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PERIODIC SOLUTIONS OF A TWO-SPECIES RATIO-DEPENDENT PREDATOR-PREY SYSTEM WITH TIME DELAY IN A TWO-PATCH ENVIRONMENT

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Abstract

By using the continuation theorem of coincidence degree theory, a sufficient condition is obtained for the existence of a positive periodic solution of a predator-prey diffusion system.

1. Introduction

Xu and Chen [4] considered a two-species ratio-dependent predator-prey diffusion model with time delay given by

$$\left.\begin{aligned} x_1'(t) &= x_1(t)\left(a_1 - a_{11}x_1(t) - \frac{a_{13}x_3(t)}{mx_3(t)+x_1(t)}\right) + D_1(x_2(t)-x_1(t)), \\ x_2'(t) &= x_2(t)(a_2 - a_{22}x_2(t)) + D_2(x_1(t)-x_2(t)), \\ x_3'(t) &= x_3(t)\left(-a_3 + \frac{a_{31}x_1(t-\tau)}{mx_3(t-\tau)+x_1(t-\tau)}\right), \end{aligned}\right\} \tag{1.1}$$

where $x_i(t)$ represents the prey population in the i^{th} patch, $i = 1, 2$, and $x_3(t)$ represents the predator population. Here $\tau > 0$ is a constant delay due to gestation, D_i is a positive constant denoting the dispersal rate, $i = 1, 2$, and a_i $(i = 1, 2, 3)$, a_{11}, a_{13}, a_{22}, a_{31} and m are positive constants.

In Xu and Chen [4], the local and global asymptotical stability of the positive equilibrium of the system (1.1) were studied. For an ecological interpretation of system (1.1), we refer to [4] and references cited therein.

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Realistic models require the inclusion of the effect of change in the environment. This motivates us to consider the following two species predator-prey diffusion model with time delay:

$$\left.\begin{aligned} x_1'(t) &= x_1(t)\left(a_1(t) - a_{11}(t)x_1(t) - \frac{a_{13}(t)x_3(t)}{m(t)x_3(t) + x_1(t)}\right) \\ &\quad + D_1(t)(x_2(t) - x_1(t)), \\ x_2'(t) &= x_2(t)(a_2(t) - a_{22}(t)x_2(t)) + D_2(t)(x_1(t) - x_2(t)), \\ x_3'(t) &= x_3(t)\left(-a_3(t) + \frac{a_{31}(t)x_1(t-\tau)}{m(t)x_3(t-\tau) + x_1(t-\tau)}\right). \end{aligned}\right\} \tag{1.2}$$

In addition, the effects of a periodically changing environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Therefore the assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment (for example, seasonal changes, food supplies, mating habits, and so on), which leads us to assume that D_i $(i = 1, 2)$, a_i $(i = 1, 2, 3)$, a_{11}, a_{13}, a_{22}, a_{31} and m are strictly positive continuous w-periodic functions.

As pointed out by Freedman and Wu [1] and Kuang [3], it is of interest to study the global existence of periodic solutions for systems representing predator-prey or competition systems. In this paper, our aim is to use the continuation theorem of coincidence degree theory which was proposed in [2] by Gaines and Mawhin to establish the existence of at least one positive w-periodic solution with $w > 0$ of system (1.2).

Let X, Z be real Banach spaces, $L : \operatorname{dom} L \subset X \to Z$ a Fredholm mapping of index zero and $P : X \to X$, $Q : Z \to Z$ continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$, $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$ and $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. Denote by $K_p : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{dom} L$ the generalised inverse (of L) and by $J : \operatorname{Im} Q \to \operatorname{Ker} L$ an isomorphism of $\operatorname{Im} Q$ onto $\operatorname{Ker} L$.

For convenience we introduce a continuation theorem [2, page 40] as follows.

LEMMA 1.1. *Let $\Omega \subset X$ be an open bounded set and $N : X \to Z$ be a continuous operator which is L-compact on $\overline{\Omega}$ (that is, $QN : \overline{\Omega} \to Z$ and $K_p(I - Q)N : \overline{\Omega} \to Y$ are compact). Assume*

(a) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \operatorname{dom} L$, $Lx \neq \lambda Nx$;*
(b) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$;*
(c) $\deg\{JQNx, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega}$.

2. Main result

For the sake of convenience we will use the notation

$$\overline{f} = \frac{1}{w}\int_0^w f(t)\,dt, \qquad f^l = \min_{t\in[0,w]} f(t) \quad \text{and} \quad f^M = \max_{t\in[0,w]} f(t),$$

where f is a strictly positive continuous w-periodic function.

We now state our fundamental theorem about the existence of a positive w-periodic solution of system (1.2).

THEOREM 2.1. *Assume the following*:

(i) $(a_1 - D_1)^l > a_{13}^M/m^l$,
(ii) $a_{31}^l > \overline{a_3}$,
(iii) $(a_2 - D_2)^l > 0$.

Then system (1.2) *has at least one positive w-periodic solution.*

PROOF. Let

$$F_1(t,s) = \frac{a_{13}(t)e^{y_3(s)}}{m(t)e^{y_3(s)} + e^{y_1(s)}} \quad \text{and} \quad F_2(t,s) = \frac{a_{31}(t)e^{y_1(s-\tau)}}{m(t)e^{y_3(s-\tau)} + e^{y_1(s-\tau)}}.$$

Consider the system

$$\left.\begin{aligned} y_1'(t) &= a_1(t) - D_1(t) - a_{11}(t)e^{y_1(t)} - F_1(t,t) + D_1(t)e^{y_2(t)-y_1(t)},\\ y_2'(t) &= a_2(t) - D_2(t) - a_{22}(t)e^{y_2(t)} + D_2(t)e^{y_1(t)-y_2(t)},\\ y_3'(t) &= -a_3(t) + F_2(t,t), \end{aligned}\right\} \tag{2.1}$$

where τ, D_i $(i = 1,2)$, a_i $(i = 1,2,3)$, a_{11}, a_{13}, a_{22}, a_{31} and m are the same as those in system (1.2). It is easy to see that if the system (2.1) has an w-periodic solution $(y_1^*(t), y_2^*(t), y_3^*(t))^T$, then $(e^{y_1^*(t)}e^{y_2^*(t)}e^{y_3^*(t)})^T$ is a positive w-periodic solution of system (1.2). Therefore for (1.2) to have at least one positive w-periodic solution it is sufficient that (2.1) has at least one w-periodic solution. In order to apply Lemma 1.1 to system (2.1), we take

$$X = \left\{(y_1(t), y_2(t), y_3(t))^T \in C^1(R,R^3) : y_i(t+w) = y_i(t),\ \text{for } i = 1,2,3\right\},$$
$$Z = \{(z_1(t), z_2(t), z_3(t))^T \in C(R,R^3) : z_i(t+w) = z_i(t),\ \text{for } i = 1,2,3\}$$

and

$$\left\|(y_1(t), y_2(t), y_3(t))^T\right\| = \max_{t\in[0,w]}|y_1(t)| + \max_{t\in[0,w]}|y_2(t)| + \max_{t\in[0,w]}|y_3(t)|.$$

With this norm, X and Z are Banach spaces. Let

$$N\begin{bmatrix} y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix} a_1(t)-D_1(t)-a_{11}(t)e^{y_1(t)}-F_1(t,t)+D_1(t)e^{y_2(t)-y_1(t)}\\ a_2(t)-D_2(t)-a_{22}(t)e^{y_2(t)}+D_2(t)e^{y_1(t)-y_2(t)}\\ -a_3(t)+F_2(t,t)\end{bmatrix},$$

$$L\begin{bmatrix} y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix} y_1'\\ y_2'\\ y_3'\end{bmatrix},\quad P\begin{bmatrix} y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix} (1/w)\int_0^w y_1(t)\,dt\\ (1/w)\int_0^w y_2(t)\,dt,\\ (1/w)\int_0^w y_3(t)\,dt\end{bmatrix},\quad \begin{bmatrix} y_1\\ y_2\\ y_3\end{bmatrix}\in X,$$

$$Q\begin{bmatrix} z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix} (1/w)\int_0^w z_1(t)\,dt\\ (1/w)\int_0^w z_2(t)\,dt\\ (1/w)\int_0^w z_3(t)\,dt\end{bmatrix},\quad \begin{bmatrix} z_1\\ z_2\\ z_3\end{bmatrix}\in Z.$$

We note that $\operatorname{Ker} L = R^3$,

$$\operatorname{Im} L=\left\{\begin{bmatrix} z_1\\ z_2\\ z_3\end{bmatrix}\;\middle|\;\begin{bmatrix} z_1\\ z_2\\ z_3\end{bmatrix}\in Z,\int_0^w z_i(t)\,dt=0,\ \text{for}\ i=1,2,3\right\}$$

is closed in Z and $\dim\operatorname{Ker} L=\operatorname{codim}\operatorname{Im} L=3$. Hence L is a Fredholm mapping of index 0. Furthermore, the generalised inverse (of L) $K_p:\operatorname{Im} L\to\operatorname{Ker} P\cap\operatorname{dom} L$ has the form

$$K_p(z)=\int_0^t z(s)\,ds-\frac{1}{w}\int_0^w\int_0^t z(s)\,ds\,dt,\quad \text{for}\quad z=\begin{bmatrix} z_1\\ z_2\\ z_3\end{bmatrix}\in Z.$$

Thus $QN: X\to Z$,

$$\begin{bmatrix} y_1\\ y_2\\ y_3\end{bmatrix}\to\begin{bmatrix} \frac{1}{w}\int_0^w\left[a_1(t)-D_1(t)-a_{11}(t)e^{y_1(t)}-F_1(t,t)+D_1(t)e^{y_2(t)-y_1(t)}\right]dt\\ \frac{1}{w}\int_0^w\left[a_2(t)-D_2(t)-a_{22}(t)e^{y_2(t)}+D_2(t)e^{y_1(t)-y_2(t)}\right]dt\\ \frac{1}{w}\int_0^w[-a_3(t)+F_2(t,t)]\,dt\end{bmatrix},$$

$K_p(I-Q)N: X\to X$ and

$$\begin{aligned}\begin{bmatrix} y_1\\ y_2\\ y_3\end{bmatrix}\to&\begin{bmatrix} \int_0^t\left[a_1(s)-D_1(s)-a_{11}(s)e^{y_1(s)}-F_1(s,s)+D_1(s)e^{y_2(s)-y_1(s)}\right]ds\\ \int_0^t\left[a_2(s)-D_2(s)-a_{22}(s)e^{y_2(s)}+D_2(s)e^{y_1(s)-y_2(s)}\right]ds\\ \int_0^t[-a_3(s)+F_2(s,s)]\,ds\end{bmatrix}\\ &-\begin{bmatrix} \frac{1}{w}\int_0^w\int_0^t[a_1(s)-D_1(s)-a_{11}(s)e^{y_1(s)}-F_1(s,s)+D_1(s)e^{y_2(s)-y_1(s)}]\,ds\,dt\\ \frac{1}{w}\int_0^w\int_0^t[a_2(s)-D_2(s)-a_{22}(s)e^{y_2(s)}+D_2(s)e^{y_1(s)-y_2(s)}]\,ds\,dt\\ \frac{1}{w}\int_0^w\int_0^t[-a_3(s)+F_2(s,s)]\,ds\,dt\end{bmatrix}\\ &-\left(\frac{1}{2}-\frac{t}{w}\right)\begin{bmatrix} \int_0^w[a_1(t)-D_1(t)-a_{11}(t)e^{y_1(t)}-F_1(t,t)+D_1(t)e^{y_2(t)-y_1(t)}]\,dt\\ \int_0^w[a_2(t)-D_2(t)-a_{22}(t)e^{y_2}+D_2(t)e^{y_1(t)-y_2(t)}]\,dt\\ \int_0^w[-a_3(t)+F_2(t,t)]\,dt\end{bmatrix}.\end{aligned}$$

Clearly QN and $K_p(I-Q)N$ are continuous by the Lebesgue theorem and moreover $QN(\overline{\Omega})$ and $K_p(I-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Hence N is $L-$compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$.

Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$\left.\begin{aligned} y_1'(t) &= \lambda\left[a_1(t) - D_1(t) - a_{11}(t)e^{y_1(t)} - F_1(t,t) + D_1(t)e^{y_2(t)-y_1(t)}\right], \\ y_2'(t) &= \lambda\left[a_2(t) - D_2(t) - a_{22}(t)e^{y_2(t)} + D_2(t)e^{y_1(t)-y_2(t)}\right], \\ y_3'(t) &= \lambda[-a_3(t) + F_2(t,t)]. \end{aligned}\right\} \tag{2.2}$$

Suppose that $(y_1(t), y_2(t), y_3(t))^T \in X$ is a solution of system (2.2) for a certain $\lambda \in (0,1)$. By integrating (2.2) over the interval $[0, w]$, we obtain

$$\int_0^w \left[a_1(t) - D_1(t) - a_{11}(t)e^{y_1(t)} - F_1(t,t) + D_1(t)e^{y_2(t)-y_1(t)}\right] dt = 0,$$

$$\int_0^w \left[a_2(t) - D_2(t) - a_{22}(t)e^{y_2(t)} + D_2(t)e^{y_1(t)-y_2(t)}\right] dt = 0$$

and

$$\int_0^w [-a_3(t) + F_2(t,t)]\, dt = 0.$$

Thus

$$\int_0^w \left[a_{11}(t)e^{y_1(t)} + F_1(t,t)\right] dt = \overline{(a_1 - D_1)}w + \int_0^w D_1(t)e^{y_2(t)-y_1(t)}\, dt, \tag{2.3}$$

$$\int_0^w a_{22}(t)e^{y_2(t)}\, dt = \overline{(a_2 - D_2)}w + \int_0^w D_2(t)e^{y_1(t)-y_2(t)}\, dt \tag{2.4}$$

and

$$\int_0^w F_2(t,t)\, dt = \overline{a_3}w. \tag{2.5}$$

From (2.2)–(2.5), it follows that

$$\begin{aligned} \int_0^w |y_1'(t)|\, dt &\le \lambda \int_0^w \left|a_1(t) - D_1(t) - a_{11}(t)e^{y_1(t)} - F_1(t,t) + D_1(t)e^{y_2(t)-y_1(t)}\right| dt \\ &< \overline{(a_1 - D_1)}w + \int_0^w [a_{11}(t)e^{y_1(t)} + F_1(t,t)]\, dt \\ &\quad + \int_0^w D_1(t)e^{y_2(t)-y_1(t)}\, dt \\ &= 2\overline{(a_1 - D_1)}w + \int_0^w D_1(t)e^{y_2(t)-y_1(t)}\, dt, \end{aligned} \tag{2.6}$$

$$\int_0^w |y_2'(t)|\, dt \le \lambda \int_0^w \left|a_2(t) - D_2(t) - a_{22}(t)e^{y_2(t)} + D_2(t)e^{y_1(t)-y_2(t)}\right| dt$$

$$< \overline{(a_2 - D_2)}w + \int_0^w a_{22}(t)e^{y_2(t)}\,dt + \int_0^w D_2(t)e^{y_1(t)-y_2(t)}\,dt$$
$$= 2\overline{(a_2 - D_2)}w + 2\int_0^w D_2(t)e^{y_1(t)-y_2(t)}\,dt \qquad (2.7)$$

and

$$\int_0^w |y_3'(t)|\,dt \le \lambda\int_0^w |-a_3(t)+F_2(t,t)|\,dt < \overline{a_3}w + \int_0^w F_2(t,t)\,dt = 2\overline{a_3}w. \qquad (2.8)$$

Multiplying the first equation and the second equation of system (2.2) by $e^{y_1(t)}$ and $e^{y_2(t)}$, respectively, and integrating both over $[0, w]$, we obtain

$$\int_0^w e^{y_1(t)}y_1'(t)\,dt = \int_0^w \big[(a_1(t)-D_1(t))e^{y_1(t)} - a_{11}(t)e^{2y_1(t)} - F_1(t,t)e^{y_1(t)} + D_1(t)e^{y_2(t)}\big]\,dt$$

and

$$\int_0^w e^{y_2(t)}y_2'(t)\,dt = \int_0^w \big[(a_2(t)-D_2(t))e^{y_2(t)} - a_{22}(t)e^{2y_2(t)} + D_2(t)e^{y_1(t)}\big]\,dt.$$

That is,

$$\int_0^w a_{11}(t)e^{2y_1(t)}\,dt + \int_0^w F_1(t,t)e^{y_1(t)}\,dt$$
$$= \int_0^w (a_1(t) - D_1(t))e^{y_1(t)}\,dt + \int_0^w D_1(t)e^{y_2(t)}\,dt \qquad (2.9)$$

and

$$\int_0^w a_{22}(t)e^{2y_2(t)}\,dt = \int_0^w (a_2(t) - D_2(t))e^{y_2(t)}\,dt + \int_0^w D_2(t)e^{y_1(t)}\,dt. \qquad (2.10)$$

Equation (2.9) implies that

$$a_{11}^l \int_0^w e^{2y_1(t)}\,dt < (a_1 - D_1)^M \int_0^w e^{y_1(t)}\,dt + D_1^M \int_0^w e^{y_2(t)}\,dt,$$

from which, using the inequality $\big(\int_0^w e^{y_1(t)}\,dt\big)^2 \le w\int_0^w e^{2y_1(t)}\,dt$, we obtain

$$\frac{a_{11}^l}{w}\left(\int_0^w e^{y_1(t)}\,dt\right)^2 < (a_1 - D_1)^M \int_0^w e^{y_1(t)}\,dt + D_1^M \int_0^w e^{y_2(t)}\,dt.$$

Thus

$$2\frac{a_{11}^l}{w}\int_0^w e^{y_1(t)}\,dt < \left[(a_1 - D_1)^M + \left[(a_1 - D_1)^M\right]^2 + 4\frac{a_{11}^l D_1^M}{w}\int_0^w e^{y_2(t)}\,dt\right]^{1/2},$$

from which, using the inequality

$$(a+b)^{1/2} < a^{1/2} + b^{1/2}, \quad \text{for } a > 0 \text{ and } b > 0, \qquad (2.11)$$

it follows that

$$\frac{a_{11}^l}{w}\int_0^w e^{y_1(t)}\,dt < (a_1-D_1)^M + \sqrt{\frac{a_{11}^l D_1^M}{w}}\left(\int_0^w e^{y_2(t)}\,dt\right)^{1/2}. \tag{2.12}$$

A similar argument to (2.12) implies from (2.10) that

$$\frac{a_{22}^l}{w}\int_0^w e^{y_2(t)}\,dt < (a_2-D_2)^M + \sqrt{\frac{a_{22}^l D_2^M}{w}}\left(\int_0^w e^{y_1(t)}\,dt\right)^{1/2}. \tag{2.13}$$

Substituting (2.13) into (2.12), we obtain

$$\begin{aligned}\frac{a_{11}^l}{w}\int_0^w e^{y_1(t)}\,dt &< (a_1-D_1)^M \\ &\quad + \sqrt{\frac{a_{11}^l D_1^M}{w}}\left[\frac{(a_2-D_2)^M w}{a_{22}^l} + \sqrt{\frac{a_{22}^l D_2^M}{w}}\,\frac{w}{a_{22}^l}\left(\int_0^w e^{y_1(t)}\,dt\right)^{1/2}\right]^{1/2},\end{aligned}$$

from which, using (2.11), it follows that

$$\begin{aligned}\frac{a_{11}^l}{w}\int_0^w e^{y_1(t)}dt &< (a_1-D_1)^M \\ &\quad + \sqrt{\frac{a_{11}^l D_1^M}{a_{22}^l}}\left[\left[(a_2-D_2)^M\right]^{1/2} + \sqrt[4]{\frac{a_{22}^l D_2^M}{w}}\left(\int_0^w e^{y_1(t)}dt\right)^{1/4}\right].\end{aligned}$$

Therefore there exists a positive constant ρ_1 such that

$$\int_0^w e^{y_1(t)}dt < \rho_1. \tag{2.14}$$

Substituting (2.14) into (2.13) implies that there exists a positive constant ρ_2 such that

$$\int_0^w e^{y_2(t)}dt < \rho_2. \tag{2.15}$$

Choose $t_i \in [0,w]$, $i=1,2$, such that $y_i(t_i) = \min_{t\in[0,w]} y_i(t)$, $i=1,2$. Then it is clear that $y_i'(t_i)=0$, $i=1,2$. In view of this and system (2.2), we obtain

$$a_1(t_1) - D_1(t_1) - a_{11}(t_1)e^{y_1(t_1)} - F_1(t_1,t_1) + D_1(t_1)e^{y_2(t_1)-y_1(t_1)} = 0 \tag{2.16}$$

and

$$a_2(t_2) - D_2(t_2) - a_{22}(t_2)e^{y_2(t_2)} + D_2(t_2)e^{y_1(t_2)-y_2(t_2)} = 0. \tag{2.17}$$

Thus

$$a_{11}^M e^{y_1(t_1)} > a_{11}(t_1)e^{y_1(t_1)} = a_1(t_1) - D_1(t_1) - F_1(t_1,t_1) + D_1(t_1)e^{y_2(t_1)-y_1(t_1)} > (a_1 - D_1)^l - a_{13}^M/m^l$$

and

$$a_{22}^M e^{y_2(t_2)} > a_{22}(t_2)e^{y_2(t_2)} = a_2(t_2) - D_2(t_2) + D_2(t_2)e^{y_1(t_2)-y_2(t_2)} > (a_2 - D_2)^l. \quad (2.18)$$

Therefore

$$y_1(t_1) > \ln \frac{(a_1 - D_1)^l - a_{13}^M/m^l}{a_{11}^M}, \quad y_2(t_2) > \ln \frac{(a_2 - D_2)^l}{a_{22}^M}. \quad (2.19)$$

Substituting (2.14), (2.15) and (2.19) into (2.6) and (2.7), we obtain

$$\int_0^w |y_1'(t)|\,dt < 2\overline{(a_1 - D_1)}w + \frac{2D_1^M \rho_2 a_{11}^M}{(a_1 - D_1)^l - a_{13}^M/m^l} \triangleq d_1 \quad (2.20)$$

and

$$\int_0^w |y_2'(t)|\,dt < 2\overline{(a_2 - D_2)}w + \frac{2D_2^M \rho_1 a_{22}^M}{(a_2 - D_2)^l} \triangleq d_2. \quad (2.21)$$

Equations (2.14) and (2.15) imply that there exist two points $\xi, \eta \in (0, w)$ such that

$$y_1(\xi) < \ln(\rho_1/w), \quad y_2(\eta) < \ln(\rho_2/w). \quad (2.22)$$

In view of this and (2.19), we have

$$|y_1(\xi)| < \max\left\{\left|\ln\frac{\rho_1}{w}\right|, \left|\ln\frac{(a_1 - D_1)^l - a_{13}^M/m^l}{a_{11}^M}\right|\right\} \quad (2.23)$$

and

$$|y_2(\eta)| < \max\left\{\left|\ln\frac{\rho_2}{w}\right|, \left|\ln\frac{(a_2 - D_2)^l}{a_{22}^M}\right|\right\}. \quad (2.24)$$

Since $\forall t \in R$

$$|y_1(t)| \le |y_1(\xi)| + \int_0^w |y_1'(s)|\,ds \quad \text{and} \quad |y_2(t)| \le |y_2(\eta)| + \int_0^w |y_2'(s)|\,ds,$$

from (2.20), (2.21) and (2.23), we obtain

$$|y_1(t)| < \max\left\{\left|\ln\frac{\rho_1}{w}\right|, \left|\ln\frac{(a_1 - D_1)^l - a_{13}^M/m^l}{a_{11}^M}\right|\right\} + d_1 \triangleq R_1$$

and

$$|y_2(t)| < \max\left\{\left|\ln\frac{\rho_2}{w}\right|, \left|\ln\frac{(a_2 - D_2)^l}{a_{22}^M}\right|\right\} + d_2 \triangleq R_2.$$

Equation (2.5) implies that there exists a point $t_3^* \in (0, w)$ such that

$$F_2(t_3^* + \tau, t_3^* + \tau) = \overline{a_3}.$$

That is, $\overline{a_3} m(t_3^* + \tau) e^{y_3(t_3^*)} = (a_{31}(t_3^* + \tau) - \overline{a_3}) e^{y_1(t_3^*)}$. Hence

$$|y_3(t_3^*)| = \left| \ln \frac{a_{31}(t_3^* + \tau) - \overline{a_3}}{m(t_3^* + \tau)\overline{a_3}} \right| + |y_1(t_3^*)| < \max_{t\in[0,w]} \left| \ln \frac{a_{31}(t) - \overline{a_3}}{m(t)\overline{a}_3} \right| + R_1. \quad (2.25)$$

Since $\forall t \in R$, $|y_3(t)| \le |y_3(t_3^*)| + \int_0^w |y_3'(s)|\, ds$, from this and (2.8), we obtain

$$|y_3(t)| < \max_{t\in[0,w]} \left| \ln \frac{a_{31}(t) - \overline{a_3}}{m(t)\overline{a}_3} \right| + R_1 + 2a_3 w \triangleq R_3.$$

Clearly R_i $(i = 1, 2, 3)$ are independent of λ. Denote $M = R_1 + R_2 + R_3 + R_0$; here R_0 is taken sufficiently large such that

$$2 \max \left\{ |\ln \delta_1|, \left| \ln \frac{\overline{(a_1 - D_1)} - \overline{(a_{13}/m)}}{\overline{a_{11}}} \right| \right\} + \left| \ln \frac{a_{31}^M - \overline{a_3}}{m^l \overline{a_3}} \right|$$
$$+ \max \left\{ \left| \ln \frac{\overline{(a_2 - D_2)} + \sqrt{\overline{a_{22}}\,\overline{D_2}\delta_1}}{\overline{a_{22}}} \right|, \left| \ln \frac{\overline{(a_2 - D_2)}}{\overline{a_{22}}} \right| \right\} < M. \quad (2.26)$$

Here $\sqrt[4]{\delta_1}$ is the only real root of the equation

$$\sqrt{\overline{a_{22}}}\,\overline{a_{11}} x^4 = \sqrt{\overline{a_{22}}}\,\overline{(a_1 - D_1)} + \sqrt{\overline{a_{11}}\,\overline{D_1 (a_2 - D_2)}} + \sqrt{\overline{a_{11}}\,\overline{D_1}}\sqrt[4]{\overline{a_{22}}\,\overline{D_2}}\, x.$$

We now take $\Omega = \left\{(y_1(t), y_2(t), y_3(t))^T \in X : \|(y_1, y_2, y_3)^T\| < M\right\}$. This satisfies condition (a) of Lemma 1.1. When $(y_1, y_2, y_3)^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^3$, $(y_1, y_2, y_3)^T$ is a constant vector in R^3 with $|y_1| + |y_2| + |y_3| = M$. We will prove that when $(y_1, y_2, y_3)^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^3$,

$$QN \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \overline{(a_1 - D_1)} - \overline{a_{11}} e^{y_1} - \frac{1}{w} \int_0^w \frac{a_{13}(t)}{m(t)e^{y_3} + e^{y_1}}\, dt\, e^{y_3} + \overline{D_1} e^{y_2 - y_1} \\ \overline{(a_2 - D_2)} - \overline{a_{22}} e^{y_2} + \overline{D_2} e^{y_1 - y_2} \\ -\overline{a_3} + \frac{1}{w} \int_0^w \frac{a_{31}(t)}{m(t)e^{y_3} + e^{y_1}}\, dt\, e^{y_1} \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

If the conclusion is not true, that is, $QN(y_1, y_2, y_3)^T = (0, 0, 0)^T$ with $|y_1| + |y_2| + |y_3| = M$. Since

$$\overline{(a_1 - D_1)} - \overline{a_{11}} e^{y_1} - \frac{1}{w} \int_0^w \frac{a_{13}(t)\, dt}{m(t)e^{y_3} + e^{y_1}} e^{y_3} + \overline{D_1} e^{y_2 - y_1} = 0, \quad (2.27)$$

we have $\overline{a_{11}} e^{2y_1} < \overline{(a_1 - D_1)} e^{y_1} + \overline{D_1} e^{y_2} < \overline{(a_1 - D_1)} e^{y_1} + \overline{D_1} e^{y_2}$. Thus

$$2\overline{a_{11}} e^{y_1} < \overline{(a_1 - D_1)} + \sqrt{\overline{(a_1 - D_1)}^2 + 4\overline{a_{11}}\,\overline{D_1} e^{y_2}} < 2\overline{(a_1 - D_1)} + 2\sqrt{\overline{a_{11}}\,\overline{D_1}}\, e^{y_2/2}.$$

That is,

$$\overline{a_{11}}e^{y_1} < \overline{(a_1 - D_1)} + \sqrt{\overline{a_{11}}\,\overline{D_1}}\, e^{y_2/2}. \tag{2.28}$$

Since

$$\overline{(a_2 - D_2)} - \overline{a_{22}}e^{y_2} + \overline{D_2}e^{y_1 - y_2} = 0, \tag{2.29}$$

we obtain $\overline{a_{22}}e^{2y_2} < \overline{(a_2 - D_2)}e^{y_2} + \overline{D_2}e^{y_1}$. Thus

$$\overline{a_{22}}e^{y_2} < \overline{(a_2 - D_2)} + \sqrt{\overline{a_{22}}\,\overline{D_2}}\, e^{y_1/2}. \tag{2.30}$$

From (2.28) and (2.30), it follows that

$$e^{y_1} < \delta_1, \quad e^{y_2} < \frac{\overline{(a_2 - D_2)} + \sqrt{\overline{a_{22}}\,\overline{D_2}\delta_1}}{\overline{a_{22}}}. \tag{2.31}$$

From (2.27) and (2.29), we obtain

$$e^{y_1} > \frac{\overline{(a_1 - D_1)} - \overline{(a_{13}/m)}}{\overline{a_{11}}} \quad \text{and} \quad e^{y_2} > \frac{\overline{(a_2 - D_2)}}{\overline{a_{22}}}. \tag{2.32}$$

Hence

$$|y_1| < \max\left\{|\ln\delta_1|, \left|\ln\frac{\overline{(a_1 - D_1)} - \overline{(a_{13}/m)}}{\overline{a_{11}}}\right|\right\} \quad \text{and}$$

$$|y_2| < \max\left\{\left|\ln\frac{\overline{(a_2 - D_2)} + \sqrt{\overline{a_{22}}\,\overline{D_2}\delta_1}}{\overline{a_{22}}}\right|, \left|\ln\frac{\overline{(a_2 - D_2)}}{\overline{a_{22}}}\right|\right\}.$$

Since $-\overline{a_3} + (1/w)\int_0^w \big(a_{31}(t)/(m(t)e^{y_3} + e^{y_1})\big)\,dt e^{y_1} = 0$, the same argument as that used for (2.25) gives

$$|y_3| \le \left|\ln\frac{a_{31}^M - \overline{a_3}}{m^l\overline{a_3}}\right| + \max\left\{|\ln\delta_1|, \left|\ln\frac{\overline{(a_1 - D_1)} - \overline{(a_{13}/m)}}{\overline{a_{11}}}\right|\right\}.$$

Therefore

$$\begin{aligned}\sum_{i=1}^{3}|y_i| &\le 2\max\left\{|\ln\delta_1|, \left|\ln\frac{\overline{(a_1 - D_1)} - \overline{(a_{13}/m)}}{\overline{a_{11}}}\right|\right\} \\ &\quad + \max\left\{\left|\ln\frac{\overline{(a_2 - D_2)} + \sqrt{\overline{a_{22}}\,\overline{D_2}\delta_1}}{\overline{a_{22}}}\right|, \left|\ln\frac{\overline{(a_2 - D_2)}}{\overline{a_{22}}}\right|\right\} + \left|\ln\frac{a_{31}^M - \overline{a_3}}{m^l\overline{a_3}}\right| \\ &< M,\end{aligned}$$

which contradicts the fact that $|y_1| + |y_2| + |y_3| = M$. So when $(y_1, y_2, y_3)^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^3$, $QN(y_1, y_2, y_3)^T \neq (0, 0, 0)^T$.

Finally we will prove that condition (c) of Lemma 1.1 is satisfied.

Define $\phi : \operatorname{Dom} L \times [0, 1] \to X$ by

$$\phi(y_1, y_2, y_3, \mu) = \begin{bmatrix} \overline{(a_1 - D_1)} - \overline{a_{11}} e^{y_1} \\ \overline{(a_2 - D_2)} - \overline{a_{22}} e^{y_2} \\ -\overline{a_3} + (1/w)\int_0^w \frac{a_{31}(t)}{m(t)e^{y_3}+e^{y_1}}\,dt e^{y_1} \end{bmatrix} + \mu \begin{bmatrix} -(1/w)\int_0^w \frac{a_{13}(t)}{m(t)e^{y_3}+e^{y_1}}\,dt e^{y_3} + \overline{D_1} e^{y_2-y_1} \\ \overline{D_2} e^{y_1-y_2} \\ 0 \end{bmatrix}.$$

When $(y_1, y_2, y_3)^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^3$, $(y_1, y_2, y_3)^T$ is a constant vector in R^3 with $|y_1|+|y_2|+|y_3| = M$. Using a similar argument to that for $QN(y_1, y_2, y_3)^T \neq 0$, when $(y_1, y_2, y_3)^T \in \partial\Omega \cap \operatorname{Ker} L$, we can show that when $(y_1, y_2, y_3)^T \in \partial\Omega \cap \operatorname{Ker} L$, $\phi(y_1, y_2, y_3, \mu) \neq (0, 0, 0)^T$. As a result, we have

$$\begin{aligned} &\deg(JQN(y_1, y_2, y_3)^T, \Omega \cap \operatorname{Ker} L, (0, 0, 0)^T) \\ &= \deg\Bigg(\bigg(\overline{(a_1 - D_1)} - \overline{a_{11}} e^{y_1}, \overline{(a_2 - D_2)} - \overline{a_{22}} e^{y_2}, \\ &\quad -\overline{a_3} + \frac{1}{w}\int_0^w \frac{a_{31}(t)}{m(t)e^{y_3} + e^{y_1}}\,dt e^{y_1}\bigg)^T, \Omega \cap \operatorname{Ker} L, (0, 0, 0)^T\Bigg) \\ &= \deg\Bigg(\bigg(\overline{(a_1 - D_1)} - \overline{a_{11}} e^{y_1}, \overline{(a_2 - D_2)} - \overline{a_{22}} e^{y_2}, \\ &\quad -\overline{a_3} + \frac{\overline{a_{31}} e^{y_1}}{m(t^*)e^{y_3} + e^{y_1}}\bigg)^T, \Omega \cap \operatorname{Ker} L, (0, 0, 0)^T\Bigg), \end{aligned}$$

where $t^* \in [0, w]$ is a constant.

Since the system of algebraic equations

$$\begin{cases} \overline{(a_1 - D_1)} - \overline{a_{11}} x = 0, \\ \overline{(a_2 - D_2)} - \overline{a_{22}} y = 0, \\ -\overline{a_3} + \overline{a_{31}} x/(m(t^*)z + x) = 0, \end{cases}$$

has a unique solution (x^*, y^*, z^*) which satisfies $x^* > 0$, $y^* > 0$ and $z^* > 0$, thus

$$\begin{aligned} &\deg\Bigg(\bigg(\overline{(a_1 - D_1)} - \overline{a_{11}} e^{y_1}, \overline{(a_2 - D_2)} - \overline{a_{22}} e^{y_2}, \\ &\quad -\overline{a_3} + \frac{\overline{a_{31}} e^{y_1}}{m(t^*)e^{y_3} + e^{y_1}}\bigg)^T, \Omega \cap \operatorname{Ker} L, (0, 0, 0)^T\Bigg) \end{aligned}$$

$$
= \operatorname{sign} \begin{vmatrix} -\overline{a_{11}} x^* & 0 & 0 \\ 0 & -\overline{a_{22}} y^* & 0 \\ \dfrac{\overline{a_{31}} m(t^*) z^*}{(m(t^*)z^* + x^*)^2} & 0 & \dfrac{-m(t^*)\overline{a_{31}} x^*}{(m(t^*)z^* + x^*)^2} \end{vmatrix}
$$

$$
= \operatorname{sign} \left[\frac{-\overline{a_{11}}\ \overline{a_{22}} m(t^*) \overline{a_{31}} y^* (x^*)^2}{(m(t^*)z^* + x^*)^2} \right] \neq 0.
$$

Consequently $\deg \big(JQN(y_1, y_2, y_3)^T, \Omega \cap \operatorname{Ker} L, (0, 0, 0)^T \big) \neq 0$. This completes the proof of condition (c) of Lemma 1.1.

By now we know that Ω verifies all the requirements of Lemma 1.1 and that system (2.1) has at least one w-periodic solution. Therefore system (2.1) has at least one positive w-periodic solution. This completes the proof of Theorem 2.1.

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