

## WATER WAVE SCATTERING BY TWO SUBMERGED NEARLY VERTICAL BARRIERS

B. N. MANDAL<sup>✉1</sup> and SOUMEN DE<sup>1</sup>

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### Abstract

The problem of surface water wave scattering by two thin nearly vertical barriers submerged in deep water from the same depth below the mean free surface and extending infinitely downwards is investigated here assuming linear theory, where configurations of the two barriers are described by the same shape function. By employing a simplified perturbational analysis together with appropriate applications of Green's integral theorem, first-order corrections to the reflection and transmission coefficients are obtained. As in the case of a single nearly vertical barrier, the first-order correction to the transmission coefficient is found to vanish identically, while the correction for the reflection coefficient is obtained in terms of a number of definite integrals involving the shape function describing the two barriers. The result for a single barrier is recovered when two barriers are merged into a single barrier.

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### 1. Introduction

Explicit solutions to water wave scattering problems involving a thin vertical plane barrier of different geometrical configurations for the case of normal incidence of a surface wave train in deep water have been obtained in the literature of linearised theory of water waves by employing various mathematical techniques (see, for example, Ursell [18], Williams [19], Evans [1], Porter [16]). Explicit solutions have also been obtained for two thin vertical barriers partially immersed to the same depth below the mean free surface or submerged from the same depth and extending infinitely downwards in deep water (see, for example, Levine and Rodemich [6], Jarvis [4]). The works

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<sup>1</sup>Physics and Applied Mathematics Unit, Indian Statistical Institute, 203, B.T. Road, Kolkata 700 108, India; e-mail: [biren@isical.ac.in](mailto:biren@isical.ac.in).

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of Levine and Rodemich [6] and Jarvis [4] employ the Schwartz-Christoffel transformation of complex variable theory. Evans and Morris [3] reinvestigated Levine and Rodemich's problem [6] of two equal surface-piercing barriers and obtained *approximate* expressions for the reflection and transmission coefficients which are simpler to compute numerically than Levine and Rodemich's *explicit* results [6] involving definite integrals whose integrands are complicated functions of elliptic integrals. For two unequal surface-piercing barriers in finite depth water, McIver [12] used the method of matched eigenfunction expansions to obtain very accurate numerical estimates for the reflection and transmission coefficients. A substantial amount of research relating to water wave scattering problems involving thin vertical barriers has been carried out during the last six decades (see Mandal and Chakrabarti [8]). Problems involving thin curved barriers or inclined straight plane barriers have also been studied mostly by using hypersingular integral equation formulations (see, for example, Parsons and Martin [14] and [15], Midya *et al.* [13], Kanoria and Mandal [5], Mandal and Gayen (Chowdhury) [9]) which essentially involve somewhat heavy numerical computations in obtaining numerical estimates for the reflection and transmission coefficients. A thin plate in the form of a circular arc symmetric about the vertical through its centre was investigated by McIver and Urka [11] by two methods, one based on the method of matched series expansions and the other based on Schwinger variational approximation.

If the thin barrier differs slightly from the vertical position and in general is of a curved nature described by a shape function, the corresponding scattering problem has no explicit solution. An integral equation formulation of the problem is always possible, whose explicit solution is almost impossible to obtain. For the case of a nearly vertical surface-piercing thin barrier, Shaw [17] employed a perturbational approach to obtain first-order corrections to the reflection and transmission coefficients  $R_1$ ,  $T_1$  in terms of some definite integrals involving the shape function, which are however somewhat difficult to evaluate. He used a physical argument to prove that  $T_1$  vanishes identically and then used this result to simplify  $R_1$ . Soon afterwards Mandal and Chakrabarti [7] devised a simplified perturbation method employed for the governing partial differential equation, the boundary and other conditions, describing the original problem. This procedure reduced to first order, and the original problem became two problems involving a vertical barrier. The solution of the first problem is well known in the literature. Without solving the second problem, the first-order corrections  $R_1$ ,  $T_1$  were obtained by simple applications of Green's integral theorem. That  $T_1$  vanishes identically was shown in a straightforward manner, and the result for  $R_1$  in the form obtained by Shaw [17] for the partially immersed barrier was also obtained without much effort. The complementary problem of a nearly vertical barrier submerged in deep water and extending infinitely downwards was also investigated by Mandal and Chakrabarti [7] by the same procedure. The case of a nearly vertical thin plate was

considered by Mandal and Kundu [10] employing Shaw's [17] method as well as the method employed by Mandal and Chakrabarti [7].

In the present paper we consider the problem of water wave scattering by two nearly thin vertical barriers submerged in deep water from the same depth below the mean free surface and extending infinitely downwards, the two barriers being described by the same shape function. As mentioned earlier, the corresponding problem of two thin vertical barriers was investigated by Jarvis [4] wherein the potential function describing the motion in water was obtained in closed form in principle and the reflection and transmission coefficients were obtained in terms of some definite integrals. The problem of two nearly vertical barriers is investigated here by employing the aforesaid simplified perturbation analysis used by Mandal and Chakrabarti [7] and Mandal and Kundu [10] for the case of single nearly vertical barriers. The first-order corrections  $R_1$ ,  $T_1$  to the reflection and transmission coefficients are obtained by appropriate use of Green's integral theorem. As in the case of a single barrier,  $T_1$  vanishes here identically while  $R_1$  is obtained in terms of definite integrals involving the shape function. The known results for a single submerged nearly vertical barrier are recovered when the two barriers are merged to a single barrier by making the distance between the two submerged edges of the barriers tend to zero.

By using the complementarity theorem that the transmission coefficient remains unaltered if the scattering body is reversed but the incident field is left unchanged, the first-order correction  $T_1$  to the transmission coefficient can be shown to vanish identically. This result holds for any number of barriers.

## 2. Statement of the problem

We consider the two-dimensional problem of water wave scattering, assuming linear theory, by two fixed nearly vertical barriers submerged in deep water from the same depth below the mean free surface whose position is given by  $y = 0$ , the  $y$ -axis being chosen vertically downwards into the fluid region, and the  $x$ -axis taken along the direction of an incoming train of surface waves. The co-ordinate system is nondimensionalised with respect to depth of the upper edges of the barriers.

Let the configurations of the barriers be described by  $x = \pm a + \epsilon c(y)$ ,  $y \geq 1$ , where  $\epsilon$  is a small nondimensional number signifying the nearness of the nearly vertical barriers and  $c(y)$  is the shape function defined for  $y \geq 1$  satisfying  $c(1) = 0$  and is a bounded and continuous function of  $y$ . Assuming the motion in the fluid to be irrotational and simple harmonic in time  $t$  with angular frequency  $\sigma$ , it can be described by a velocity potential  $\text{Re}\{\phi(x, y)e^{-i\sigma t}\}$ .

Then  $\phi(x, y)$  satisfies the Laplace equation

$$\nabla^2 \phi = 0 \quad \text{in the fluid region,} \quad (2.1)$$

the linearised free surface condition

$$k\phi + \phi_y = 0 \quad \text{on } y = 0, \quad (2.2)$$

with  $k = \sigma^2/g$ ,  $g$  being the acceleration due to gravity, the barrier conditions

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } x = \pm a + \epsilon c(y), \quad y \geq 1, \quad (2.3)$$

where  $n$  denotes the normal to the surface of the curved barriers, the edge conditions

$$r^{1/2}\nabla\phi \quad \text{is bounded as } r \rightarrow 0, \quad (2.4)$$

where  $r$  is the distance from the points  $(\pm a, 1)$ , the deep water conditions

$$\phi, \nabla\phi \rightarrow 0 \quad \text{as } y \rightarrow \infty, \quad (2.5)$$

and finally, the infinity requirements given by

$$\phi(x, y) \sim \begin{cases} T\phi^{\text{inc}}(x, y) & \text{as } x \rightarrow \infty, \\ \phi^{\text{inc}}(x, y) + R\phi^{\text{inc}}(-x, y) & \text{as } x \rightarrow -\infty. \end{cases} \quad (2.6)$$

In the condition (2.6),  $R$  and  $T$  respectively denote the (complex) reflection and transmission coefficients to be determined and

$$\phi^{\text{inc}}(x, y) = e^{-ky+ikx} \quad (2.7)$$

denotes the incident wave potential propagating from the direction of  $x = -\infty$ .

### 3. Method of solution

For nearly vertical barriers, the parameter  $\epsilon$  can be assumed to be very small. The boundary conditions (2.3) on the nearly vertical barriers can be expressed as

$$\frac{\partial\phi(\pm a \pm 0, y)}{\partial x} - \epsilon \frac{d}{dy} \left\{ c(y) \frac{\partial\phi(\pm a \pm 0, y)}{\partial y} \right\} + O(\epsilon^2) = 0 \quad \text{for } y > 1, \quad (3.1)$$

where  $\pm 0$  denote values on two sides of each barrier. The forms of the approximate boundary conditions (3.1) suggest that  $\phi$ ,  $R$  and  $T$  have the following perturbational expansions, in terms of the small parameter  $\epsilon$ :

$$\begin{aligned} \phi(x, y; \epsilon) &= \phi_0(x, y) + \epsilon\phi_1(x, y) + o(\epsilon^2), \\ R(\epsilon) &= R_0 + \epsilon R_1 + o(\epsilon^2), \\ T(\epsilon) &= T_0 + \epsilon T_1 + o(\epsilon^2). \end{aligned} \quad (3.2)$$

Substituting the expansions (3.2) into the basic partial differential equation (2.1), the free surface condition (2.2), the approximate boundary conditions (3.1) and the condition (2.4)–(2.6), we find, after equating the coefficients of identical powers of  $\epsilon$  on both sides of the results, that  $\phi_0$  and  $\phi_1$  satisfy the following two problems  $P_1$  and  $P_2$  respectively.

**PROBLEM  $P_1$ .** The function  $\phi_0$  satisfies

- (i)  $\nabla^2 \phi_0 = 0$  in  $y > 0$ ,  $-\infty < x < \infty$ ;
- (ii)  $k\phi_0 + \phi_{0y} = 0$  on  $y = 0$ ;
- (iii)  $\phi_{0x} = 0$  on  $x = \pm a$ ,  $y > 1$ ;
- (iv)  $r^{1/2} \nabla \phi_0$  is bounded as  $r = \{(x \pm a)^2 + (y - 1)^2\}^{1/2} \rightarrow 0$ ;
- (v)  $\phi_0, \nabla \phi_0 \rightarrow 0$  as  $y \rightarrow \infty$ ;
- (vi)  $\phi_0(x, y) \sim \begin{cases} T_0 e^{-ky+ikx} & \text{as } x \rightarrow \infty, \\ e^{-ky+ikx} + R_0 e^{-ky-ikx} & \text{as } x \rightarrow -\infty. \end{cases}$

**PROBLEM  $P_2$ .** The function  $\phi_1$  satisfies

- (i)  $\nabla^2 \phi_1 = 0$  in  $y > 0$ ,  $-\infty < x < \infty$ ;
- (ii)  $k\phi_1 + \phi_{1y} = 0$  on  $y = 0$ ;
- (iii)  $\frac{\partial \phi_1(\pm a \pm 0, y)}{\partial x} = \frac{d}{dy} \left\{ c(y) \frac{\partial \phi_0(\pm a \pm 0, y)}{\partial y} \right\}$ ,  $y > 1$ ;
- (iv)  $r^{1/2} \nabla \phi_1$  is bounded as  $r = \{(x \pm a)^2 + (y - 1)^2\}^{1/2} \rightarrow 0$ ;
- (v)  $\phi_1, \nabla \phi_1 \rightarrow 0$  as  $y \rightarrow \infty$ ;
- (vi)  $\phi_1(x, y) \sim \begin{cases} T_1 e^{-ky+ikx} & \text{as } x \rightarrow \infty, \\ R_1 e^{-ky-ikx} & \text{as } x \rightarrow -\infty. \end{cases}$

Problem  $P_1$  corresponds to water wave scattering by two thin vertical barriers submerged from the same depth below the mean free surface of a deep water. Its explicit solution was obtained by Jarvis [4] by using complex variable theory. His results are reproduced in Appendix A in an equivalent form for the purpose of their use in obtaining the first-order corrections  $R_1$  and  $T_1$  to the reflection and transmission coefficients appearing in Problem  $P_2$ .

Without solving Problem  $P_2$  fully,  $R_1$  and  $T_1$  can be obtained by employing Evans's [2] idea. To find  $R_1$ , we apply Green's integral theorem to the functions  $\phi_0(x, y)$  and  $\phi_1(x, y)$  in the region bounded by the lines

$$\begin{aligned} y = 0 & \quad \text{and} \quad y = Y, & x \in [-X, X]; \\ x = X & \quad \text{and} \quad x = -X, & y \in [0, Y]; \\ x = a \pm 0 & \quad \text{and} \quad x = -a \pm 0, & y \in [1, Y]; \end{aligned}$$

and circles of small radius  $\delta$  with centres at  $(\pm a, 1)$  and ultimately make  $X, Y$  tend to infinity and  $\delta$  tend to zero. Using arguments similar to Evans [2], we obtain

$$iR_1 = \int_1^\infty \left\{ \phi_0(a_+, y) \frac{\partial \phi_1(a_+, y)}{\partial x} - \phi_0(a_-, y) \frac{\partial \phi_1(a_-, y)}{\partial x} \right\} dy + \int_1^\infty \left\{ \phi_0(-a_+, y) \frac{\partial \phi_1(-a_+, y)}{\partial x} - \phi_0(-a_-, y) \frac{\partial \phi_1(-a_-, y)}{\partial x} \right\} dy, \quad (3.3)$$

where  $a_-$  and  $a_+$  stand for  $a - 0$  and  $a + 0$  to shorten the notation. Using condition (iii) of  $\mathbf{P}_2$  in (3.3), integrating by parts and using  $c(1) = 0$ , we find that

$$iR_1 = \int_1^\infty c(y) \left[ \frac{d}{dy} \{ \phi_0(a_-, y) + \phi_0(-a_+, y) \} \frac{d}{dy} \{ \phi_0(a_-, y) - \phi_0(-a_+, y) \} - \frac{d}{dy} \{ \phi_0(a_+, y) + \phi_0(-a_-, y) \} \frac{d}{dy} \{ \phi_0(a_+, y) - \phi_0(-a_-, y) \} \right] dy. \quad (3.4)$$

Using the expressions for  $\phi_0(\pm a \pm 0, y)$  given in (A.5), the relation (3.4) produces

$$\begin{aligned} \frac{iR_1}{4} &= \int_1^\infty c(y) [A \{ kP_2(y)e^{-ky} - \xi_2(-y)W(\xi_2(-y)) \} - Hke^{-ky}] \\ &\quad \times [B \{ kQ_2(y)e^{-ky} - W(\xi_2(-y)) \} - Gke^{-ky}] dy \\ &\quad - \int_1^\infty c(y) [A \{ kP_1(y)e^{-ky} + \xi_1(-y)W(\xi_1(-y)) \} + Hke^{-ky}] \\ &\quad \times [B \{ kQ_1(y)e^{-ky} + W(\xi_1(-y)) \} + Gke^{-ky}] dy, \end{aligned} \quad (3.5)$$

where the constants  $G$  and  $H$  are given by

$$G = BM_1 + D \sin ka, \quad H = AL_1 + C \cos ka.$$

The functions  $P_n(y)$  and  $Q_n(y)$ ,  $n = 1, 2$ , and the various constants are given in Appendix A. The integrals appearing in (3.5) can be evaluated numerically, once the form of  $c(y)$  is known.

To determine  $T_1$ , we next use Green’s integral theorem for the functions  $\psi_0(x, y)$  ( $= \phi_0(-x, y)$ ) and  $\phi_1(x, y)$  in the region mentioned above and we find, on using condition (iii) of  $\mathbf{P}_2$ , and integrating by parts, that  $T_1 \equiv 0$ , since  $c(1) = 0$ .

This holds good irrespective of the shape of the barriers. Thus the first-order correction to the transmission coefficient vanishes identically for the two nearly vertical barriers also, as was the case for a single nearly vertical barrier. It is rather obvious to observe that the above idea of using Green’s theorem for the functions  $\hat{\psi}_0(x, y)$  ( $= \hat{\phi}_0(-x, y)$ ) and  $\hat{\phi}_1(x, y)$  in a suitably designed region, gives rise to the conclusion that  $\hat{T}_1 \equiv 0$ , where the notation  $\hat{\phi}_0, \hat{\phi}_1, \hat{T}_1$  has obvious meanings which correspond to the problem of scattering by an arbitrary number  $n$  (say), of nearly vertical barriers represented by  $x = a_j + \epsilon c(y)$ ,  $y \in (d, \infty)$  with  $c(d) = 0$ ,  $j = 1, 2, \dots, n$ .

## 4. Conclusion

A simplified perturbational analysis together with appropriate use of Green's integral theorem is employed to obtain first-order corrections to the reflection and transmission coefficients for two nearly vertical barriers completely submerged in deep water. The first-order correction to the transmission coefficient is found to be identically zero as was the case for a single barrier. The first-order correction to the reflection coefficient is obtained in terms of definite integrals involving the shape function describing the barriers. If the two barriers are made to assume the shape of a single barrier, the results for first-order correction to the reflection coefficient for a single barrier are recovered.

In fact, this last result involving the transmission coefficient holds good for an arbitrary number of nearly vertical barriers, that is,  $\hat{T}_1 \equiv 0$ , for an arbitrary number of barriers. That such a result is in fact valid can be expected because of the *complementarity theorem* stated (without proof) by Shaw [17]. The analysis of the present paper has thus established the complementarity theorem of Shaw [17] in a somewhat constructive manner, as explained above.

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## Appendix A.

To reproduce the results of Jarvis [4], we introduce a second complex unit  $j$  which does not interact with the complex unit  $i$ , and is used to denote the complex variable  $z = x + jy$ . Then  $\phi_0(x, y)$  is given by

$$\phi_0(x, -y) = \text{Re}_j w(z), \quad (\text{A.1})$$

where

$$w(z) = e^{-jkz} \int_{j\infty}^z \{A\zeta(u) + B\} f(\zeta(u)) e^{jku} du + (C + jD) e^{-jkz}, \quad (\text{A.2})$$

$\zeta$  and  $z$  being related by

$$z = \frac{2a}{\pi} \left[ \frac{(\alpha^2 - 1)\zeta}{(1 - \zeta^2)^{1/2}} - j \log \{(1 - \zeta^2)^{1/2} + j\zeta\} \right], \quad (\text{A.3})$$

where  $\alpha$  is a real number satisfying

$$\alpha(\alpha^2 - 1)^{1/2} + \log \{ \alpha + (\alpha^2 - 1)^{1/2} \} = \frac{\pi}{2a} \quad (\text{A.4})$$

and  $A, B, C, D$  are real constants with respect to  $j$ .

It may be noted that relation (A.3) denotes a mapping of the whole complex  $z$ -plane cut from  $\pm a \pm j$  to  $\pm a \pm j\infty$  into the complex  $\zeta$ -plane cut from  $\pm 1$  to  $\pm\infty$ , and that a branch of  $(1 - \zeta^2)^{1/2}$  is chosen which is real and positive when  $\zeta = 0$ . The cuts in the  $z$ -plane are mapped into those in the  $\zeta$ -plane and the  $x$ -axis is mapped onto the section  $-1 \leq \text{Re}_j \zeta \leq 1$  of the real axis in the  $\zeta$ -plane. Also the points  $z = \pm a \pm j$  in the  $z$ -plane correspond to the points  $\zeta = \pm\alpha \pm j0$  in the  $\zeta$ -plane.

The constants  $A, B, C, D$  (real with respect to  $j$ ) are given by

$$\begin{aligned} A &= (Le^{ika} - I \operatorname{cosec} ka)^{-1}, & B &= (Me^{ika} + iJ \sec ka)^{-1}, \\ C &= -AI \operatorname{cosec} ka, & D &= BJ \sec ka, \end{aligned}$$

where

$$(I, J) = \operatorname{Re}_j \int_{-1}^1 (\zeta, 1) f(\zeta) e^{-kv} dv$$

with  $\zeta = \zeta(a + jv)$ ,  $-1 \leq v \leq 1$ , and

$$(L, M) = e^{-jka} \int_{\Gamma} (\zeta, 1) f(\zeta) e^{jkz} dz,$$

where  $\Gamma$  is a loop around the cut  $z = a + jv$ , ( $v > 1$ ) in the  $z$ -plane. It may be noted that  $L, M$  are real constants.

The reflection and transmission coefficients  $R_0, T_0$  for the problem of vertical barriers are given by

$$R_0 = -iLA + MB \cos ka, \quad T_0 = (L \cos ka - I \sin ka)A - MB \cos ka,$$

for which the energy identity  $|R_0|^2 + |T_0|^2 = 1$  is satisfied.

Explicit expressions for  $\phi_0$  on the two sides of the lines  $x = \pm a$ ,  $y > 1$  can be obtained from (A.1) and (A.2), and are given by

$$\begin{aligned} \phi_0(a + 0, y) &= -e^{-ky} [AP_1(y) + BQ_1(y) - C \cos ka \\ &\quad - D \sin ka - AL_1 - BM_1], \\ \phi_0(-a + 0, y) &= -e^{-ky} [AP_2(y) - BQ_2(y) - C \cos ka \\ &\quad + D \sin ka - AL_1 + BM_1]. \end{aligned} \quad (\text{A.5})$$

Here  $\phi_0(a - 0, y)$  is obtained from  $\phi_0(a + 0, y)$  with  $P_1(y)$  replaced by  $P_2(y)$  and  $Q_1(y)$  replaced by  $Q_2(y)$ , while  $\phi_0(-a - 0, y)$  is obtained from  $\phi_0(-a + 0, y)$  with

$P_2(y)$  replaced by  $P_1(y)$  and  $Q_2(y)$  replaced by  $Q_1(y)$ . The functions  $P_n(y)$ ,  $Q_n(y)$ ,  $n = 1, 2$ , and the constants  $L_1$ ,  $M_1$  appearing in (A.5) are given by

$$\begin{aligned} (P_n(y), Q_n(y)) &= \int_{-1}^{-y} (\xi_n(v), 1) W(\xi_n(v)) e^{-kv} dv, \quad n = 1, 2, \\ (L_1, M_1) &= \int_1^{\infty} (\xi_1(v), 1) W(\xi_1(v)) e^{-kv} dv, \end{aligned} \quad (\text{A.6})$$

where  $W(\xi) = (\xi^2 - 1)^{1/2}/(\xi^2 - \alpha^2)$ ,  $\xi_1(v) (> \alpha)$  and  $\xi_2(v) (1 < \xi_2(v) < \alpha)$  being the two real roots of

$$(\alpha^2 - 1)\xi(\xi^2 - 1)^{-1/2} + \log \left\{ \xi + (\xi^2 - 1)^{1/2} \right\} = \frac{\pi v}{2a}, \quad v > 1. \quad (\text{A.7})$$

## Appendix B.

**Approximation as  $a \rightarrow 0$ .** We first note from (A.4) that for small  $a$ ,

$$\alpha \approx \left( \frac{\pi}{2a} \right)^{1/2},$$

so that  $\alpha \rightarrow \infty$  as  $a \rightarrow 0$ .

For small values of  $a$ , that is, large values of  $\alpha$ , the real root  $\xi_1(v)$ , ( $v > 1$ ), of (A.7) can be approximated as

$$\xi_1(v) \approx \frac{1}{2} e^{1+\alpha^2(v-1)}, \quad v > 1 \quad (\text{B.1})$$

and the other real root  $\xi_2(v)$ ,  $v > 1$ , can be approximated as

$$\xi_2(v) \approx \frac{v}{(v^2 - 1)^{1/2}}, \quad v > 1. \quad (\text{B.2})$$

Using (B.1) and (B.2) it can be shown that for small values of  $a$ , ( $v > 1$ )

$$\begin{aligned} W(\xi_1) &\approx \frac{2}{e^{1+\alpha^2(v-1)}}, & \xi_1 W(\xi_1) &\approx 1, \\ W(\xi_2) &\approx \frac{-1}{\alpha^2(v^2 - 1)^{1/2}}, & \xi_2 W(\xi_2) &\approx \frac{-v}{\alpha^2(v^2 - 1)}, \end{aligned}$$

also  $M \approx K_0(k)/\alpha^2$ ,  $J \approx \pi I_0(k)/\alpha^2$ ,  $L \approx e^{-k}/k + U/\alpha^2$ ,  $I \approx V/\alpha^2$ , where  $I_0(k)$  and  $K_0(k)$  are modified Bessel functions and

$$U = \int_1^{\infty} \frac{ye^{-ky}}{y^2 - 1} dy, \quad V = \int_{-1}^1 \frac{ye^{-ky}}{y^2 - 1} dy.$$

Again for small values of  $a$ ,

$$A \approx \frac{\pi k}{\pi e^{-k} - 2V}, \quad B \approx \frac{\alpha^2}{K_0(k) + i\pi I_0(k)},$$

$$C \approx \frac{-2V}{\pi e^{-k} - 2V}, \quad D \approx \frac{\pi I_0(k)}{K_0(k) + i\pi I_0(k)},$$

also

$$P_1(y) \approx \frac{1}{k}(e^k - e^{ky}), \quad Q_1(y) \approx \frac{2}{e(\alpha^2 - k)} \left[ \frac{1}{e^{\alpha^2(y-1)-ky}} - \frac{1}{e^{2\alpha^2-k}} \right],$$

$$P_2(y) \approx \frac{1}{\alpha^2} \int_{-1}^{-y} \frac{ve^{-kv}}{v^2 - 1} dv, \quad Q_2(y) \approx \frac{-1}{\alpha^2} \int_{-1}^{-y} \frac{e^{-kv}}{(v^2 - 1)^{1/2}} dv,$$

$$H \approx 1 \quad \text{and} \quad G \approx \frac{1}{K_0(k) + i\pi I_0(k)} \frac{\pi^2 k I_0(k)}{2\alpha^2}.$$

Using these expressions in (3.5), and noting that the second integral contributes nothing as  $a \rightarrow 0$  and using the contribution of the first integral as  $a \rightarrow 0$ , we obtain

$$\frac{iR_1}{4k} (K_0(k) + i\pi I_0(k)) \approx k \int_1^\infty c(y) e^{-2ky} \left( \int_1^y \frac{e^{kv}}{(v^2 - 1)^{1/2}} dv \right) dy$$

$$- \int_1^\infty c(y) \frac{e^{-ky}}{(y^2 - 1)^{1/2}} dy.$$

This result coincides with the result obtained by Shaw [17] and Mandal and Chakrabarti [7] for a single completely submerged nearly vertical barrier.

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