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SINGULAR PROBLEMS MODELLING PHENOMENA IN THE THEORY OF PSEUDOPLASTIC FLUIDS

RAVI P. AGARWAL[1] and DONAL O'REGAN[2]

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Abstract

Existence criteria are presented for nonlinear singular initial and boundary value problems. In particular our theory includes a problem arising in the theory of pseudoplastic fluids.

1. Introduction

This paper is motivated by the boundary value problem

$$\begin{cases} y^{1/n}y'' + nt = 0, & 0 < t < 1 \\ y'(0) = y(1) = 0 \end{cases}$$

which arises in the theory of pseudoplastic fluids. In particular we present existence theory for the mixed boundary value problem

$$\begin{cases} \frac{1}{p}(py')' + q(t)f(t,y) = 0, & 0 < t < 1 \\ \lim_{t\to 0^+} p(t)y'(t) = y(1) = 0 \end{cases}$$

where $f : [0,1] \times (0,\infty) \to \mathbf{R}$ is continuous. Notice f may be singular at $y = 0$. Problems of the above form have been discussed extensively in the literature (see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]) usually when f is positone, that is, $f : (0,1) \times (0,\infty) \to (0,\infty)$. Only a handful of papers (see [3, 4, 5] and the references therein) have appeared where the nonlinearity f is allowed to change sign. This paper presents a new theory, with the idea being to approximate the singular problem by a sequence

[1]Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901–6975, USA; e-mail: agarwal@fit.edu.
[2]Department of Mathematics, National University of Ireland, Galway, Ireland; e-mail: donal.oregan@nuigalway.ie.

of nonsingular problems each of which has a lower solution α_m and a upper solution β, and then use a limiting argument. This seems to be more natural and more general than the theory presented in [3, 4, 5] since the study of lower solutions to nonsingular problems is well documented. Also in this paper we discuss the singular initial value problem

$$\begin{cases} y' = q(t)\, f(t, y), & 0 < t < T(< \infty) \\ y(0) = 0. \end{cases}$$

For the remainder of this section we describe the physical problem which motivates our study. The boundary layer equations for steady flow over a semi-infinite plate [1] are

$$\begin{aligned} U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} &= \frac{1}{\rho}\frac{\partial \tau_{XY}}{\partial Y}, \\ \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} &= 0, \end{aligned}$$

where the X and Y axes are taken along and perpendicular to the plate, ρ is the density, U and V are the velocity components parallel and normal to the plate and the shear stress $\tau_{XY} = K(\partial U/\partial Y)^n$. The case $n = 1$ corresponds to a Newtonian fluid and for $0 < n < 1$ the power law relation between shear stress and rate of strain describes pseudoplastic non-Newtonian fluids. The fluid has zero velocity on the plate and the flow approaches stream conditions far from the plate, that is,

$$U(X, 0) = V(X, 0) = 0, \quad U(X, \infty) = U_\infty,$$

where U_∞ is the uniform potential flow. The above results (if we use stream function-similarity variables) [1, 9] in a third-order infinite interval problem

$$F''' + F(F'')^{2-n} = 0, \quad F(0) = F'(0) = 0, \quad F'(\infty) = 1.$$

Now use the Crocco-type transformation $u = F'$ and $G = F''$ to obtain

$$G^n G'' + (n-1)G^{n-1}(G')^2 + u = 0, \quad G'(0) = 0, \quad G(1) = 0.$$

Setting $y = G^n$ we obtain

$$\begin{cases} y^{1/n} y'' + nu = 0, & 0 < u < 1 \\ y'(0) = y(1) = 0. \end{cases}$$

2. Mixed boundary value problems

Motivated by the example in Section 1 concerning non-Newtonian fluids, we consider the mixed boundary value problem

$$\begin{cases} \frac{1}{p}(py')' + q(t)f(t,y) = 0, & 0 < t < 1 \\ \lim_{t\to 0^+} p(t)y'(t) = y(1) = 0. \end{cases} \tag{2.1}$$

We note also that we do *not* assume $\int_0^1 ds/p(s) < \infty$. For our first result in this section we will assume the following conditions are satisfied:

$$p \in C[0,1] \cap C^1(0,1) \quad \text{with } \; p > 0 \text{ on } (0,1) \tag{2.2}$$

$$q \in C(0,1) \quad \text{with } \; q > 0 \text{ on } (0,1) \tag{2.3}$$

$$\int_0^1 p(s)q(s)ds < \infty \quad \text{and} \quad \int_0^1 \frac{1}{p(t)} \int_0^t p(s)q(s)\,ds\,dt < \infty \tag{2.4}$$

$$f : [0,1] \times (0,\infty) \to \mathbf{R} \text{ is continuous} \tag{2.5}$$

$$\begin{cases} \exists n_0 \in \{1,2,\ldots\} \text{ and associated with each } m \in N_0 = \{n_0, n_0+1, \ldots\}, \\ \exists \alpha_m \in C[0,1] \cap C^2(0,1),\ p\alpha_m' \in AC[0,1], \\ \text{with } \; p(t)q(t)f(t,\alpha_m(t)) + (p(t)\alpha_m'(t))' \ge 0 \; \text{ for } \; t \in (0,1), \\ \lim\limits_{t\to 0^+} p(t)\alpha_m'(t) \ge 0 \; \text{ and } \; 0 < \alpha_m(1) \le 1/m \end{cases} \tag{2.6}$$

$$\begin{cases} \exists \alpha \in C[0,1], \alpha > 0 \; \text{ on } \; [0,1) \; \text{ and } \; \alpha(t) \le \alpha_m(t), \\ t \in [0,1] \; \text{ for each } \; m \in N_0 \end{cases} \tag{2.7}$$

$$\begin{cases} \exists \beta \in C[0,1] \cap C^2(0,1),\ p\beta' \in AC[0,1] \; \text{ with} \\ p(t)q(t)f(t,\beta(t)) + (p(t)\beta'(t))' \le 0 \; \text{ for } \; t \in (0,1), \\ \lim\limits_{t\to 0^+} p(t)\beta'(t) \le 0 \; \text{ and } \; \beta(1) \ge \beta_0 > 0 \end{cases} \tag{2.8}$$

and

$$\alpha_m(t) \le \beta(t), \quad t \in [0,1] \; \text{ for each } \; m \in N_0. \tag{2.9}$$

THEOREM 2.1. (I) *Suppose* (2.2)–(2.9) *hold and in addition assume the following condition is satisfied*:

$$\begin{cases} 0 \le f(t,y) \le g(y) \; \textit{ on } \; [0,1] \times (0,a_0] \; \textit{ with } \; g > 0 \\ \textit{continuous and nonincreasing on } \; (0,\infty); \end{cases} \tag{2.10}$$

here $a_0 = \sup_{t\in[0,1]} \beta(t)$. *Then* (2.1) *has a solution* $y \in C[0,1] \cap C^2(0,1)$ *with* $y(t) \ge \alpha(t)$ *for* $t \in [0,1]$.

(II) *Suppose* (2.2)–(2.9) *hold and in addition assume the following condition is satisfied*:

$$f(t,x) - f(t,y) > 0 \ \textit{for} \ 0 < x < y, \ \textit{for each fixed} \ t \in (0,1). \tag{2.11}$$

Then (2.1) *has a solution* $y \in C[0,1] \cap C^2(0,1)$ *with* $y(t) \ge \alpha(t)$ *for* $t \in [0,1]$.

PROOF. Without loss of generality assume $\beta_0 \ge 1/n_0$. Fix $m \in N_0$ and consider the boundary value problem

$$\begin{cases} (py')' + pq f_m^\star(t,y) = 0, & 0 < t < 1 \\ \lim_{t \to 0^+} p(t)y'(t) = 0 \\ y(1) = 1/m, \end{cases} \tag*{$(2.12)^m$}$$

where

$$f_m^\star(t,y) = \begin{cases} f(t,\beta(t)) + r(\beta(t) - y), & y > \beta(t) \\ f(t,y), & \alpha_m(t) \le y \le \beta(t) \\ f(t,\alpha_m(t)) + r(\alpha_m(t) - y), & y < \alpha_m(t) \end{cases}$$

with $r : \mathbf{R} \to [-1,1]$ the radial retraction defined by

$$r(u) = \begin{cases} u, & |u| \le 1 \\ u/|u|, & |u| > 1. \end{cases}$$

It is immediate from Schauder's fixed point theorem (see [10]) that $(2.12)^m$ has a solution $y_m \in C[0,1]$ (in fact $y_m \in C[0,1] \cap C^2(0,1)$ with $py_m' \in AC[0,1]$). A standard argument (see [10, Chapter 5]; note $f_m^\star : [0,1] \times \mathbf{R} \to \mathbf{R}$ is continuous) guarantees that

$$\alpha_m(t) \le y_m(t) \le \beta(t) \quad \text{for} \ t \in [0,1]. \tag{2.13}$$

As a result y_m is a solution of

$$\begin{cases} (py')' + pq f(t,y) = 0, & 0 < t < 1 \\ \lim_{t \to 0^+} p(t)y'(t) = 0 \\ y(1) = 1/m. \end{cases} \tag{2.14}$$

In addition (2.7) guarantees that

$$\alpha(t) \le \alpha_m(t) \le y_m(t) \le \beta(t) \quad \text{for} \ t \in [0,1]. \tag{2.15}$$

The proof is now broken into two cases.

Case (A). Suppose (2.10) holds.

We first show

$$\{y_m\}_{m\in N_0} \text{ is a bounded, equicontinuous family on } [0,1]. \tag{2.16}$$

First notice from (2.10) that $(py'_m)' \le 0$ on $(0,1)$, so $py'_m \le 0$ on $(0,1)$. In addition $-(p(t)y'_m(t))' \le p(t)q(t)g(y_m(t))$ for $t \in (0,1)$, so integration from 0 to t yields

$$-p(t)y'_m(t) \le g(y_m(t)) \int_0^t p(s)q(s)\,ds \quad \text{for } t \in (0,1).$$

As a result

$$0 \le \frac{-y'_m(t)}{g(y_m(t))} \le \frac{1}{p(t)} \int_0^t p(s)q(s)\,ds \quad \text{for } t \in (0,1).$$

Now consider $I(z) = \int_0^z du/g(u)$. For $t, s \in [0,1]$ we have

$$|I(y_m(t)) - I(y_m(s))| = \left| \int_s^t \frac{y'_m(x)}{g(y_m(x))} \right| \le \left| \int_s^t \frac{1}{p(x)} \int_0^x p(z)q(z)\,dz\,dx \right|,$$

so

$$\{I(y_m)\}_{m\in N_0} \text{ is a bounded, equicontinuous family on } [0,1]. \tag{2.17}$$

The uniform continuity of I^{-1} on $[0, I(a_0)]$ together with (2.17) and

$$|y_m(t) - y_m(s)| = |I^{-1}(I(y_m(t))) - I^{-1}(I(y_m(s)))|$$

guarantees (2.16). A standard argument [2, page 90] using the Arzela-Ascoli theorem (and (2.15)) completes the proof.

Case (B). Suppose (2.11) holds.

We begin by showing

$$y_{m+1}(t) \le y_m(t) \text{ for } t \in [0,1] \text{ for each } m \in N_0. \tag{2.18}$$

Suppose (2.18) is false. Then for some $m \in N_0$, $y_{m+1} - y_m$ would have a positive absolute maximum at say $\tau_0 \in [0,1)$. Suppose to begin with $\tau_0 \in (0,1)$, so $(y_{m+1} - y_m)'(\tau_0) = 0$ and $(p(y_{m+1} - y_m)')'(\tau_0) \le 0$. On the other hand, (2.11) implies

$$(p(y_{m+1} - y_m)')'(\tau_0) = -p(\tau_0)q(\tau_0)[f(\tau_0, y_{m+1}(\tau_0)) - f(\tau_0, y_m(\tau_0))] > 0,$$

a contradiction. If $\tau_0 = 0$ then $\lim_{t\to 0^+} p(t)[y_{m+1} - y_m]'(t) = 0$ and there exists $\mu > 0$ with $y_{m+1}(s) - y_m(s) > 0$ for $s \in (0,\mu)$. Thus for $t \in (0,\mu)$ we have from (2.11) that

$$p(y_{m+1} - y_m)'(t) = \int_0^t p(s)q(s)[f(s, y_m(s)) - f(s, y_{m+1}(s))]ds > 0,$$

a contradiction since $y_{m+1} - y_m$ has a positive absolute maximum at 0. As a result (2.18) holds.

Lets look at the interval $[0, 1 - 1/n_0]$. Let

$$R_{n_0} = \sup \left\{ |f(t, y)| : t \in [0, 1 - 1/n_0] \text{ and } \alpha(t) \le y \le a_0 \right\}; \tag{2.19}$$

here $a_0 = \sup_{t \in [0,1]} \beta(t)$. In addition

$$|y_m'(t)| \le \frac{R_{n_0}}{p(t)} \int_0^t p(s)q(s)\, ds \quad \text{for } t \in (0, 1 - 1/n_0).$$

Thus $\{y_m\}_{m \in N_0}$ is a bounded, equicontinuous family on $[0, 1 - 1/n_0]$. The Arzela-Ascoli theorem guarantees the existence of a subsequence N_{n_0} of N_0 and a function $z_{n_0} \in C[0, 1 - 1/n_0]$ with y_m converging uniformly on $[0, 1 - 1/n_0]$ to z_{n_0} as $m \to \infty$ through N_{n_0}. Proceed inductively to obtain subsequences of integers

$$N_{n_0} \supseteq N_{n_0+1} \supseteq \cdots \supseteq N_k \supseteq \cdots$$

and functions $z_k \in C[0, 1 - 1/k]$ with y_m converging uniformly on $[0, 1 - 1/k]$ to z_k as $m \to \infty$ through N_k, and $z_{k+1} = z_k$ on $[0, 1 - 1/k]$.

Define a function $y : [0, 1] \to [0, \infty)$ by $y(x) = z_k(x)$ on $[0, 1 - 1/k]$ and $y(1) = 0$. Notice y is well-defined and $\alpha(t) \le y(t) \le a_0$ for $t \in [0, 1)$. Next fix $t \in (0, 1)$ and let $k \in \{n_0, n_0 + 1, \dots\}$ be such that $0 < t < 1 - 1/k$. Let $N_k^\star = \{n \in N_k : n \ge k\}$. Now y_m, $m \in N_k^\star$, satisfies

$$y_m(t) = y_m(0) - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x) f(x, y_m(x))\, dx\, ds.$$

Let $m \to \infty$ through $N_k^\star$ to obtain

$$y(t) = y(0) - \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x) f(x, y(x))\, dx\, ds.$$

We can do this argument for each $t \in (0, 1)$, so $(py')'(t) + p(t)q(t) f(t, y(t)) = 0$ for $t \in (0, 1)$ and $\lim_{t \to 0^+} p(t)y'(t) = 0$.

It remains to show y is continuous at 1. Let $\epsilon > 0$ be given. Now since $\lim_{m \to \infty} y_m(1) = 0$ there exists $n_1 \in N_0$ with $y_{n_1}(1) < \epsilon/2$. Also since $y_{n_1} \in C[0, 1]$ there exists $\delta_{n_1} > 0$ with $y_{n_1}(t) < \epsilon/2$ for $t \in [1 - \delta_{n_1}, 1]$. From (2.18) for $m \ge n_1$ we have $y_m(t) \le y_{n_1}(t) < \epsilon/2$ for $t \in [1 - \delta_{n_1}, 1]$. As a result for $m \ge n_1$ we have

$$0 \le \alpha(t) \le y_m(t) < \epsilon/2 \quad \text{for } t \in [1 - \delta_{n_1}, 1].$$

Consequently

$$0 \le \alpha(t) \le y(t) \le \epsilon/2 < \epsilon \quad \text{for } t \in [1 - \delta_{n_1}, 1),$$

so y is continuous at 1.

REMARK 2.1. In Theorem 2.1 (I) we can replace (2.10) with

$$\begin{cases} |f(t,y)| \le g(y) \ \text{ on } \ [0,1]\times(0,a_0] \ \text{ with } g>0 \\ \text{continuous and nonincreasing on } \ (0,\infty) \end{cases} \tag{2.20}$$

and

$$\int_0^1 \frac{1}{p(s)} \int_0^s p(x)q(x)g(\alpha(x))\,dx\,ds < \infty; \tag{2.21}$$

here $a_0 = \sup_{t\in[0,1]} \beta(t)$. Notice we only used (2.10) to show (2.16). If we assume (2.20) and (2.21) then (2.16) is immediate since

$$\pm(p(t)y_m'(t))' \le p(t)q(t)g(y_m(t)) \le p(t)q(t)g(\alpha(t)) \quad \text{for } t\in(0,1),$$

so

$$|y_m'(t)| \le \frac{1}{p(t)} \int_0^t p(s)q(s)g(\alpha(s))\,ds \quad \text{for } t\in(0,1).$$

We next state and prove a more general result motivated from Theorem 2.1 (II).

THEOREM 2.2. *Suppose* (2.2)–(2.7) *hold and in addition assume the following conditions are satisfied*:

$$\begin{cases} \text{for each } m\in N_0, \exists\beta_m\in C[0,1]\cap C^2(0,1),\ p\beta_m'\in AC[0,1] \\ \text{with } p(t)q(t)f(t,\beta_m(t)) + (p(t)\beta_m'(t))' \le 0 \ \text{ for } \ t\in(0,1), \\ \lim_{t\to 0^+} p(t)\beta_m'(t) \le 0 \ \text{ and } \ \beta_m(1)\ge 1/m \end{cases} \tag{2.22}$$

$$\alpha_m(t) \le \beta_m(t), \qquad t\in[0,1] \ \text{for each } \ m\in N_0 \tag{2.23}$$

and

$$\begin{cases} \text{for each } \ t\in[0,1] \ \text{ we have that } \{\beta_m(t)\}_{m\in N_0} \ \text{ is a} \\ \text{nonincreasing sequence and } \ \lim_{m\to\infty} \beta_m(1) = 0. \end{cases} \tag{2.24}$$

Then (2.1) *has a solution* $y\in C[0,1]\cap C^2(0,1)$ *with* $y(t)\ge\alpha(t)$ *for* $t\in[0,1]$.

PROOF. Fix $m\in N_0$. Proceed as in Theorem 2.1 with β_m replacing β in $f_m^\star$. The same reasoning as in Theorem 2.1 guarantees that there exists a solution $y_m\in C[0,1]$ to (2.14) with $\alpha(t)\le\alpha_m(t)\le y_m(t)\le\beta_m(t)$ for $t\in[0,1]$. Also as in Theorem 2.1 (from (2.19) onwards) there exists $y\in C[0,1)$ (as described in Theorem 2.1 (II)) with

$$\alpha(t) \le y(t) \le a_0 = \sup_{t\in[0,1]} \beta_{n_0}(t) \quad \text{for } t\in[0,1), \tag{2.25}$$

with $(py')'(t) + p(t)q(t)f(t,y(t)) = 0$, $0<t<1$ and $\lim_{t\to 0^+} p(t)y'(t) = 0$.

It remains to show y is continuous at 1. Let $\epsilon > 0$ be given. Now since $\lim_{m\to\infty}\beta_m(1) = 0$ there exists $n_1 \in N_0$ with $\beta_{n_1}(1) < \epsilon/2$, and so there exists $\delta_{n_1} > 0$ with $\beta_{n_1}(t) < \epsilon/2$ for $t \in [1-\delta_{n_1}, 1]$. From (2.24) for $m \geq n_1$ we have

$$\alpha(t) \leq \alpha_m(t) \leq y_m(t) \leq \beta_m(t) \leq \beta_{n_1}(t) < \epsilon/2 \quad \text{for } t \in [1-\delta_{n_1}, 1].$$

That is, for $m \geq n_1$ we have $0 \leq \alpha(t) \leq y_m(t) < \epsilon/2$ for $t \in [1-\delta_{n_1}, 1]$. Consequently $0 \leq \alpha(t) \leq y(t) \leq \epsilon/2 < \epsilon$ for $t \in [1-\delta_{n_1}, 1)$, so y is continuous at 1.

EXAMPLE (Fluid problem). Consider the boundary value problem

$$\begin{cases} y'' + \nu t/y^{1/\nu} = 0, & 0 < t < 1 \\ y'(0) = y(1) = 0 \end{cases} \tag{2.26}$$

where $0 < \nu \leq 1$. We will show using Theorem 2.1 (part (I) or (II)) that (2.26) has a solution.

First we choose $n_0 \in \{1, 2, \dots\}$ so that

$$\frac{\nu}{6} + \frac{1}{n_0} \leq 1 \quad \text{and} \quad \left(\frac{\nu}{6} - 1\right)\frac{1}{\nu+1} + \frac{1}{n_0} \leq 0. \tag{2.27}$$

Let $p = 1$, $q(t) = 2t$ and clearly (2.2)–(2.5) hold. Also let

$$\begin{aligned} \alpha_m(t) &= \nu(1-t^3)/6 + 1/m, \\ \alpha(t) &= \nu(1-t^3)/6 \end{aligned} \tag{2.28}$$

and $\beta(t) = 1 - \nu t^3/(\nu+1)$. To check (2.6), for $m \in N_0 = \{n_0, n_0+1, \dots\}$, notice $\alpha_m(1) = 1/m$, $\alpha_m'(0) = 0$ and

$$\alpha_m'' + qf(t, \alpha_m) = -\nu t + \frac{\nu t}{[\alpha_m(t)]^{1/\nu}} \geq -\nu t + \nu t = 0 \quad \text{for } t \in (0, 1),$$

since $\alpha_m(t) \leq \nu/6 + 1/n_0 \leq 1$, $t \in [0, 1]$ from (2.27). Thus (2.6) holds and (2.7) is immediate. To check (2.8) notice $\beta(1) = 1 - \nu/\nu + 1 \equiv \beta_0$, $\beta'(0) = 0$ and

$$\begin{aligned} \beta'' + qf(t, \beta) &= \frac{-6\nu t}{\nu+1} + \frac{\nu t}{[\beta(t)]^{1/\nu}} \leq \frac{-6\nu t}{\nu+1} + \nu t(\nu+1)^{1/\nu} \\ &= \nu t\left\{\frac{-6}{\nu+1} + (\nu+1)^{1/\nu}\right\} \leq 0 \quad \text{for } t \in (0, 1), \end{aligned}$$

since $\beta(t) \geq 1/(\nu+1)$ for $t \in [0, 1]$, and $(\nu+1)^{(\nu+1)/\nu} \leq 4 \leq 6$ for $0 < \nu \leq 1$ (note with $f(x) = (x+1)^{(x+1)/x}$ we have $f(0^+) = e$, $f(1) = 4$ and $f'(x) \geq 0$ on $(0, 1)$).

Thus (2.8) holds. In addition (2.9) is true since (2.27) implies for $m \in N_0$ that

$$\begin{aligned}\alpha_m(t) &= \frac{\nu}{6}(1-t^3) + \frac{1}{m} \le \frac{\nu}{6}\left(1 - \frac{\nu}{\nu+1}t^3\right) + \frac{1}{n_0} \\ &= \frac{\nu}{6}\beta(t) + \frac{1}{n_0} = \beta(t) + \left\{\frac{1}{n_0} + \left(\frac{\nu}{6} - 1\right)\beta(t)\right\} \\ &\le \beta(t) + \left\{\frac{1}{n_0} + \left(\frac{\nu}{6} - 1\right)\frac{1}{\nu+1}\right\} \le \beta(t) \quad \text{for } t \in (0,1)\end{aligned}$$

since $\nu/(\nu+1) \le 1$ and $(\nu/6 - 1)/(\nu+1) + 1/n_0 \le 0$. Finally (2.10) with $g(y) = 1/y^{1/\nu}$ (or (2.11) since if $0 < x < y$ then $x^{1/\nu} < y^{1/\nu}$) holds. The existence of a solution y to (2.26) follows from Theorem 2.1 (I) (or (II)). Note as well that $y(t) \ge \alpha(t)$ for $t \in [0,1]$ where α is given in (2.28).

3. Initial value problems

In this section we consider the initial boundary value problem

$$\begin{cases} y' = qf(t,y), & 0 < t < T(<\infty) \\ y(0) = 0. \end{cases} \tag{3.1}$$

Our results in this section differ from those in [4], that is, instead of assuming the existence of a lower solution to the singular problem (which is difficult to construct in practice) as in [4] we assume only the existence of a lower solution to the "approximating nonsingular problem". For our first result in this section we assume the following conditions are satisfied:

$$f : [0,T] \times (0,\infty) \to \mathbf{R} \text{ is continuous} \tag{3.2}$$

$$q \in C(0,T], \quad q > 0 \text{ on } (0,T] \quad \text{and} \quad \int_0^T q(x)\,dx < \infty \tag{3.3}$$

$$\begin{cases} \exists n_0 \in \{1,2,\dots\} \text{ and associated with each } m \in N_0 = \{n_0, n_0+1, \dots\}, \\ \exists \alpha_m \in C[0,T] \cap C^1(0,T] \text{ with} \\ q(t)f(t,\alpha_m(t)) \ge \alpha_m'(t) \text{ for } t \in (0,T) \text{ and } 0 < \alpha_m(0) \le 1/m \end{cases} \tag{3.4}$$

$$\begin{cases} \exists \alpha \in C[0,T], \ \alpha > 0 \text{ on } (0,T] \text{ and } \alpha(t) \le \alpha_m(t), \\ t \in [0,T] \text{ for each } m \in N_0 \end{cases} \tag{3.5}$$

$$\begin{cases} \exists \beta \in C[0,T] \cap C^1(0,T] \text{ with } q(t)f(t,\beta(t)) \le \beta'(t) \\ \text{for } t \in (0,T) \text{ and } \beta(0) \ge \beta_0 > 0 \end{cases} \tag{3.6}$$

and

$$\alpha_m(t) \le \beta(t), \quad t \in [0,T] \text{ for each } m \in N_0. \tag{3.7}$$

THEOREM 3.1. (I) *Suppose* (3.2)–(3.7) *hold and in addition assume the following condition is satisfied*:

$$\begin{cases} |f(t,y)| \le g(y) \ \text{ on } \ [0,T]\times(0,a_0] \ \text{ with } \ g>0 \\ \text{continuous and nonincreasing on } \ (0,\infty); \end{cases} \tag{3.8}$$

here $a_0 = \sup_{t\in[0,T]} \beta(t)$. *Then* (3.1) *has a solution* $y \in C[0,T]\cap C^1(0,T]$ *with* $y(t) \ge \alpha(t)$ *for* $t \in [0,T]$.

(II) *Suppose* (3.2)–(3.7) *hold and in addition assume the following condition is satisfied*:

$$f(t,x) - f(t,y) \ge 0 \quad \text{for } \ 0 < x < y, \ \text{for each fixed } \ t \in (0,T). \tag{3.9}$$

Then (3.1) *has a solution* $y \in C[0,T]\cap C^1(0,T]$ *with* $y(t) \ge \alpha(t)$ *for* $t \in [0,T]$.

PROOF. Without loss of generality assume $\beta_0 \ge 1/n_0$. Fix $m \in N_0$ and consider

$$\begin{cases} y' = q f_m^\star(t,y), \quad 0<t<T \\ y(0) = 1/m, \end{cases} \tag{3.10^m}$$

where

$$f_m^\star(t,y) = \begin{cases} f(t,\beta(t)), & y > \beta(t) \\ f(t,y), & \alpha_m(t) \le y \le \beta(t) \\ f(t,\alpha_m(t)), & y < \alpha_m(t). \end{cases}$$

It is immediate from Schauder's fixed point theorem (see [10]) that $(3.10)^m$ has a solution $y_m \in C[0,T]$. A standard argument (see [11, Chapter 3]; note $f_m^\star : [0,1]\times\mathbf{R}\to\mathbf{R}$ is continuous) guarantees that

$$\alpha_m(t) \le y_m(t) \le \beta(t) \quad \text{for } \ t \in [0,T]. \tag{3.11}$$

As a result y_m is a solution of

$$\begin{cases} y' = q f(t,y), \quad 0<t<T \\ y(0) = 1/m \end{cases} \tag{3.12}$$

with

$$\alpha(t) \le \alpha_m(t) \le y_m(t) \le \beta(t) \quad \text{for } \ t \in [0,T]. \tag{3.13}$$

The proof is now broken into two cases.

Case (A). Suppose (3.8) holds.

We first show

$$\{y_m\}_{m\in N_0} \ \text{ is a bounded, equicontinuous family on } \ [0,T]. \tag{3.14}$$

To see this notice (3.8) guarantees that $|y_m'(t)|/g(y_m(t)) \leq q(t)$ for $t \in (0,T)$, and so $\pm v_m'(t) \leq q(t)$ for $t \in (0,T)$; here

$$v_m(t) = \int_0^{y_m(t)} \frac{du}{g(u)} = G(y_m(t)).$$

For $t, s \in [0,T]$ we have

$$|v_m(t) - v_m(s)| = \left| \int_s^t v_m'(\tau)\, d\tau \right| \leq \left| \int_s^t q(\tau)\, d\tau \right|.$$

This together with the uniform continuity of G^{-1} on $[0, G(a_0)]$ and

$$|y_m(t) - y_m(s)| = |G^{-1}(G(y_m(t))) - G^{-1}(G(y_m(s)))|$$

immediately guarantees (3.14). A standard argument [4, page 53] using the Arzela-Ascoli theorem completes the proof.

Case (B). Suppose (3.9) holds.

We begin by showing

$$y_{m+1}(t) \leq y_m(t) \quad \text{for } t \in [0,T] \text{ for each } m \in N_0. \tag{3.15}$$

Suppose (3.15) is false. Then for some $m \in N_0$ there exists $\tau_1 < \tau_2$ with $y_{m+1}(\tau_1) = y_m(\tau_1)$, $y_{m+1}(\tau_2) > y_m(\tau_2)$ and $y_{m+1}(t) > y_m(t)$ for $t \in (\tau_1, \tau_2)$. As a result from (3.9) we have

$$0 < y_{m+1}(\tau_2) - y_m(\tau_2) = \int_{\tau_1}^{\tau_2} q(s)[f(s, y_{m+1}(s)) - f(s, y_m(s))]\, ds \leq 0,$$

a contradiction. As a result (3.15) holds.

Essentially the same reasoning as in Theorem 2.1 guarantees that there exist subsequences of integers $N_{n_0} \supseteq N_{n_0+1} \supseteq \cdots \supseteq N_k \supseteq \cdots$ and functions $z_k \in C[T/k, T]$ with y_m converging uniformly on $[T/k, T]$ to z_k as $m \to \infty$ through N_k, and $z_{k+1} = z_k$ on $[T/k, T]$.

Define a function $y : [0,T] \to [0, \infty)$ by $y(x) = z_k(x)$ on $[T/k, T]$ and $y(0) = 0$. Notice y is well-defined and $\alpha(t) \leq y(t) \leq a_0$ for $t \in (0,T]$. Next fix $t \in (0,T)$ and let $k \in \{n_0, n_0+1, \dots\}$ be such that $T/k < t < T$. Let $N_k^\star = \{n \in N_k : n \geq k\}$. Now y_m, $m \in N_k^\star$, satisfies

$$y_m(t) = y_m(T) - \int_t^T q(s) f(s, y_m(s))\, ds.$$

Let $m \to \infty$ through $N_k^\star$ to obtain $y(t) = y(T) - \int_t^T q(s) f(s, y(s))\, ds$. We can do this argument for each $t \in (0,T)$. It remains to show y is continuous at 0. Let $\epsilon > 0$

be given. Then there exists $n_1 \in N_0$ with $y_{n_1}(0) < \epsilon/2$, so there exists $\delta_{n_1} > 0$ with $y_{n_1}(t) < \epsilon/2$ for $t \in [0, \delta_{n_1}]$. From (3.15) for $m \geq n_1$ we have

$$\alpha(t) \leq y_m(t) \leq y_{n_1}(t) < \epsilon/2 \quad \text{for } t \in [0, \delta_{n_1}].$$

As a result $0 \leq \alpha(t) \leq y(t) \leq \epsilon/2 < \epsilon$ for $t \in (0, \delta_{n_1}]$, so y is continuous at 0.

In fact one can obtain a more general result motivated from Theorem 3.1 (II).

THEOREM 3.2. *Suppose* (3.2)–(3.5) *hold and in addition assume the following conditions are satisfied*:

$$\begin{cases} \text{for each } m \in N_0, \exists \beta_m \in C[0,T] \cap C^1(0,T] \text{ with} \\ q(t) f(t, \beta_m(t)) \leq \beta_m'(t) \text{ for } t \in (0,T) \text{ and } \beta_m(0) \geq 1/m \end{cases} \tag{3.16}$$

$$\alpha_m(t) \leq \beta_m(t), \quad t \in [0,T] \text{ for each } m \in N_0 \tag{3.17}$$

and

$$\begin{cases} \text{for each } t \in [0,T] \text{ we have that } \{\beta_m(t)\}_{m \in N_0} \text{ is a} \\ \text{nonincreasing sequence and } \lim_{m \to \infty} \beta_m(0) = 0. \end{cases} \tag{3.18}$$

Then (3.1) *has a solution* $y \in C[0,T] \cap C^1(0,T]$ *with* $y(t) \geq \alpha(t)$ *for* $t \in [0,T]$.

PROOF. Fix $m \in N_0$. Proceed as in Theorem 3.1 with β_m replacing β in $f_m^\star$. The same reasoning as in Theorem 3.1 guarantees that there exists a solution $y_m \in C[0,T]$ to (3.12) with $\alpha(t) \leq \alpha_m(t) \leq y_m(t) \leq \beta_m(t)$ for $t \in [0,T]$. Also as in Theorem 3.1 there exists $y \in C(0,T]$ (as described in Theorem 3.1 (II)) with

$$\alpha(t) \leq y(t) \leq a_0 = \sup_{t \in [0,T]} \beta_{n_0}(t) \quad \text{for } t \in (0,T],$$

with $y' = q f(t,y)$ for $0 < t < T$. It is easy to see (using (3.18)) that y is continuous at 0.

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