ANZIAM J. 47(2006), 359–366

EXPLICIT BOUNDS FOR THIRD-ORDER DIFFERENCE EQUATIONS

KENNETH S. BERENHAUT[1], EVA G. GOEDHART[1] and STEVO STEVIĆ[2]

(Received 30 April, 2005)

Abstract

This paper gives explicit, applicable bounds for solutions of a wide class of third-order difference equations with nonconstant coefficients. The techniques used are readily adaptable for higher-order equations. The results extend recent work of the authors for second-order equations.

2000 *Mathematics subject classification*: 39A10, 39A11.
Keywords and phrases: explicit bounds, applicable bounds, third-order linear difference equations, growth rates, nonconstant coefficients.

1. Introduction

This paper studies explicit, applicable growth rates for third-order difference equations. In particular, we will consider solutions $\{b_n\} = \{b_n(b_0, b_1, b_2)\}$ of equations of the form

$$\Delta^3 b_{n-2} = p_n b_n - q_n b_{n-1} + r_n b_{n-2}, \tag{1.1}$$

where for a sequence $\{a_i\}$, Δ is the forward difference operator and $\Delta a_i = a_{i+1} - a_i$. That is,

$$b_{n+1} = (3 + p_n)b_n - (3 + q_n)b_{n-1} + (1 + r_n)b_{n-2}, \tag{1.2}$$

for $n \geq 2$. We provide sharp inequalities for $\{b_i\}$ in terms of the sequences $\{p_i\}$, $\{q_i\}$ and $\{r_i\}$, and the initial values b_0, b_1 and b_2. Solutions of difference equations of the

[1]Wake Forest University, Department of Mathematics, Winston-Salem, NC 27109, USA; e-mail: berenhks@wfu.edu.
[2]Mathematical Institute of Serbian Academy of Science, Knez Mihailova 35/I 11000 Beograd, Serbia; e-mail: sstevic@ptt.yu and sstevo@matf.bg.ac.yu.

form in (1.1) have been studied by many authors (see, for example, [2, 3, 4, 5, 6, 7, 8, 9, 12, 10, 11]). Often these studies have focused on the understanding of oscillatory or asymptotic behaviour.

In what follows, it will be convenient to have the following notation.

For a sequence $a = \{a_i\}$, define the linear operator $\mathscr{L}$ by

$$\mathscr{L}(a)_i \stackrel{\text{def}}{=} p_{i+1}a_{i+1} - q_{i+1}a_i + r_{i+1}a_{i-1}, \quad \text{for } i \geq 1.$$

We now state our main result which extends recent results for second-order equations (see, for example, [1] and [13]). Closely related results can also be found in [14].

Theorem 1.1. *Suppose $\{B_i\}$, $\{p_i\}$, $\{q_i\}$ and $\{r_i\}$ are real-valued sequences such that $\{B_i\}$ is positive, nondecreasing and convex, and for each $i \geq 2$, either*

$$p_i \geq \max(q_i, 0) \quad \textit{and} \quad r_i \geq 0, \quad \underline{\textit{or}} \quad q_i \leq \min(r_i, 0) \quad \textit{and} \quad p_i \geq 0. \tag{1.3}$$

In addition, suppose there exist positive constants, c_0, c_1 and c_2, satisfying

$$\Delta^2 B_i \geq \begin{cases} c_0 \Delta^2 b_i(1,0,0) = \Delta^2 b_i(c_0, 0, 0) \geq 0, \\ c_1 \Delta^2 b_i(0,-1,0) = \Delta^2 b_i(0, -c_1, 0) \geq 0, \\ c_2 \Delta^2 b_i(0,0,1) = \Delta^2 b_i(0, 0, c_2) \geq 0, \end{cases} \tag{1.4}$$

for $i = 0, 1, 2$. Now, define the sequence $\{V(i)\}$ via

$$V(i) \stackrel{\text{def}}{=} \Delta^3 B_{i-1} - \mathscr{L}(B)_i, \qquad \textit{for } i \geq 1. \tag{1.5}$$

If

$$V(i) \geq 0, \qquad \textit{for } i \geq 3, \tag{1.6}$$

then

$$|b_n| \leq \left(\frac{|b_0|}{c_0} + \frac{|b_1|}{c_1} + \frac{|b_2|}{c_2} \right) B_n, \quad \textit{for } n \geq 3. \tag{1.7}$$

The key to employing Theorem 1.1 is to determine a positive, nondecreasing sequence B satisfying (1.4) and (1.6). While this can be done inductively for many $\{(p_j, q_j, r_j)\}$, it is particularly convenient when the third derivative of an extension, $\tilde{B}$, to $[0, \infty)$ of the bounding sequence B exists. The next lemma follows directly from the fact that $\Delta^3 B_{n-1} = B'''(\zeta)$, for some $\zeta \in [n-1, n+2]$.

Lemma 1.2. *Suppose $\tilde{B}'''$ exists.*

(1) *If $\tilde{B}'''$ is nondecreasing, and for $n \geq n_0$, $\mathscr{L}(B)_n \leq \tilde{B}'''(n-1)$, then $V(n) \geq 0$ for $n \geq n_0$.*

(2) *If B''' is nonincreasing, and for $n \geq n_0$, $\mathscr{L}(B)_n \leq \tilde{B}'''(n+2)$, then $V(n) \geq 0$ for $n \geq n_0$.*

It will be helpful to have the following notation, which will be useful when demonstrating that (1.4) holds for particular examples.

For given $\{b_i\}$ and $\{B_i\}$, define $\boldsymbol{G}$ and $\boldsymbol{h}$ via

$$\boldsymbol{G} = \begin{bmatrix} g_{0,0} & g_{0,1} & g_{0,2} \\ g_{1,0} & g_{1,1} & g_{1,2} \\ g_{2,0} & g_{2,1} & g_{2,2} \end{bmatrix}$$
$$\stackrel{\text{def}}{=} \begin{bmatrix} \Delta^2 b_0(1,0,0) & \Delta^2 b_1(1,0,0) & \Delta^2 b_2(1,0,0) \\ \Delta^2 b_0(0,-1,0) & \Delta^2 b_1(0,-1,0) & \Delta^2 b_2(0,-1,0) \\ \Delta^2 b_0(0,0,1) & \Delta^2 b_1(0,0,1) & \Delta^2 b_2(0,0,1) \end{bmatrix}$$

and $\boldsymbol{h} = (h_0, h_1, h_2) \stackrel{\text{def}}{=} (\Delta^2 B_0, \Delta^2 B_1, \Delta^2 B_2)$. Note that (1.4) can be rewritten as $h_i \geq c_j g_{j,i} \geq 0$, for $0 \leq i, j \leq 2$. In fact, if $h_i > 0$ and $g_{j,i} > 0$, for $0 \leq i \leq 2$, we may take $c_j = \min_{0\leq i\leq 2}\{h_i/g_{j,i}\}$.

We now give some examples of applications for Theorem 1.1.

EXAMPLE 1 (Power-type rate bounds). Consider $\{B_n\}$ defined by $B_n = n^k$ (with $k \in \mathbb{R}$), and note that $\tilde{B}$ given by $\tilde{B}(x) = x^k$, is positive, nondecreasing and convex for $k \geq 1$. Taking derivatives gives $\tilde{B}'''(x) = k(k-1)(k-2)x^{k-3}$ and $\tilde{B}^{(4)}(x) = k(k-1)(k-2)(k-3)x^{k-4}$, and hence $\tilde{B}'''$ is nondecreasing for $1 \leq k \leq 2$ and $k \geq 3$, and nonincreasing for $2 \leq k \leq 3$.

Now, set $c = k(k-1)(k-2)$. Employing Lemma 1.2, each of the following satisfy (1.6) of Theorem 1.1:

(i) $p \equiv q \equiv 0$, $k \geq 3$, and for $n \geq 3$,

$$0 \leq r_{n+1} \leq \frac{c}{(n-1)^3}\,; \tag{1.8}$$

(ii) $p \equiv q \equiv 0$, $k \in [2,3]$, and for $n \geq 3$,

$$0 \leq r_{n+1} \leq \left(\frac{n+2}{n-1}\right)^k \frac{c}{(n+2)^3}\,; \tag{1.9}$$

(iii) $q \equiv r \equiv 0$, $k \geq 3$, and for $n \geq 3$,

$$0 \leq p_{n+1} \leq \frac{c(n-1)^{k-3}}{(n+1)^k}\,; \quad \text{and} \tag{1.10}$$

(iv) $q \equiv r \equiv 0$, $k \in [2,3]$, and for $n \geq 3$,

$$0 \leq p_{n+1} \leq \frac{c(n+2)^{k-3}}{(n+1)^k}\,. \tag{1.11}$$

For $k \geq 2$, c is nonnegative, and hence the sequences in (i)–(iv) all satisfy (1.3). Now, note that (a) r_n defined by $r_n = c/n^3$ satisfies both (1.8) and (1.9), and (b) p_n defined by $p_n = c/(n+1)^3$ satisfies (1.11). We will consider these two instances in some detail.

(a) ($r_n = c/n^3$) That $r_n = c/n^3$ satisfies (1.8) is immediate. To see that the right-hand inequality in (1.9) also holds, note that

$$(n+2)^{3-k}(n-1)^k \leq (n+2)(n-1)^2 = n^3 - 3n + 2 < (n+1)^3.$$

Now, employing the formulae in Table 2 below, we have the values

$$\boldsymbol{G} = \begin{bmatrix} 1 & 1 + c/8 & 1 + c/8 \\ 2 & 2 & 2 - c/27 \\ 1 & 1 & 1 \end{bmatrix}.$$

Hence there exist $c_0 > 0$, $c_1 > 0$ and $c_2 > 0$ satisfying (1.4), whenever $0 < c/27 < 2$, that is, $2 < k < k_0$, where $k_0 \approx 4.867936$. For example, when $k = 3$ ($c = 6$), we have $\boldsymbol{h} = (6, 12, 18)$ and

$$\boldsymbol{G} = \begin{bmatrix} 1 & 7/4 & 7/4 \\ 2 & 2 & 16/9 \\ 1 & 1 & 1 \end{bmatrix}.$$

Thus, taking ratios as suggested earlier, we may use $c_0 = c_2 = 6$ and $c_1 = 3$ in (1.7).

(b) ($p_n = c/(n+1)^3$) Here we have

$$p_{n+1} = \frac{c}{(n+2)^3} \leq \frac{c}{(n+2)^3}\left(\frac{n+2}{n+1}\right)^k$$

and (1.11) is satisfied. In addition,

$$\boldsymbol{G} = \begin{bmatrix} 1 & 1 & 1 + p_3 \\ 2 & 2 & 2 + 3p_3 \\ 1 & 1 + p_2 & 1 + 3p_3 + p_2 + p_3 p_2 \end{bmatrix}, \tag{1.12}$$

and since each entry in (1.12) is strictly positive, there exist $c_0 > 0$, $c_1 > 0$ and $c_2 > 0$ satisfying (1.4), for all $2 < k \leq 3$. For example, when $k = 2.5$ ($c = 1.875$), we have

$$\boldsymbol{h} = \left(4\sqrt{2} - 2,\ 9\sqrt{3} - 8,\ 32 - 18\sqrt{3} + 4\sqrt{2}\right)$$
$$\approx (3.656854248,\ 5.27474877,\ 6.479939708).$$

Hence, employing (1.12) with $p_2 = 5/72 \approx 0.06944444444$ and $p_3 = 15/512 \approx 0.02929687500$, we may take $c_0 = c_2 = 3.65$ and $c_1 = 1.82$ in (1.7).

EXAMPLE 2 (Exponential rate bounds). Consider $B = \{B_n\}$ and $\tilde{B}$ defined by $B_n = ne^n$ and $\tilde{B}(x) = xe^x$, respectively. We then have $\tilde{B}'''(x) = (x+3)e^x$, and hence $\tilde{B}'''$ is nondecreasing. Employing Lemma 1.2, each of the following satisfy the requirements of Theorem 1.1:

(i) $p \equiv q \equiv 0$ and for $n \geq 3$,

$$0 \leq r_{n+1} \leq \frac{n+2}{n-1} \quad \text{and} \tag{1.13}$$

(ii) $q \equiv r \equiv 0$, and for $n \geq 3$,

$$0 \leq p_{n+1} \leq \frac{(n+2)e^{-2}}{n+1}. \tag{1.14}$$

As an example of r_n satisfying (1.13), we have $r_n = (n+1)/(n-1)$. Here

$$\begin{aligned} \boldsymbol{h} &= \left(2\,e^2 - 2\,e,\ 3\,e^3 - 4\,e^2 + e,\ 4\,e^4 - 6\,e^3 + 2\,e^2\right) \\ &\approx (9.341548544,\ 33.41866819,\ 112.6574908), \end{aligned}$$

$r_2 = 3$, $r_3 = 2$ and

$$\boldsymbol{G} = \begin{bmatrix} 1 & 4 & 4 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}. \tag{1.15}$$

Thus $c_0 = 8.35$, $c_1 = 4.67$ and $c_2 = 9.34$ satisfy (1.4), and Theorem 1.1 is applicable.

We now turn to a proof of Theorem 1.1.

2. Proof of Theorem 1.1

In this section we will prove Theorem 1.1.

Prior to proving Theorem 1.1 we quote the following two tables which we use in the proof of the theorem.

TABLE 1. Values for $\{b_i\}$.

Case	b_0	b_1	b_2	b_3	b_4
1	c_0	0	0	$(1+r_2)c_0$	$(3+p_3)(1+r_2)c_0$
2	0	$-c_1$	0	$(3+q_2)c_1$	$(3+p_3)(3+q_2)c_1 - (1+r_3)c_1$
3	0	0	c_2	$(3+p_2)c_2$	$(3+p_3)(3+p_2)c_2 - (3+q_3)c_2$

TABLE 2. Second-order differences for $\{b_i\}$.

Case	$\Delta^2 b_0$	$\Delta^2 b_1$	$\Delta^2 b_2$
1	c_0	$(1+r_2)c_0$	$(1+p_3)(1+r_2)c_0$
2	$2c_1$	$(2+q_2)c_1$	$(1+p_3)(3+q_2)c_1-(1+r_3)c_1$
3	c_2	$(1+p_2)c_2$	$(1+p_3)(3+p_2)c_2-(2+q_3)c_2$

PROOF OF THEOREM 1.1. Suppose $\{p_i\}$, $\{q_i\}$, $\{r_i\}$, $\{B_i\}$ and (c_0, c_1, c_2) satisfy the hypotheses of the theorem. We will consider three cases for $\{b_i(b_0, b_1, b_2)\}$, namely Case 1: $\{b_i(c_0, 0, 0)\}$, Case 2: $\{b_i(0, -c_1, 0)\}$ and Case 3: $\{b_i(0, 0, c_2)\}$. The values in Tables 1 and 2 follow directly from (1.2).

Now, note that, for each case, $b_2 \geq 0$, $\Delta b_1 \geq 0$, and by (1.4), $\Delta^2 b_i \geq 0$, for $i = 0, 1, 2$. Also, for $n \geq 2$, expanding b_{n+1} via (1.2) and simplifying, gives

$$\Delta^2 b_{n-1} = b_{n+1} - 2b_n + b_{n-1} = \Delta^2 b_{n-2} + \mathscr{L}(b)_{n-1}. \tag{2.1}$$

Assuming that $\Delta^2 b_i \geq 0$ for $i < N-1$, gives $b_i \geq 0$ for $2 \leq i < N+1$ and $\Delta b_i \geq 0$ for $1 \leq i < N$. Hence (1.3) implies that either

$$\mathscr{L}(b)_{N-1} = p_N b_N - q_N b_{N-1} + r_N b_{N-2} \geq (p_N - q_N) b_{N-1} + r_N b_{N-2} \geq 0$$

or $\mathscr{L}(b)_{N-1} \geq p_N b_N + (-q_N + r_N) b_{N-2} \geq 0$. Thus, combining this with the induction hypothesis and (2.1) gives $\Delta^2 b_{N-1} \geq 0$, and the induction is complete. In particular, we have $\Delta b_i \geq 0$ for $i \geq 1$ and $b_i \geq 0$, for $i \geq 2$.

Now, for $i \geq 0$, define ϵ_i by $\epsilon_i \stackrel{\text{def}}{=} B_i - b_i$. The values of ϵ_i, for the first few i, are given in Table 3.

TABLE 3. Values for $\{\epsilon_i\}$.

Case	ϵ_0	ϵ_1	ϵ_2
1	$B_0 - c_0$	B_1	B_2
2	B_0	$B_1 + c_1$	B_2
3	B_0	B_1	$B_2 - c_2$

We will show that $\epsilon_i \geq 0$ for all $i \geq 3$; the result in (1.7) then follows, since for general b_0, b_1 and b_2, we then have

$$\begin{aligned} |b_n(b_0, b_1, b_2)| &= \left| \frac{b_0}{c_0} b_n(c_0, 0, 0) - \frac{b_1}{c_1} b_n(0, -c_1, 0) + \frac{b_2}{c_2} b_n(0, 0, c_2) \right| \\ &\leq \frac{|b_0|}{c_0} B_n + \frac{|b_1|}{c_1} B_n + \frac{|b_2|}{c_2} B_n. \end{aligned}$$

Note that (1.4) guarantees that $\Delta^2\epsilon_i \geq 0$, for $i = 0, 1, 2$ and the assumptions on B give $\Delta\epsilon_0 > 0$ and $\epsilon_1 > 0$ (see Table 3). Now, assume $\Delta^2\epsilon_n \geq 0$, for $n < N$. It then follows immediately that

$$\epsilon_n \geq \epsilon_{n-1} \geq 0, \tag{2.2}$$

for $1 \leq n < N + 2$. Hence we have

$$\begin{aligned}
\Delta^2\epsilon_N &= \Delta^2 B_N - \Delta^2 b_N \\
&= \Delta^3 B_{N-1} + \Delta^2 B_{N-1} - \Delta^2 b_N \\
&= \Delta^3 B_{N-1} + \Delta^2 B_{N-1} - b_{N+2} + 2b_{N+1} - b_N \\
&= \Delta^3 B_{N-1} + \Delta^2 B_{N-1} - \big((3 + p_{N+1})b_{N+1} \\
&\quad - (3 + q_{N+1})b_N + (1 + r_{N+1})b_{N-1}\big) + 2b_{N+1} - b_N \\
&= (\Delta^3 B_{N-1} - p_{N+1}B_{N+1} + q_{N+1}B_N - r_{N+1}B_{N-1}) \\
&\quad + p_{N+1}\epsilon_{N+1} - q_{N+1}\epsilon_N + r_{N+1}\epsilon_{N-1} + (\Delta^2 B_{N-1} - \Delta^2 b_{N-1}) \\
&\geq V(N) + \Delta^2\epsilon_{N-1} \\
&\geq 0.
\end{aligned} \tag{2.3}$$

The second to last inequality in (2.3) follows from (2.2) and (1.3). The final inequality follows from (1.6) and the induction hypothesis. Thus $\{\epsilon_i\}$ is positive (and convex), and as mentioned, (1.7) now follows. □

Acknowledgements

The first author acknowledges financial support from a Sterge Faculty Fellowship.

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