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THE STRUCTURE OF SOLUTIONS FOR EQUATIONS RELATED TO THE MOTIONS OF PLANE CURVES

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Abstract

We study the structure of solutions of an initial value problem arising in the study of steadily rotating spiral waves in the kinematic theory of excitable media. In particular, we prove that under certain conditions there is a unique global positive monotone increasing solution.

1. Introduction

In this paper, we study the initial value problem (P):

$$w'' - a(w)w' + \eta = 0, \quad y > 0, \tag{1.1}$$

$$w(0) = 0, \quad w'(0) = 1, \tag{1.2}$$

where $\eta \in \mathbb{R}$ is a parameter and $a(s)$ is a piecewise C^1 function satisfying $a(0) = 0$, $a'(s) > 0$ for $s \neq 0$, $a(s) \to -\infty$ as $s \to -\infty$, and

$$\int_0^\infty \frac{ds}{1 + A(s)} < \infty, \tag{1.3}$$

where $A(s) := \int_0^s a(t)\,dt$. Note that $A(s) \to \infty$ as $s \to -\infty$. A typical example is

$$a(s) = |s|^{\alpha-1}s, \quad \alpha > 0.$$

In particular, when $a(s) = s$ the initial value problem (P) arises in the study of steadily rotating spiral waves in the kinematic theory of excitable media. See, for

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example, [1, 4, 5], especially the nice review article of Meron [4] in 1992 and the references cited therein. Let κ be the curvature of a steadily rotating spiral wave whose tip neither grows nor retracts, κ_c be the curvature of the spiral arm at the tip, c_r be the normal velocity, s be the arc length, and ω be the constant frequency. Then we have the following relation (see, for example, (4.48) and (4.49) in [4]):

$$\kappa\left(\int_0^s \kappa(\xi)c_r(\xi)\,d\xi\right)+\frac{dc_r}{ds}=\omega,\quad s>0, \tag{1.4}$$
$$\lim_{s\to 0}\kappa(s)=\kappa_c,\quad \lim_{s\to\infty}\kappa(s)=0.$$

Using the following curvature-speed relation $c_r = c - D\kappa$, where c and D are positive constants, and by neglecting the higher order term of κ in the integral of (1.4), we have

$$c\kappa\left(\int_0^s \kappa(\xi)\,d\xi\right)-D\frac{d\kappa}{ds}=\omega.$$

Set

$$q(s)=\int_0^s \kappa(\xi)\,d\xi. \tag{1.5}$$

Then $q'(s)=\kappa(s)$ and q satisfies

$$Dq''-cqq'+\omega=0,\quad s>0.$$

Now we introduce the change of variables:

$$y=\sqrt{\frac{c\kappa_c}{D}}\,s,\quad w(y)=\sqrt{\frac{c}{D\kappa_c}}\,q(s),\quad \eta=\frac{\omega}{\sqrt{cD\kappa_c^3}}.$$

Then $w(y)$ satisfies (1.1) with $a(s)=s$. Also $w(0)=0$, $w'(0)=1$ and $w'(y)\to 0$ as $y\to\infty$.

For more references on spiral waves, we refer the reader to the paper of Ikota, Ishimura and Yamaguchi [3], the recent paper of Giga, Ishimura and Kohsaka [2], and the references cited therein.

The local existence and uniqueness of solutions of (P) for any given $\eta\in\mathbb{R}$ is trivial. We denote this solution by $w(y;\eta)$. Note that $w>0$ and $w'>0$ for small $y>0$.

The main theorem of this paper is as follows.

THEOREM 1.1. *There is a unique positive constant η^* such that the following hold*:

(i) *if $\eta>\eta^*$, then $w(y;\eta)$ has exactly one critical point and w exists globally such that $w(y;\eta)\to-\infty$ and $w'(y;\eta)\to 0$ as $y\to\infty$;*

(ii) *if $\eta = \eta^*$, then $w^*(y) := w(y;\eta^*)$ is a global positive monotone increasing solution such that $w^*(y) \to \infty$ and $(w^*)'(y) \to 0$ as $y \to \infty$;*

(iii) *if $\eta < \eta^*$, then $w(y;\eta)$ is monotone increasing in $[0, R)$ such that $w(y;\eta) \to \infty$ and $w'(y;\eta) \to \infty$ as $y \to R^-$ for some $R < \infty$.*

We give an interpretation of Theorem 1.1 for the case when $a(s) = s$ as follows. In this case, we have found a family of steadily rotating spiral waves. In this family there is a unique frequency $\omega^* := \eta^*\sqrt{cD\kappa_c^3}$ such that any curve in the plane with this frequency is a spiral wave with positive curvature. If we increase the frequency ω so that $\omega > \omega^*$, then the curvature of the curve would change sign exactly once. Furthermore, it follows from (1.5) that the rotation number defined by

$$\frac{1}{2\pi}\int_0^\infty \kappa(s)\,ds$$

is $+\infty$ when $\omega = \omega^*$ and is $-\infty$ when $\omega > \omega^*$. Hence they are indeed spiral waves.

This paper is organised as follows. In Section 2, we study the properties of various solutions of (P) for different values of the parameter η. After deriving a comparison principle, we prove Theorem 1.1 in Section 3. In particular, there is a unique global positive monotone increasing solution of (P). Finally, we give a discussion in Section 4.

2. Structure of solutions

We first note that, by (1.3), $A(s) \to \infty$ as $s \to \infty$. Moreover, $a(s) \to \infty$ as $s \to \infty$. Otherwise $A(s)$ grows at most linearly and so the first integral in (1.3) diverges.

Let $w(y;\eta)$ be the solution of (P) for a given $\eta \in \mathbb{R}$. Let

$$F(y) = w'(y) - A(w(y)), \quad y \geq 0. \tag{2.1}$$

By (1.1) and (1.2), it is easy to see that $F'(y) = -\eta$ and $F(0) = 1$. Hence we obtain that

$$w'(y) - A(w(y)) = 1 - \eta y. \tag{2.2}$$

We shall see that the above relation plays an important role in this study.

LEMMA 2.1. *Suppose that $\eta \leq 0$. Then $w(y;\eta)$ is monotone increasing in $[0, R)$ such that $w(y;\eta) \to \infty$ and $w'(y;\eta) \to \infty$ as $y \to R^-$ for some $R < \infty$.*

PROOF. Let $w(y) = w(y;\eta)$. By assumption, we have

$$w'(y) \geq A(w(y)) + 1 > 0 \tag{2.3}$$

for $y \geq 0$. Hence $w > 0$ in its existence interval $[0, R)$.

We claim that $R < \infty$. For contradiction, we suppose that $R = \infty$. By (2.3), we have

$$\frac{w'(y)}{1 + A(w(y))} \geq 1, \quad \forall y \geq 0,$$

and so

$$\int_0^{w(y)} \frac{ds}{1 + A(s)} \geq y \to \infty$$

as $y \to \infty$, contradicting (1.3). Hence $R < \infty$. It is trivial that $w(y) \to \infty$ and $w'(y) \to \infty$ as $y \to R^-$. This completes the proof.

In what follows, we always assume that $\eta > 0$. Since $w'' < 0$ for any critical point of w, there is at most one critical point of w.

LEMMA 2.2. *Suppose that for some $\eta > 0$ the corresponding solution $w(y; \eta)$ has exactly one critical point. Then w is global such that $w(y; \eta) \to -\infty$ and $w'(y; \eta) \to 0$ as $y \to \infty$.*

PROOF. Note that by assumption there is a unique $y_0 > 0$ such that $w'(y_0) = 0$. Then $w' > 0$ in $[0, y_0)$ and $w' < 0$ for $y > y_0$ as long as w exists.

Suppose that w is not global. Then $w(y) \to -\infty$ as $y \to R^-$ for some $R < \infty$. Otherwise, if $\lim_{y \to R^-} w(y)$ is finite, then there is a sequence $\{y_n\}$ such that $y_n \to R$ and $w'(y_n) \to 0$ as $n \to \infty$. By (2.2), the limit $\lim_{y \to R^-} w'(y)$ exists. Then we have $w'(y) \to 0$ as $y \to R^-$. This implies that w can be continued beyond R, a contradiction. Hence $w(y) \to -\infty$ as $y \to R^-$. But from (2.2) again it follows that $w'(y) = A(w(y)) + 1 - \eta y \to +\infty$ as $y \to R^-$, another contradiction. Hence w is global.

Note that the limit $l := \lim_{y \to \infty} w(y)$ exists. If $l > -\infty$, then, by (2.2), $\lim_{y \to \infty} w'(y) = -\infty$. This is impossible. Hence $w(y) \to -\infty$ as $y \to \infty$.

Next, we claim that either $w'' < 0$ in $[0, \infty)$ or $w'' > 0$ in (y_1, ∞) for some $y_1 > 0$ with $w''(y_1) = 0$. Indeed, if there exists $y_1 > 0$ such that $w'' < 0$ in $[0, y_1)$ and $w''(y_1) = 0$, then, by (1.1), $w(y_1) \neq 0$ and $w'(y_1) \neq 0$. By differentiating (1.1) with respect to y, we obtain that

$$w''' = a(w)w'' + a'(w)(w')^2. \tag{2.4}$$

Hence $w'''(y_1) > 0$ and so $w''(y) > 0$ for $y \in (y_1, y_1 + \delta)$ for some $\delta > 0$. It follows from (2.4) and the fact that $a'(s) > 0$, $\forall\, s \neq 0$, that $w''' > 0$ when $w'' = 0$. Therefore $w'' > 0$ in (y_1, ∞).

In any case, the limit $L := \lim_{y \to \infty} w'(y)$ exists and $L \leq 0$. If $L < 0$, then, by (1.1), $w''(y) \to \infty$ as $y \to \infty$. This contradicts the assumption $w' < 0$ in (y_0, ∞). We conclude that $w'(y) \to 0$ as $y \to \infty$. This proves the lemma.

From now on we assume that $w' > 0$ in its existence interval. By differentiating (1.1) with respect to y, we obtain for $v(y) := w'(y)$ that

$$v'' = a(w)v' + a'(w)v^2, \quad v(0) = 1. \tag{2.5}$$

Hence by (2.5) the function v has at most one critical point. Note that $v'(0) = w''(0) = -\eta < 0$.

LEMMA 2.3. *Given $\eta > 0$, suppose that $w'(y;\eta) > 0$ in its existence interval and that $w'(y;\eta)$ has exactly one critical point. Then $w(y;\eta) \to \infty$ and $w'(y;\eta) \to \infty$ as $y \to R^-$ for some $R < \infty$.*

PROOF. Suppose that w is global. We claim that $w(y) \to \infty$ and $w'(y) \to \infty$ as $y \to \infty$. Since $w' > 0$, the limit $l := \lim_{y\to\infty} w(y)$ exists and $l > 0$. By (2.2), $A(w(y)) = w'(y) + \eta y - 1 \geq \eta y - 1 \to \infty$ as $y \to \infty$. Hence $l = \infty$. Suppose that $v'(y_0) = 0$ for some $y_0 > 0$. Then by assumption $v' > 0$ for $y > y_0$. Hence the limit $L := \lim_{y\to\infty} w'(y)$ exists and $L > 0$. If $L < \infty$, then, by (1.1), $w''(y) \to \infty$ as $y \to \infty$, a contradiction. Therefore $L = \infty$. It follows from l'Hôpital's rule that

$$\lim_{y\to\infty} \frac{y}{1 + A(w(y))} = \lim_{y\to\infty} \frac{1}{a(w(y))w'(y)} = 0. \tag{2.6}$$

Now, from (2.2) and (2.6) it follows that

$$\lim_{y\to\infty} \frac{w'(y)}{1 + A(w(y))} = 1 - \eta \lim_{y\to\infty} \frac{y}{1 + A(w(y))} = 1.$$

Thus there exists $y_1 > y_0$ such that

$$\frac{w'(y)}{1 + A(w(y))} \geq \frac{1}{2}, \quad \forall y \geq y_1. \tag{2.7}$$

By integrating (2.7) from y_1 to y and letting $y \to \infty$, this leads to a contradiction. Hence w cannot be global.

It is clear that $w(y;\eta) \to \infty$ and $w'(y;\eta) \to \infty$ as $y \to R^-$ for some $R < \infty$. The lemma follows.

The following lemma is about the case when v has no critical point.

LEMMA 2.4. *Suppose that $w(y;\eta)$ is a global solution such that $w'(y;\eta) > 0$ for all $y \in (0,\infty)$ for some $\eta > 0$. Then $w'' < 0$ in $(0,\infty)$, $w(y;\eta) \to \infty$ and $w'(y;\eta) \to 0$ as $y \to \infty$. Conversely, if $w'(y;\eta) > 0$ and $w''(y;\eta) < 0$ as long as w exists for some $\eta > 0$, then $w(y;\eta)$ is global.*

PROOF. From $A(w(y)) = w'(y) + \eta y - 1$, it follows that $w(y) \to \infty$ as $y \to \infty$. Since w is global, by Lemma 2.3, $w'' < 0$ in $(0, \infty)$. For the contradiction, suppose that the limit $l := \lim_{y\to\infty} w'(y) > 0$. Then, by (1.1), $w''(y) \to \infty$ as $y \to \infty$, a contradiction. Hence $w'(y;\eta) \to 0$ as $y \to \infty$.

Now, suppose that there is an $\eta > 0$ such that $w'(y;\eta) > 0$ and $w''(y;\eta) < 0$ in the existence interval and that $w(y;\eta)$ is not global. Then $w(y) \to \infty$ as $y \to R^-$ for some $R < \infty$. Otherwise, by (2.2), both $w(y)$ and $w'(y)$ have finite limits and then w can be continued beyond R, a contradiction. Hence $w(y) \to \infty$ as $y \to R^-$ for some $R < \infty$. Therefore, by (2.2) again, $w'(y) \to \infty$ as $y \to R^-$, which contradicts the assumption that $w'' < 0$. Hence $w(y;\eta)$ is global if $w'(y;\eta) > 0$ and $w''(y;\eta) < 0$ as long as w exists. This proves the lemma.

3. Proof of Theorem 1.1

For convenience, we denote the existence interval of $w(y;\eta)$ by $[0, R)$ for some $R = R(\eta) > 0$. We begin with the following comparison principle.

LEMMA 3.1. *Let $w_i(y) = w(y;\eta_i)$, $R_i = R(\eta_i)$, $i = 1, 2$, and $R = \min\{R_1, R_2\}$. Suppose that $w_2' > 0$ in $[0, R_2)$ and $\eta_1 < \eta_2$. Then $w_1' > w_2'$ in $(0, R)$ and $w_1'' > w_2''$ in $[0, R)$.*

PROOF. Note that, by (1.1), the lemma holds for y small, since $\eta_1 < \eta_2$. Let $v_i = w_i'$. It follows from (1.1) that

$$\begin{aligned}(v_1 - v_2)' &= a(w_1)v_1 - a(w_2)v_2 + (\eta_2 - \eta_1)\\ &= a(w_1)(v_1 - v_2) + [a(w_1) - a(w_2)]v_2 + (\eta_2 - \eta_1). \qquad (3.1)\end{aligned}$$

Therefore to prove the lemma it suffices to prove that $w_1' > w_2'$ in $(0, R)$.

For the contradiction, suppose that there is $y_0 \in (0, R)$ such that $w_1' > w_2'$ in $(0, y_0)$ and $w_1'(y_0) = w_2'(y_0)$. Then $w_1(y_0) > w_2(y_0) > 0$ and $v_1'(y_0) \le v_2'(y_0)$, contradicting (3.1). This completes the proof.

The following lemma gives the uniqueness of a global monotone increasing solution of (P).

LEMMA 3.2. *There is at most one global monotone increasing solution of* (P) *such that $w(y) \to \infty$ and $w'(y) \to 0$ as $y \to \infty$.*

PROOF. Suppose that there are two distinct global monotone increasing solutions w_1, w_2 of (P) such that $w_i(y) \to \infty$ and $w_i'(y) \to 0$ as $y \to \infty$, $i = 1, 2$. Suppose that

$\eta_1 < \eta_2$, where $\eta_i = -w_i''(0)$, $i = 1, 2$. Then, by Lemma 3.1, $[w_1'(y) - w_2'(y)]' > 0$ in $[0, \infty)$. Since $w_1'(1) - w_2'(1) > 0$, the limit $\lim_{y\to\infty}[w_1'(y) - w_2'(y)]$ is strictly bigger than zero, a contradiction. The lemma follows.

We introduce the following two sets:

$$I_1 = \{\eta > 0 \mid w(y;\eta) \to \infty \text{ and } w'(y;\eta) \to \infty \text{ as } y \to R^- \text{ for some } R < \infty\},$$
$$I_2 = \{\eta > 0 \mid w(y;\eta) \text{ has exactly one critical point}\}.$$

By the continuous dependence on the parameter η in (1.1), it is trivial that the set I_2 is open.

LEMMA 3.3. *The set I_2 is nonempty.*

PROOF. Suppose that $I_2 = \emptyset$. Then $w'(y;\eta) > 0$, $\forall y \geq 0$, for all $\eta > 0$. We claim that $R(\eta) \geq R(0)$ for all $\eta > 0$. From (2.2) it follows that

$$\frac{w'(y)}{1 + A(w(y))} \leq 1, \quad \forall y \in [0, R(\eta)). \tag{3.2}$$

By an integration of (3.2), we get $w(y) \leq u(y)$ as long as w and u exist, where $u(y)$ is the solution of (1.1) with $\eta = 0$. Note that $u' > 0$ in $[0, R_0)$ and $u(y) \to \infty$ as $y \to R_0^-$ for some $R_0 := R(0) \in (0, \infty)$. Hence $R(\eta) \geq R_0$ for all $\eta > 0$. Notice that $\{w(R_0/2;\eta)\}_{\eta>0}$ is uniformly bounded.

On the other hand, from (2.2) it follows that

$$w'(R_0/2;\eta) = A(w(R_0/2;\eta)) + 1 - \eta R_0/2 \to -\infty$$

as $\eta \to \infty$, a contradiction. Thus I_2 is nonempty and the proof is complete.

We observe that for any $\eta \in (0, \infty) \setminus I_2$ the solution w satisfies $w'(y) > 0$ in $[0, R(\eta))$. Hence I_2 is connected by the comparison principle (Lemma 3.1). Therefore $I_2 = (\eta^*, \infty)$ for some $\eta^* \geq 0$. Indeed, $\eta^* = \inf I_2$.

By Lemmas 2.3 and 2.4, the solution $v(y;\eta)$ of (2.5) has one critical point if and only if $\eta \in I_1$. By the continuous dependence on the parameter η in (2.5), the set I_1 is open. Hence $\eta^* \notin I_1 \cup I_2$.

Let w^* be the corresponding solution of (1.1) for $\eta = \eta^*$. Then $(w^*)' > 0$ as long as w^* exists. Moreover, w^* is global. Otherwise, if $R(\eta^*) < \infty$, then $w^*(y) \to \infty$ as $y \to R^-(\eta^*)$. Hence $(w^*)'(y) \to \infty$ as $y \to R^-(\eta^*)$, by (2.2). This implies that $\eta^* \in I_1$, a contradiction. Hence w^* is global. Also, it follows from Lemma 2.4 that $w^*(y) \to \infty$ and $(w^*)'(y) \to 0$ as $y \to \infty$. We conclude from Lemma 2.1 that $\eta^* > 0$.

Observe that any solution $w(y;\eta)$ of (P) with $\eta \notin I_1 \cup I_2$ is global and satisfies $w(y) \to \infty$ and $w'(y) \to 0$ as $y \to \infty$. Hence $I_1 = (0, \eta^*)$. This completes the proof of Theorem 1.1.

4. Discussion

In this paper, we first reduce the initial value problem (P) for a second-order ODE to a first-order ODE (2.2) by an integration. It is trivial that the problem (P) is equivalent to (2.2) with the initial value $w(0) = 0$. Using this important relationship, we are able to classify the structure of solutions of (P). Then in Theorem 1.1 we prove the characterisation of solutions of (P) with respect to the parameter η.

In the case of excitable media, that is, when $a(s) = s$, we have found a family of steadily rotating spiral waves. In this family there is a unique frequency ω^* such that any curve in the plane with this frequency is a spiral wave with positive curvature. If we increase the frequency ω so that $\omega > \omega^*$, then the curvature of the wave would change sign exactly once.

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