}'(t)$, $t \in J$. Using the same argument we can prove that $z_{k+1}(t) \le z_k(t)$ and $z_{k+1}'(t) \le z_k'(t)$, $t \in J$.

Let $p = y_{k+1} - z_{k+1}$, so $p(0) = 0$. Then we get

$$\begin{aligned} p'(t) &= (A+B)^{-1}\{f(t, y_k, y_k') + By_k'(t) + N[y_{k+1}(t) - y_k(t)] - f(t, z_k, y_k') \\ &\qquad + f(t, z_k, y_k') - f(t, z_k, z_k') - Bz_k'(t) - N[z_{k+1}(t) - z_k(t)]\} \\ &\le (A+B)^{-1}\{-N[z_k(t) - y_k(t)] + B[z_k'(t) - y_k'(t)] \\ &\qquad + N[y_{k+1}(t) - y_k(t) - z_{k+1}(t) + z_k(t)] + B[y_k'(t) - z_k'(t)]\} \\ &= (A+B)^{-1}Np(t), \quad t \in J. \end{aligned}$$

Thus $y_{k+1}(t) \le z_{k+1}(t)$ and $y_{k+1}'(t) \le z_{k+1}'(t)$, $t \in J$, so (2.2) holds.

Hence, by induction, we have

$$\begin{cases} y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t), & t \in J, \\ y_0'(t) \le y_1'(t) \le \cdots \le y_n'(t) \le z_n'(t) \le \cdots \le z_1'(t) \le z_0'(t), & t \in J \end{cases}$$

for all n.

We now show that the sequences $\{y_n\}$ and $\{z_n\}$ converge uniformly and monotonically to y and z, respectively, where y and z are solutions of (1.2). The sequences $\{y_n\}$ and $\{y_n'\}$ are uniformly bounded because

$$y_0(t) \le y_n(t) \le z_0(t) \quad \text{and} \quad y_0'(t) \le y_n'(t) \le z_0'(t), \quad t \in J$$

for all n, where y_0 and $z_0 \in C^1(J, \mathbb{R}^m)$. Note that the sequences $\{y_n\}$ and $\{z_n\}$ are well-defined because y_n and z_n are unique solutions of the corresponding linear IVP's. Moreover, $y_n \in C^1(J, \mathbb{R}^m)$ and

$$y_n(t) = e^{Kt}\left[k_0 + \int_0^t G_{n-1}(s)\,ds\right], \quad t \in J,$$

with $K = (A+B)^{-1}N$ and

$$G_j(s) = e^{-Ks}(A+B)^{-1}\left\{f(s, y_j(s), y_j'(s)) + By_j'(s) - Ny_j(s)\right\}.$$

It is easy to see that $\{y_n\}$ is a sequence of equicontinuous functions. Indeed, $\{z_n\}$ is a sequence of equicontinuous functions too.

Note that $\{y_n'\}$ and $\{z_n'\}$ are sequences of continuous functions on the interval $[0, b]$, so uniform continuity implies that for any $\epsilon > 0$ there exists $\delta > 0$ such that for all n and $t_1, t_2 \in J$ and $|t_1 - t_2| < \delta$ we have

$$\begin{aligned}\left\|(A+B)^{-1}[f(t_1, y_{n-1}(t_1), y_{n-1}'(t_1)) - f(t_2, y_{n-1}(t_2), y_{n-1}'(t_2))]\right\|_* &< \epsilon/3,\\ \left\|(A+B)^{-1}B[y_{n-1}'(t_1) - y_{n-1}'(t_2)]\right\|_* &< \epsilon/3,\\ \left\|(A+B)^{-1}N[y_n(t_1) - y_n(t_2) - y_{n-1}(t_1) + y_{n-1}(t_2)]\right\|_* &< \epsilon/3\end{aligned}$$

because f is continuous on a closed set. Here we used the norm:

$$\|u\|_* = \max_{i=1,2,\dots,m} |u_i|.$$

From the above and the relation

$$\begin{aligned}y_n'(t_1) - y_n'(t_2) = (A+B)^{-1}\{&f(t_1, y_{n-1}(t_1), y_{n-1}'(t_1))\\ &- f(t_2, y_{n-1}(t_2), y_{n-1}'(t_2)) + B[y_{n-1}'(t_1) - y_{n-1}'(t_2)]\\ &+ N[y_n(t_1) - y_{n-1}(t_1) - y_n(t_2) + y_{n-1}(t_2)]\}\end{aligned}$$

we see that $\{y_n'\}$ is a sequence of equicontinuous functions. Hence $y_n \to y$, $y_n' \to y'$ and $y \in C^1(J, \mathbb{R}^m)$, by Arzeli's theorem. Similarly we have $z_n \to z$, $z_n' \to z'$ and $z \in C^1(J, \mathbb{R}^m)$. The Lebesgue theorem yields that

$$\begin{aligned}y(t) &= k_0 + (A+B)^{-1}\left\{\int_0^t \left[f(s, y(s), y'(s)) + By'(s)\right] ds\right\}, \quad t \in J,\\ z(t) &= k_0 + (A+B)^{-1}\left\{\int_0^t \left[f(s, z(s), z'(s)) + Bz'(s)\right] ds\right\}, \quad t \in J.\end{aligned}$$

Thus y and z are solutions of problem (1.2).

In the next step we will show that y and z are minimal and maximal solutions of (1.2). Let x be any solution of problem (1.2) such that $y_0(t) \le x(t) \le z_0(t)$ and $y_0'(t) \le x'(t) \le z_0'(t)$, $t \in J$.

We are going to show that

$$y_n(t) \le x(t) \le z_n(t) \quad \text{and} \quad y_n'(t) \le x'(t) \le z_n'(t), \quad t \in J \tag{2.3}$$

for all natural n.

Put $p = y_1 - x$ on J. Then

$$\begin{aligned}p'(t) &= (A+B)^{-1}\{f(t, y_0, y_0') + By_0'(t) + N[y_1(t) - y_0(t)] - f(t, x, y_0')\\ &\quad + f(t, x, y_0') - f(t, x, x') - Bx'(t)\}\\ &\le (A+B)^{-1}\{-N[x(t) - y_0(t)] + B[x'(t) - y_0'(t)] + By_0'(t)\\ &\quad + N[y_1(t) - y_0(t)] - Bx'(t)\} = (A+B)^{-1}Np(t), \quad p(0) = 0.\end{aligned}$$

Hence $y_1(t) \le x(t)$ and $y_1'(t) \le x'(t)$, $t \in J$, by Lemma 2.1.

Let $p = x - z_1$, $t \in J$. Then

$$\begin{aligned} p'(t) &= (A+B)^{-1}\{f(t,x,x') + Bx'(t) - f(t,z_0,x') + f(t,z_0,x') \\ &\qquad - f(t,z_0,z_0') - Bz_0'(t) - N[z_1(t) - z_0(t)]\} \\ &\le (A+B)^{-1}\{-N[z_0(t) - x(t)] + B[z_0'(t) - x'(t)] \\ &\qquad + B[x'(t) - z_0'(t)] - N[z_1(t) - z_0(t)]\} \\ &= (A+B)^{-1}Np(t). \end{aligned}$$

Lemma 2.1 yields $x(t) \le z_1(t)$ and $x'(t) \le z_1'(t)$, $t \in J$. Thus (2.3) holds for $n = 1$.

Assume that (2.3) holds for some $k \ge 1$. Put $p = y_{k+1} - x$. Then

$$\begin{aligned} p'(t) &= (A+B)^{-1}\{f(t,y_k,y_k') + By_k'(t) + N[y_{k+1}(t) - y_k(t)] \\ &\qquad - f(t,x,y_k') + f(t,x,y_k') - f(t,x,x') - Bx'(t)\} \\ &\le (A+B)^{-1}\{-N[x(t) - y_k(t)] + B[x'(t) - y_k'(t)] + By_k'(t) \\ &\qquad + N[y_{k+1}(t) - y_k(t)] - Bx'(t)\} = (A+B)^{-1}Np(t), \qquad p(0) = 0. \end{aligned}$$

Hence $y_{k+1}(t) \le x(t)$ and $y_{k+1}'(t) \le x'(t)$, $t \in J$, by Lemma 2.1.

Let $p = x - z_{k+1}$, $t \in J$. Then

$$\begin{aligned} p'(t) &= (A+B)^{-1}\{f(t,x,x') + Bx'(t) - f(t,z_k,x') + f(t,z_k,x') \\ &\qquad - f(t,z_k,z_k') - Bz_k'(t) - N[z_{k+1}(t) - z_k(t)]\} \\ &\le (A+B)^{-1}\{-N[z_k(t) - x(t)] + B[z_k'(t) - x'(t)] \\ &\qquad + B[x'(t) - z_k'(t)] - N[z_{k+1}(t) - z_k(t)]\} \\ &= (A+B)^{-1}Np(t). \end{aligned}$$

Lemma 2.1 yields $x(t) \le z_{k+1}(t)$ and $x'(t) \le z_{k+1}'(t)$, $t \in J$. Thus (2.3) holds for all natural n.

Now, if $n \to \infty$, then (2.3) yields $y(t) \le x(t) \le z(t)$ and $y'(t) \le x'(t) \le z'(t)$, $t \in J$, showing that y and z are minimal and maximal solutions of problem (1.2), respectively.

This ends the proof.

3. A special case of (1.2)

Let $m = 2$ and $A = \left[\begin{smallmatrix} 1 & -b \\ 0 & 0 \end{smallmatrix}\right]$, $b \ge 0$. Then problem (1.2) takes the form

$$\begin{cases} x_1'(t) - bx_2'(t) = f_1(t, x_1(t), x_2(t), x_1'(t), x_2'(t)), & x_1(0) = x_{0,1}, \\ \qquad\qquad\quad 0 = f_2(t, x_1(t), x_2(t), x_1'(t), x_2'(t)), & x_2(0) = x_{0,2}. \end{cases} \tag{3.1}$$

Assume that f_1 and f_2 satisfy the following conditions:

$$f_i(t,x_1,x_2,y_1,y_2) - f_i(t,x_1,x_2,\bar{y}_1,\bar{y}_2) \le b_{i,1}[\bar{y}_1 - y_1] + b_{i,2}[\bar{y}_2 - y_2] \tag{3.2}$$

if $\bar{y}_i \ge y_i$, and

$$f_i(t,x_1,x_2,y_1,y_2) - f_i(t,\bar{x}_1,\bar{x}_2,y_1,y_2) \le -c_{i,1}[\bar{x}_1 - x_1] - c_{i,2}[\bar{x}_2 - x_2] \tag{3.3}$$

if $\bar{x}_i \ge x_i$ for $i = 1,2$ with $b_{i,j}, c_{i,j} \ge 0$, $i = 1,2$, $j = 1,2$.

Note that in this case

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad N = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

$$(A+B)^{-1} = \frac{1}{\det(A+B)} \begin{bmatrix} b_{22} & b - b_{12} \\ -b_{21} & 1 + b_{11} \end{bmatrix}$$

provided that $\det(A+B) = b_{22} + b_{21}b + \det(B) \neq 0$. Note that if $b \ge b_{12}$ and $b_{21} = 0$, then $(A+B)^{-1} \ge 0$, so assumptions (ii) and (iii) of Theorem 2.2 hold.

The following system

$$\begin{cases} x_1'(t) = f_1(t, x_1(t), x_2(t), x_1'(t)), & x_1(0) = x_{0,1}, \\ \quad 0 = f_2(t, x_1(t), x_2(t), x_2'(t)), & x_2(0) = x_{0,2} \end{cases} \tag{3.4}$$

is a special case of problem (3.1). Note that in this case we have $b_{12} = b_{21} = 0$, so

$$(A+B)^{-1} = \begin{bmatrix} 1/(1+b_{11}) & 0 \\ 0 & 1/b_{22} \end{bmatrix}.$$

It is quite simple to formulate corresponding theorems to Theorem 2.2 for problems (3.1) and (3.4).

EXAMPLE. Let us consider the following problem:

$$\begin{cases} x_1'(t) - 2x_2'(t) = 2x_1(t) + 3x_2(t) - 8[x_1'(t)]^2 - [x_2'(t)]^2 + t, & t \in J, \\ \qquad 0 = x_1(t) + [x_1(t)]^2 + 5x_2(t) - 10[x_2'(t)]^2, & t \in J, \\ x_1(0) = x_2(0) = 0. \end{cases} \tag{3.5}$$

Comparing this with (3.1) we have $A = \left[\begin{smallmatrix} 1 & -2 \\ 0 & 0 \end{smallmatrix}\right]$, $b = 2$ and

$$f_1(t,x_1,x_2,y_1,y_2) = 2x_1 + 3x_2 - 8y_1^2 - y_2^2 + t,$$
$$f_2(t,x_1,x_2,y_1,y_2) = x_1 + x_1^2 + 5x_2 - 10y_2^2.$$

It is simple to check that

$$\begin{cases} y_{01}(t) = 0, & t \in J, \\ y_{02}(t) = 0, & t \in J, \end{cases} \quad \text{and} \quad \begin{cases} z_{01}(t) = t, & t \in J, \\ z_{02}(t) = t, & t \in J, \end{cases}$$

are lower and upper solutions of problem (3.5), respectively.

Let $0 \le x_1 \le \bar{x}_1 \le t$, $0 \le x_2 \le \bar{x}_2 \le t$, $0 \le y_1 \le \bar{y}_1 \le 1$ and $0 \le y_2 \le \bar{y}_2 \le 1$. Then conditions (3.2) and (3.3) hold with

$$B = \begin{bmatrix} 16 & 2 \\ 0 & 20 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}.$$

Indeed we have

$$A + B = \begin{bmatrix} 17 & 0 \\ 0 & 20 \end{bmatrix} \quad \text{and} \quad (A+B)^{-1} = \frac{1}{340} \begin{bmatrix} 20 & 0 \\ 0 & 17 \end{bmatrix}.$$

Thus all assumptions of Theorem 2.2 are satisfied. By Theorem 2.2, problem (3.5) has minimal and maximal solutions.

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