

E. R. LOVE'S INTEGRAL EQUATION FOR THE CIRCULAR PLATE CONDENSER

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*In memory of Eric Russell Love (1912–2001)*²

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Abstract

In a classical paper, E. R. Love considered a certain function defined by a singular integral which is harmonic outside a circular disk. Love's objective was to derive a simple integral equation whose solution leads to a useful formula for the capacitance of the condenser consisting of two parallel circular plates. We close a gap in Love's derivation by finding a new nonsingular representation of Love's singular integral which permits one to draw the required conclusions about its boundary values and thereby establishes the correctness of Love's expression for the capacitance.

1. Introduction

Let D , D' denote two coaxial parallel unit disks lying a distance $\kappa > 0$ apart in R^3 ,

$$D = \{(x, y, z) : x^2 + y^2 \leq 1, z = 0\},$$

$$D' = \{(x, y, z) : x^2 + y^2 \leq 1, z = -\kappa\}.$$

In an important and frequently quoted paper, E. R. Love [7] considered the problem of determining the potential in space when D and D' form a condenser $\mathcal{C} = D \cup D'$ whose upper plate D is kept at potential 1 and whose lower plate D' is kept at potential -1 . We shall denote the potential by $L(x, y, z)$. Mathematically, L is

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²See John F. Clark, "Eric Russell Love, 1912–2001", *Austral. Math. Soc. Gaz.* **28** (2001), 249–251. In addition to the information in this obituary, the author of the present article, who himself once had the pleasure of meeting Prof. Love, would like to call attention to the reference to Love in the preface to the seventh edition of G. H. Hardy's famous "A Course of Pure Mathematics", Cambridge, 1938.

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uniquely determined by the requirements that it can be extended to all of R^3 in such a manner that it is harmonic in $R^3 \setminus \mathcal{C}$, assumes the values $+1$ on D and -1 on D' , is of order $1/\sqrt{x^2 + y^2 + z^2}$ as $x^2 + y^2 + z^2 \rightarrow \infty$, and is continuous in R^3 .

Our object is to review Love's method for determining $L(x, y, z)$, while at the same time filling a gap in Love's work brought about by the fact that since he considered only a normal approach to D and D' , Love failed to establish continuity of the potential at all points of \mathcal{C} . Especially when establishing the continuity of L at the centres of the plates, the question turns out to be technically nontrivial. It is not difficult to show by example that, unless continuity is proved at the centres of the plates allowing an unrestricted manner of approach, L is not uniquely determined. It is to be noted that Love, himself, was motivated by the desire to put on a firm basis formally derived and in part erroneous as well as even meaningless expressions in Nicholson's work [8]. Except for the aforementioned gap, which we fill here, he did succeed in doing this.

For the sake of readability we duplicate Love's work in part, especially in Section 3.

2. The function $V(x, y, z)$

Following [7], we focus attention on the transformation

$$V(x, y, z) = \frac{1}{\pi} \int_{-1}^1 \frac{f(t) dt}{\sqrt{r^2 + (z + it)^2}}, \quad (x, y, z) \in R^3 \setminus D, \quad r = \sqrt{x^2 + y^2}, \quad (2.1)$$

where $f(t)$ is a given function, real and continuous for $-1 \leq t \leq 1$, and $f(t) = f(-t)$. The square root in the denominator of the integrand is determined by the requirement that $\operatorname{Re} \sqrt{r^2 + (z + it)^2} > 0$. The resulting function V is real and is an even function of z . Differentiating under the integral sign, it is easily seen that the Laplacian of V vanishes; that is, V is harmonic in $R^3 \setminus D$. At a large distance from the origin,

$$V(x, y, z) \sim \left(\frac{1}{\pi} \int_{-1}^1 f(t) dt \right) / \sqrt{x^2 + y^2 + z^2}.$$

We will now proceed to find two alternative representations for V . The first one will have the advantage of avoiding the singularity of the kernel of (2.1) and thereby greatly simplify the problem of determining the limit behaviour of V as D is approached. Both representations are needed for Section 3.

THEOREM 2.1. For $(x, y, z) \in R^3 \setminus D$, let μ denote the positive root of the equation

$$\frac{x^2 + y^2}{1 + \mu} + \frac{z^2}{\mu} = 1, \quad (2.2)$$

and let

$$P(x, y, z) = \frac{2}{\pi} \arctan\left(\frac{1}{\sqrt{\mu}}\right) = 1 - \frac{2}{\pi} \arctan(\sqrt{\mu}). \quad (2.3)$$

Then

$$V(x, y, z) = \frac{2}{\pi} \int_0^{\pi P(x, y, z)/2} f\left(\sqrt{(x^2 + y^2) \sin^2 \alpha + z^2 \tan^2 \alpha}\right) d\alpha. \quad (2.4)$$

PROOF. With $r = \sqrt{x^2 + y^2}$, let $F(r, z, t) = r^2 + z^2 - t^2$, $G(r, z, t) = 2zt$,

$$Q(r, z, t) = \sqrt{F^2 + G^2} = \sqrt{(r^2 + z^2)^2 + 2(z^2 - r^2)t^2 + t^4},$$

$H(r, z, t) = \sqrt{(F + Q)/2}$ and

$$\begin{aligned} R(r, z, t) &= H/Q = \sqrt{F + Q}/(\sqrt{2}Q) \\ &= \frac{1}{\sqrt{2}} \frac{\sqrt{r^2 + z^2 - t^2} + \sqrt{(r^2 + z^2)^2 + 2(z^2 - r^2)t^2 + t^4}}{\sqrt{(r^2 + z^2)^2 + 2(z^2 - r^2)t^2 + t^4}}. \end{aligned}$$

Then

$$V(x, y, z) = V(r, z) = \frac{2}{\pi} \int_0^1 R(r, z, t) f(t) dt. \quad (2.5)$$

We will first consider the special case, $f(t) \equiv 1$, $-1 \leq t \leq 1$, and denote the value of $V(r, z)$ by $W(r, z)$ in this special case. We proceed to compute $W(r, z)$ explicitly by real variable transformations. This involves a number of changes of variable.

First, letting $a = z/r$ and replacing t by the new integration variable $u = t/r$ in (2.5), we obtain

$$W(r, z) = \frac{\sqrt{2}}{\pi} \int_0^{1/r} \frac{\sqrt{1 + a^2 - u^2} + \sqrt{(1 + a^2)^2 + 2(a^2 - 1)u^2 + u^4}}{\sqrt{(1 + a^2)^2 + 2(a^2 - 1)u^2 + u^4}} du. \quad (2.6)$$

Next, replacing u by the integration variable v , where $u = (\sqrt{1 + a^2})v$, and setting

$$b = \frac{a^2 - 1}{a^2 + 1} = \frac{z^2 - r^2}{z^2 + r^2}, \quad s = r\sqrt{1 + a^2} = \sqrt{r^2 + z^2} = \sqrt{x^2 + y^2 + z^2},$$

we obtain

$$W(r, z) = \frac{\sqrt{2}}{\pi} \int_0^{1/s} \frac{\sqrt{1 - v^2} + \sqrt{1 + 2bv^2 + v^4}}{\sqrt{1 + 2bv^2 + v^4}} dv. \quad (2.7)$$

Now let $c = \sqrt{1 - b^2} = 2|z|r/(z^2 + r^2)$. Before proceeding to the next change of variable, we must consider the possibilities, $r > 1$, $z = 0$, and $r = 0$, $z \neq 0$, separately. In either situation, $c = 0$, and by (2.7), we have

$$W(r, 0) = \frac{2}{\pi} \int_0^{1/r} \frac{dv}{\sqrt{1 - v^2}} = \frac{2}{\pi} \arcsin(1/r), \quad (r > 1), \quad (2.8)$$

while

$$W(0, z) = \frac{2}{\pi} \arctan(1/|z|). \quad (2.9)$$

Excluding these possibilities, we have $c > 0$ and can introduce a new integration variable η defined by $\eta = (v^2 + b)/c$, that is, $v = \sqrt{c\eta - b}$. The result is the following:

$$W(r, z) = \frac{1}{\sqrt{2\pi}} \int_{(z^2 - r^2)/2|z|r}^{(z^2 - r^2 + 1)/2|z|r} \frac{\sqrt{1 + b + c(\sqrt{1 + \eta^2} - \eta)}}{\sqrt{c\eta - b} \sqrt{1 + \eta^2}} d\eta. \quad (2.10)$$

For the next change of variable of integration, we set

$$\zeta = \eta + \sqrt{1 + \eta^2} > 0, \quad \text{that is, } \eta = (\zeta^2 - 1)/2\zeta.$$

This results in

$$\frac{\sqrt{1 + b + c(\sqrt{1 + \eta^2} - \eta)}}{\sqrt{c\eta - b} \sqrt{1 + \eta^2}} d\eta = \frac{\sqrt{2} \sqrt{c + (b + 1)\zeta}}{\zeta \sqrt{c\zeta^2 - 2b\zeta - c}} d\zeta.$$

Setting $h = (b + 1)/c = (b + 1)/\sqrt{1 - b^2} = |z|/r$, we can factor

$$c\zeta^2 - 2b\zeta - c = c \left(\zeta - \frac{b-1}{c} \right) \left(\zeta - \frac{b+1}{c} \right) = c(\zeta - h) \left(\zeta + \frac{1}{h} \right).$$

Hence

$$\frac{\sqrt{c + (b + 1)\zeta}}{\sqrt{c\zeta^2 - 2b\zeta - c}} = \frac{\sqrt{1 + h\zeta}}{\sqrt{(\zeta - h)(\zeta + 1/h)}} = \frac{\sqrt{h}}{\sqrt{\zeta - h}}$$

and

$$W(r, z) = \frac{\sqrt{h}}{\pi} \int_{\zeta_1}^{\zeta_2} \frac{d\zeta}{\zeta \sqrt{\zeta - h}}, \quad \text{where} \quad (2.11)$$

$$\zeta_1 = \frac{|z|}{r}, \quad \zeta_2 = \frac{z^2 - r^2 + 1 + \sqrt{4z^2 r^2 + (z^2 - r^2 + 1)^2}}{2|z|r}.$$

Finally, let $\lambda = \sqrt{\zeta - h}/\sqrt{h}$, that is, $\zeta = h(1 + \lambda^2)$. The result is

$$W(r, z) = \frac{2}{\pi} \int_0^{\lambda_2} \frac{d\lambda}{\lambda^2 + 1}, \quad \lambda_2 = \frac{\sqrt{1 - z^2 - r^2 + \sqrt{4z^2 + (1 - z^2 - r^2)^2}}}{\sqrt{2}|z|}. \quad (2.12)$$

When $r = 0, z \neq 0$, this agrees with (2.9). Therefore

$$W(r, z) = \frac{2}{\pi} \arctan \sqrt{\frac{1 - z^2 - r^2 + \sqrt{4z^2 + (1 - z^2 - r^2)^2}}{2z^2}}, \quad (z \neq 0). \quad (2.13)$$

By (2.8),

$$W(r, 0) = \frac{2}{\pi} \arctan \left(\frac{1}{\sqrt{r^2 - 1}} \right), \quad (r > 1). \quad (2.14)$$

Note that (2.14) is the limiting value of (2.13). Comparing (2.3) with (2.13) and (2.14), we see that $W(r, z) \equiv P(x, y, z)$ in $R^3 \setminus D$. One recognises ([3, 4]) $P(x, y, z)$ as the potential in $R^3 \setminus D$ due to a potential of +1 on the single disk D . The equipotential surfaces are the ellipsoids (2.2), parametrised by $\mu, 0 < \mu < \infty$. As $\mu \rightarrow 0$, the ellipsoids shrink down to D , while as $\mu \rightarrow \infty$, the ellipsoids approach spherical shells of radius $\sqrt{\mu}$. The corresponding value of $P(x, y, z)$ varies continuously from 1 to 0.

If we go back to (2.1) and carry out the same changes of variable that we have used for the evaluation of $W(r, z)$, we find in place of (2.12),

$$V(r, z) = \frac{2}{\pi} \int_0^{\lambda_2} \frac{1}{\lambda^2 + 1} f \left[\sqrt{z^2(1 + \lambda^2) - \frac{r^2}{1 + \lambda^2} - z^2 + r^2} \right] d\lambda. \quad (2.15)$$

Changing the variable of integration from λ to $\alpha = \arctan \lambda$, and noting that by (2.12),

$$\arctan(\lambda_2) = \frac{\pi}{2} W(r, z) = \frac{\pi}{2} P(r, z),$$

our formula (2.4) follows.

Note that a continuous extension of $P(x, y, z)$ to all of R^3 is obtained by defining $P(x, y, z)$ as 1 for $(x, y, z) \in D$. With this definition of $P(x, y, z)$, (2.4) yields a continuous extension of $V(x, y, z)$ to all of R^3 . Thus we can state the following result.

COROLLARY. $V(x, y, z)$ has boundary values for an unrestricted approach to D ; namely, if $0 \leq r = \sqrt{x^2 + y^2} \leq 1, z = 0$, then

$$V(x, y, z) = \frac{2}{\pi} \int_0^{\pi/2} f(r \sin \alpha) d\alpha, \quad (x, y, z) \in D, \quad (2.16)$$

provides a continuous extension of (2.1) to D .

The third representation of V is valid outside the plane containing D . It is as follows.

THEOREM 2.2. (Compare [7, pp. 439–440], for a less elementary derivation of essentially the same fact.) For $z \neq 0$, $r = \sqrt{x^2 + y^2}$, we have

$$V(x, y, z) = \frac{|z|}{\pi^2} \int_{-1}^1 \left[\int_0^\pi \frac{d\phi}{z^2 + (t - r \cos \phi)^2} \right] f(t) dt. \quad (2.17)$$

PROOF. We make use of the formula

$$\int_0^\pi \frac{d\phi}{w + i \cos \phi} = \frac{\pi}{\sqrt{1 + w^2}}, \quad \left(\operatorname{Re} w > 0, \operatorname{Re} \sqrt{1 + w^2} > 0 \right), \quad (2.18)$$

which is easily verified by writing the left-hand side as a contour integral over the unit circle and evaluating the contour integral by residues. By (2.1) and (2.18), if $z > 0$,

$$V(x, y, z) = \operatorname{Re} V(x, y, z) = \frac{1}{\pi^2} \int_{-1}^1 f(t) dt \int_0^\pi \operatorname{Re} \frac{1}{z + i(t + r \cos \phi)} d\phi.$$

Replacing ϕ by $\pi - \phi$ and recalling that V is an even function of z yields (2.17).

3. The integral equation

To find $L(x, y, z)$, the basic idea is to try to represent it as the difference of two functions of type (2.1), namely

$$L(x, y, z) = V_1(x, y, z) - V_2(x, y, z), \quad (x, y, z) \in \mathcal{C} = D \cup D', \quad (3.1)$$

where V_1, V_2 have been determined relative to D, D' , as the extensions to R^3 of the respective functions, ($r = \sqrt{x^2 + y^2}$),

$$V_1(x, y, z) = \frac{1}{\pi} \int_{-1}^1 \frac{f(t)}{\sqrt{r^2 + (z + it)^2}} dt, \quad (x, y, z) \in R^3 \setminus D \quad (3.2)$$

and

$$V_2(x, y, z) = \frac{1}{\pi} \int_{-1}^1 \frac{f(t)}{\sqrt{r^2 + (z + \kappa + it)^2}} dt, \quad (x, y, z) \in R^3 \setminus D'. \quad (3.3)$$

We do not know *a priori* that such an $f(t)$ exists, but the existence is established once we prove that L can be made to have the appropriate boundary values on the condenser \mathcal{C} .

LOVE'S THEOREM. *There exists a function $f(t)$ such that the function $L(x, y, z)$ defined by (3.1)–(3.3) has the correct boundary values on \mathcal{C} . The function $f(t)$ is the unique solution of the integral equation*

$$f(\tau) = 1 + \frac{1}{\pi} \int_{-1}^1 \frac{\kappa}{\kappa^2 + (\tau - t)^2} f(t) dt, \quad (-1 \leq \tau \leq 1). \quad (3.4)$$

PROOF. On the basis of Fredholm theory (see, for example, [7]), the integral equation (3.4) has a unique solution $f(t)$, and the solution is continuous and an even function.

For $(x, y, z) \in D$, we have, by (2.16), ($r = \sqrt{x^2 + y^2}$),

$$V_1(x, y, z) = \frac{1}{\pi} \int_0^\pi f(r \cos \phi) d\phi,$$

while, by (2.17),

$$V_2(x, y, z) = \frac{\kappa}{\pi^2} \int_{-1}^1 \left[\int_0^\pi \frac{d\phi}{\kappa^2 + (t - r \cos \phi)^2} \right] f(t) dt,$$

and therefore, assuming L is determined by (3.1) and f satisfies (3.4),

$$L(x, y, z) = \frac{1}{\pi} \int_0^\pi \left\{ f(r \cos \phi) - \frac{\kappa}{\pi} \int_{-1}^1 \frac{f(t) dt}{\kappa^2 + (t - r \cos \phi)^2} \right\} d\phi = 1.$$

By symmetry, we conclude, similarly, that $L(x, y, z) = -1$ on D' .

4. Capacitance

The capacitance $C(\kappa)$ of the condenser \mathcal{C} depends on the plate separation κ . Since the potential difference between the plates is 2, we have $C(\kappa) = q/2$, where q is the charge on the plate D . The charge can be calculated as the surface integral

$$q = -\frac{1}{4\pi} \iint \text{grad } V_1 dS,$$

with the integration over any smooth surface enclosing D , for example, over an equipotential surface near infinity. Referring to the behaviour of $V_1(x, y, z)$ near infinity, we conclude that

$$q = \frac{1}{\pi} \int_{-1}^1 f(t) dt,$$

where $f(t)$ is the solution of (3.4). Hence [7]

$$C(\kappa) = \frac{1}{\pi} \int_0^1 f(t) dt. \quad (4.1)$$

The case $\kappa \rightarrow \infty$ can be checked easily. It is tantamount to the condenser's consisting of D alone, with D' absent, and corresponds to $f(t) \equiv 1$. The well-known fact that

$$\lim_{\kappa \rightarrow \infty} C(\kappa) = 1/\pi \quad (4.2)$$

follows [3, 4, 13]. On the other extreme is the interesting case, $\kappa \rightarrow 0$, for which, as a first crude approximation, the capacitance behaves like $1/(4\pi\kappa)$ times the area of either of a pair of parallel plates. Since, in our case, the plates are disks of unit radius, this means that

$$\lim_{\kappa \rightarrow 0} \kappa C(\kappa) = 1/4. \quad (4.3)$$

Attempts to find more precise estimates go back to Kirchhoff and others. (See [10] and the bibliography there.) Hutson [2] used Love's theorem and (4.1) to prove that

$$C(\kappa) = \frac{1}{4\kappa} + \frac{1}{4\pi} \log\left(\frac{1}{\kappa}\right) + \frac{1}{4\pi} (\log 16\pi - 1) + o(1) \quad \text{as } \kappa \rightarrow 0. \quad (4.4)$$

In recent work of Soibelman's ([11, 12]) a different method is used to obtain further information on the $o(1)$ term of (4.4). (See also [5] and [6] for treatment of the asymptotic capacity when the electrode surfaces have more general shapes.)

In [9], the current author used Love's theorem and (4.1) to relate $C(\kappa)$ to a one-dimensional random walk.

Nicholson himself [8, p. 364] claimed to have found an explicit integral for the capacitance, namely

$$C(\kappa) = \frac{6}{\pi^3} \int_{-1}^1 Q_0(t) Q_0\left(\frac{\tanh(\pi t/\kappa)}{\tanh(\pi/\kappa)}\right) dt, \quad Q_0(x) = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right),$$

but this is incorrect. After some changes of variable of integration the expression becomes

$$C(\kappa) = \frac{6}{\pi^3} \int_0^1 \log\left(\frac{x}{1-x}\right) \log \sinh\left(\frac{2\pi x}{\kappa}\right) dx. \quad (4.5)$$

It does turn out that this satisfies (4.2). However, after some manipulation one finds that (4.5) implies that $\lim_{\kappa \rightarrow 0} [C(\kappa) - 6/(\pi^2\kappa)] = 0$, contradicting (4.3).

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