ANZIAM J. 48(2006), 179–197

SUB-SUPERSOLUTIONS IN A VARIATIONAL INEQUALITY RELATED TO A SANDPILE PROBLEM

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(Received 29 April, 2005; revised 12 October, 2006)

Abstract

In this paper we study a variational inequality in which the principal operator is a generalised Laplacian with fast growth at infinity and slow growth at 0. By defining appropriate sub- and super-solutions, we show the existence of solutions and extremal solutions of this inequality above the subsolutions or between the sub- and super-solutions.

2000 *Mathematics subject classification*: primary 35B05, 35J85; secondary 47J20, 35J60.
Keywords and phrases: sub-supersolution, sandpile problem, variational inequality, extremal solution.

1. Introduction

In this paper, we study a variational inequality in which the principal operator is a generalised Laplacian (ϕ-Laplacian) with fast growth at infinity and slow growth at 0 and where the lower order term is nonlinear. An example of such a variational inequality is the following:

$$\begin{cases} \displaystyle\int_\Omega \Phi(|\nabla v|)\,dx - \int_\Omega \Phi(|\nabla u|)\,dx \geq \int_\Omega f(x,u)(v-u)\,dx, \quad \forall\, v \in W_0^1 L_\Phi, \\ u \in W_0^1 L_\Phi, \end{cases} \tag{1.1}$$

where Φ is the Young function given by

$$\Phi(t) = e^{|t|^p - |t|^{-p}} \quad (\Phi(0) = 0), \tag{1.2}$$

for $p \geq 1$. Here, Ω is a bounded open set in $\mathbb{R}^N$ ($N \geq 1$) with Lipschitz boundary $\partial\Omega$, $W_0^1 L_\Phi$ is the first-order Orlicz-Sobolev space of functions vanishing on $\partial\Omega$ (see,

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for example, Section 2.1 for more details), and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with a certain growth condition to be specified later.

Note that for Φ given in (1.2) we have

$$\Phi(t) = o(t^q) \qquad \text{as } \ t \to 0^+$$

and

$$t^q = o(\Phi(t)) \quad \text{as } \ t \to \infty,$$

for any power t^q $(q > 0)$. Inequality (1.1) is the weak (variational) form of the nonlinear degenerate elliptic boundary value problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\phi(|\nabla u|)}{|\nabla u|}\nabla u\right) = f(x, u) & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{1.3}$$

where $\phi = \Phi'$. Problem (1.3) and the variational inequality (1.1) are related to a sandpile problem studied recently by Aronsson *et al.* [2], Evans *et al.* [15] and Prigozhin [21, 20]. In these works, the (dynamic) problem is formulated as a parabolic equation that contains the p-Laplacian with large p:

$$\begin{cases} u_t - \Delta_p u = f & \text{in } \ \mathbb{R}^N \times (0, \infty), \\ u = g & \text{on } \ \mathbb{R}^N \times \{t = 0\}, \end{cases} \tag{1.4}$$

and also their limits when $p \to \infty$. The limit problem is in fact equivalent to the following variational inequality:

$$\begin{cases} f - u_t \in \partial I_K(u) & \text{for } \ t > 0, \\ u = g & \text{when } \ t = 0. \end{cases} \tag{1.5}$$

Here, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $\partial I_K(u)$ is the subdifferential of the indicator function I_K of the convex set $K = \{v : |\nabla v| \le 1 \text{ a.e.}\}$. The motivation of (1.4) and (1.5) is the consideration of fast/slow diffusion operators such that within the region $\{|\nabla u| < 1 - \delta\}$ $(\delta > 0$, small), the diffusion coefficient $|\nabla u|^{p-2}$ is very small, whereas within $\{|\nabla u| < 1 + \delta\}$, $|\nabla u|^{p-2}$ is very large. The limit variational inequality (1.5) is also closely related to the elastic-plastic torsion problem (see, for example, [9, 8] or [22]). In the variational form, Δ_p is the derivative of the functional

$$I_p(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx \tag{1.6}$$

with the integrand $|\nabla u|^p$ being very small in $\{|\nabla u| < 1 - \delta\}$ and very large in $\{|\nabla u| < 1 + \delta\}$. Because I_p is convex, the equation in (1.4) is (in the weak form) equivalent to the variational inequality

$$\int_\Omega u_t(v-u)\,dx + I_p(v) - I_p(u) \ge \int_\Omega f(v-u)\,dx, \tag{1.7}$$

(for all v in a certain space of admissible functions). The limit inclusion in (1.5) is equivalent to the inequality

$$\int_\Omega u_t(v-u)\,dx + I_K(v) - I_K(u) \ge \int_\Omega f(v-u)\,dx. \tag{1.8}$$

The functional I_K in (1.8) and (1.5) can be written formally as an integral functional similar to (1.6):

$$I_K(u) = \int_\Omega Q(|\nabla u|)\,dx,$$

where $Q : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is the maximal graph:

$$Q(t) = \begin{cases} 0 & \text{for } t \in [-1,1], \\ \infty & \text{for } |t| > 1. \end{cases}$$

Note that the functions Q_p given by $Q_p(t) = |t|^p/p$, $t \in \mathbb{R}$, have limit Q (pointwise) as $p \to \infty$. When f and u are independent of t, the stationary inequality associated with (1.5) is $f \in \partial I_K(u)$, which is not always solvable.

We propose here to study an intermediate problem between the p-Laplacian problems (1.7) and their limit variational inequality (1.8). We consider the inequality

$$\int_\Omega u_t(v-u)\,dx + \int_\Omega \Phi(|\nabla v|)\,dx - \int_\Omega \Phi(|\nabla u|)\,dx \ge \int_\Omega f(v-u)\,dx, \tag{1.9}$$

for all v in some appropriate function space (to be defined later). Here, Φ is a convex function such that, for every $p \ge 1$,

$$Q_p(t) = o(\Phi(t)) \quad \text{and} \quad \Phi(t) = o(Q(t)) \quad \text{for large } t, \tag{1.10}$$

and

$$Q(t) = o(\Phi(t)) \quad \text{and} \quad \Phi(t) = o(Q_p(t)) \quad \text{for small } t. \tag{1.11}$$

Thus, the function Φ plays an intermediate role between *all* functions Q_p and their limit Q. An example of Young functions that satisfy (1.10) and (1.11) is

$$\Phi(t) = \begin{cases} e^{|t|-1/|t|}, & t \neq 0, \\ 0, & t = 0, \end{cases}$$

or, more generally, the function given in (1.2) above. In what follows, we consider the case where Φ is given by (1.2); however, the arguments can be adapted in a straightforward manner to problems with Φ satisfying (1.10) and (1.11).

Furthermore, we propose to investigate in this paper the stationary inequality associated with (1.9), that is, the variational inequality

$$\int_\Omega \Phi(|\nabla v|)\,dx - \int_\Omega \Phi(|\nabla u|)\,dx \geq \int_\Omega f(v-u)\,dx,$$

that is, the inequality (1.1) above. The evolutionary problem will be investigated in a future project.

We are concerned here with the existence and properties of solutions of the variational inequality (1.1). In the case where the lower order term is linear, that is, $f = f(x)$ does not depend on u, (1.1) has a unique solution, as can be proved by classical existence theory for variational inequalities. In the general case where f also depends on u, the problem is no longer coercive and thus may not have solutions. We study (1.1) in that general case by a sub-supersolution approach. This approach, when applicable, usually gives useful information not only on the existence of solutions of the problem but also on the structure of the solution sets, such as their compactness, directedness, or the existence of extremal solutions. The method was developed recently in [17, 16] for variational inequalities and has been extended to other types of inequalities such as variational-hemivariational inequalities or systems of variational inequalities in first-order Sobolev spaces $W^{1,p}$ (see, for example, [3, 4, 6, 5, 7, 18] and the references therein). However, this technique has not been extended so far to apply to equations or inequalities in nonreflexive Banach spaces such as Orlicz-Sobolev spaces. Another point is that, in most previous works so far, the potential functionals for the principal operators are smooth or at least Lipschitz continuous. In our problem here, the principal functional is not differentiable and even not defined on the whole associated function space. Therefore, a new sub-supersolution approach is needed for the present problem. The sub- and super-solution approach for variational inequalities where the potentials of their principal operators are nonsmooth do not appear to have been studied. Also, since our functionals here do not satisfy Δ_2 conditions, working in nonreflexive Orlicz-Sobolev spaces also requires new arguments and techniques. In this paper, we shall define the appropriate concepts of sub- and super-solutions for (1.1). Next, we prove the existence of solutions and study some properties of solutions of (1.1), between sub- and super-solutions.

The paper is organised as follows. In the second section, after a short review of the basic properties of Orlicz-Sobolev spaces, we define sub- and super-solutions of the inequalitiy (1.1). Existence and enclosure properties of solutions of (1.1) above subsolutions and between the sub- and super-solutions are established in Section 3. Section 4 is devoted to the existence of extremal solutions, that is, of the smallest and greatest solutions of (1.1), between the sub- and super-solutions.

2. Sub- and super-solutions

First, let us recall some basic definitions and notation concerning Orlicz-Sobolev spaces.

2.1. Preliminaries on Orlicz-Sobolev spaces Let Φ be a Young function (or N-function). We denote by $\bar{\Phi}$ the Hölder conjugate function of Φ, which is defined by $\bar{\Phi}(t) = \sup\{ts - \Phi(s) : s \in \mathbb{R}\}$, and by Φ^* the Sobolev conjugate of Φ (in $\mathbb{R}^N$), with

$$(\Phi^*)^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}}\, ds,$$

provided that

$$\int_1^\infty \frac{\Phi^{-1}(s)}{s^{(N+1)/N}}\, ds = \infty, \tag{2.1}$$

(we refer to [1], [13] or [14] for the properties of Young functions). The Orlicz class $\tilde{L}_\Phi := \tilde{L}_\Phi(\Omega)$ is the set of all (equivalence classes of) measurable functions u defined on Ω such that $\int_\Omega \Phi(|u(x)|)\, dx < \infty$. The Orlicz space $L_\Phi := L_\Phi(\Omega)$ is the linear hull of $\tilde{L}_\Phi$, that is, the set of all measurable functions u on Ω such that

$$\int_\Omega \Phi\left(\frac{|u(x)|}{k}\right) dx < \infty, \quad \text{for some } k > 0.$$

Then L_Φ is a Banach space when equipped with the (Luxemburg) norm

$$\|u\|_\Phi = \|u\|_{L_\Phi} = \inf\left\{k > 0 : \int_\Omega \Phi\left(\frac{|u|}{k}\right) dx \le 1\right\}.$$

It is clear that $L^\infty(\Omega) \subset L_\Phi \subset L^1(\Omega)$. The closure of $L^\infty(\Omega)$ in L_Φ is denoted by E_Φ, which is a separable Banach space. The first-order Orlicz-Sobolev space $W^1L_\Phi := W^1L_\Phi(\Omega)$ is the set of all $u \in L_\Phi$ such that the distributional derivatives $\partial_i u = \partial u/\partial x_i$, $i = 1, \ldots, N$, are also in L_Φ. We note that W^1L_Φ is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} = \|u\|_{W^1L_\Phi} = \|u\|_\Phi + \sum_{i=1}^N \|\partial_i u\|_\Phi.$$

The Orlicz-Sobolev space W^1E_Φ is defined similarly. It is known (see, for example, [13, 14]) that L_Φ is the dual space of $E_{\bar{\Phi}}$, that is, $L_\Phi = (E_{\bar{\Phi}})^*$ and $L_{\bar{\Phi}} = (E_\Phi)^*$.

The spaces W^1L_Φ and W^1E_Φ can be identified with closed subspaces of the products $\prod_{i=0}^N L_\Phi$ and $\prod_{i=0}^N E_\Phi$, respectively. It is the case that

$$\prod_{i=0}^N L_\Phi = \Big(\prod_{i=0}^N E_{\bar\Phi}\Big)^*$$

and if we denote by $\tau = \sigma\big(\prod L_\Phi, \prod E_{\bar\Phi}\big)$ the weak* topology in $\prod L_\Phi$ and also the restriction of τ to the closed subspace W^1L_Φ, then W^1L_Φ is closed under weak* convergence of $\prod L_\Phi$. Since $\prod E_{\bar\Phi}$ is separable, we have the following properties of W^1L_Φ, which shall be used frequently in what follows (see, for example, [11]).

If $\{u_n\}$ is a bounded sequence in W^1L_Φ (with respect to $\|\cdot\|_{1,\Phi}$), then $\{u_n\}$ has a subsequence which converges with respect to the topology τ to some $u \in W^1L_\Phi$, that is, a bounded set in W^1L_Φ is relatively sequentially compact with respect to the weak* topology τ.

We denote by $W_0^1L_\Phi$ the closure of $C_0^\infty(\Omega)$ with respect to the weak* topology τ. By a Poincaré inequality for Orlicz-Sobolev spaces (see [11]), we know that on $W_0^1L_\Phi$ the norm $\|\cdot\|_{W^1L_\Phi}$ is equivalent to the norm $\|\cdot\|_{W_0^1L_\Phi}$ given by $\|u\|_{W_0^1L_\Phi} = \||\nabla u|\|_{L_\Phi}$. We define an ordering on L_Φ and thus on W^1L_Φ and $W_0^1L_\Phi$ in a natural way as follows. For $u, v \in L_\Phi, u \le v \iff u(x) \le v(x)$ for almost all $x \in \Omega$. It is clear that "$\le$" is a partial ordering among functions in L_Φ.

A Young function Φ_1 is said to grow essentially more slowly than another Young function Φ_2 (at infinity) (see, for example, [1, 13, 14]), abbreviated by $\Phi_1 \ll \Phi_2$, if

$$\lim_{t\to\infty} \frac{\Phi_1(t)}{\Phi_2(kt)} = 0, \quad \text{for all } k > 0.$$

We have the following embeddings, similar to those among Sobolev spaces:

- The embedding $W_0^1L_\Phi \hookrightarrow L_{\Phi^*}$ is continuous.
- If $\Psi \ll \Phi^*$, then the embedding $W^1L_\Phi \hookrightarrow L_\Psi$ is compact. In particular, since $\Phi \ll \Phi^*$ (see, for example, [11]), the embedding $W^1L_\Phi \hookrightarrow L_\Phi$ is compact.

Moreover, in the case

$$\int_1^\infty \frac{\Phi^{-1}(s)}{s^{(N+1)/N}}\, ds < \infty$$

in (2.1) (which is the case we study in this paper), it is shown that W^1L_Φ is continuously embedded in $L^\infty(\Omega)$ (see [1, 11]).

A Young function Φ is said to satisfy a Δ_2 condition (at infinity) if there exist $K > 0$ and $t_0 \ge 0$ such that $\Phi(2t) \le K\Phi(t)$ for all $t \ge t_0$. Properties of the Orlicz space L_Φ and of the Orlicz-Sobolev spaces W^1L_Φ and $W_0^1L_\Phi$ when Φ and/or $\bar\Phi$ satisfies a Δ_2 condition are presented in detail in the references [1, 13, 14, 11].

2.2. Definitions of sub- and super-solutions In what follows we assume that Φ is given by (1.2). All results here are extended in a straightforward manner to the case where Φ satisfies (1.10)–(1.11). Let us denote

$$J(u) = \int_\Omega \Phi(|\nabla u|)\, dx, \quad u \in W^1 L_\Phi,$$

and let $D(J) = \{u \in W^1 L_\Phi : J(u) < \infty\} = \{u \in W^1 L_\Phi : |\nabla u| \in \tilde{L}_\Phi\}$ be the effective domain of J.

We are now ready to define sub- and super-solutions for (1.1).

DEFINITION 2.1. (a) A function $\underline{u} \in W^1 L_\Phi$ is a subsolution of (1.1) if

$$\begin{cases} \text{(i)} & \underline{u} \le 0 \ \text{ on } \ \partial\Omega, \\ \text{(ii)} & f(\cdot, \underline{u}) \in L^1(\Omega), \\ \text{(iii)} & \int_\Omega \Phi(|\nabla \underline{u}|)\, dx < \infty \quad (\text{that is, } |\nabla \underline{u}| \in \tilde{L}_\Phi), \end{cases} \tag{2.2}$$

and for all $v \in \underline{u} \wedge [W_0^1 L_\Phi \cap D(J)]$

$$\int_\Omega \Phi(|\nabla v|)\, dx - \int_\Omega \Phi(|\nabla \underline{u}|)\, dx \ge \int_\Omega f(x, \underline{u})(v - \underline{u})\, dx. \tag{2.3}$$

(b) A function $\bar{u} \in W^1 L_\Phi$ is a supersolution of (1.1) if

$$\begin{cases} \text{(i)} & \bar{u} \ge 0 \ \text{ on } \ \partial\Omega, \\ \text{(ii)} & f(\cdot, \bar{u}) \in L^1(\Omega), \\ \text{(iii)} & \int_\Omega \Phi(|\nabla \bar{u}|)\, dx < \infty \quad (\text{that is, } |\nabla \bar{u}| \in \tilde{L}_\Phi), \end{cases} \tag{2.4}$$

and for all $v \in \bar{u} \vee [W_0^1 L_\Phi \cap D(J)]$

$$\int_\Omega \Phi(|\nabla v|)\, dx - \int_\Omega \Phi(|\nabla \bar{u}|)\, dx \ge \int_\Omega f(x, \bar{u})(v - \bar{u}) dx. \tag{2.5}$$

In these definitions, we use the following notation:

$$u \vee v = \max\{u, v\}, \qquad A * B = \{a * b : a \in A, b \in B\},$$
$$u \wedge v = \min\{u, v\}, \qquad u * A = \{u\} * A,$$

where $u, v \in W^1 L_\Phi$, $A, B \subset W^1 L_\Phi$, and $* \in \{\vee, \wedge\}$.

Let us illustrate Definition 2.1 by simple examples of constant sub- and super-solutions. Let $a \in \mathbb{R}$, $a \le 0$. Then $\underline{u} \equiv a$ is a subsolution of (1.1) if $f(x, a)$ is in $L^1(\Omega)$ and $f(x, a) \ge 0$ for a.e. $x \in \Omega$. In fact, by the conditions in (2.2), Definition 2.1 is trivial. Moreover, for any $v \in \underline{u} \wedge [W_0^1 L_\Phi \cap D(J)]$, we have $v - \underline{u} \le 0$ and thus

$$\int_\Omega \Phi(|\nabla v|)\, dx - \int_\Omega \Phi(|\nabla \underline{u}|)\, dx = \int_\Omega \Phi(|\nabla v|)\, dx \ge 0 \ge \int_\Omega f(x, \underline{u})(v - \underline{u})\, dx,$$

that is, $\underline{u}$ also satisfies (2.3) in Definition 2.1. Similarly, if $b \ge 0$ is a number such that $f(\cdot, b) \in L^1(\Omega)$ and $f(x, b) \le 0$ for a.e. $x \in \Omega$, then $\bar{u} \equiv b$ is a supersolution of (1.1).

3. Existence of solutions above subsolutions or between sub- and super-solutions

First, let us prove the following lattice property of Orlicz-Sobolev spaces W^1L_Φ and $W_0^1L_\Phi$, which extends that of first-order Sobolev spaces.

LEMMA 3.1. *W^1L_Φ and $W_0^1L_\Phi$ are closed under the operations $\vee$ and $\wedge$, that is, if $u, v \in W^1L_\Phi$ (respectively $W_0^1L_\Phi$), then $u \vee v, u \wedge v \in W^1L_\Phi$ (respectively $W_0^1L_\Phi$).*

PROOF. Assume $u, v \in W^1L_\Phi$. We have $u, v \in W^{1,1}(\Omega)$ and from Stampacchia's theorem (see, for example, [10] or [12]),

$$\nabla(u \vee v) = \begin{cases} \nabla u & \text{in } \{x \in \Omega : u(x) \geq v(x)\}, \\ \nabla v & \text{in } \{x \in \Omega : u(x) < v(x)\}. \end{cases} \tag{3.1}$$

There exists $\varepsilon > 0$ such that $\varepsilon u, \varepsilon v, |\nabla(\varepsilon u)|, |\nabla(\varepsilon v)| \in \tilde{L}_\Phi$. Because

$$\varepsilon(u \vee v) = \begin{cases} \varepsilon u & \text{in } \{x \in \Omega : u(x) \geq v(x)\}, \\ \varepsilon v & \text{in } \{x \in \Omega : u(x) < v(x)\}, \end{cases}$$

we have

$$\int_\Omega \Phi(\varepsilon(u \vee v))\, dx \leq \int_\Omega \Phi(\varepsilon u)\, dx + \int_\Omega \Phi(\varepsilon v)\, dx < \infty.$$

This proves that $u \vee v \in L_\Phi$. Similarly, by using (3.1) on $\nabla(\varepsilon u \vee \varepsilon v) = \nabla(\varepsilon(u \vee v))$, one obtains

$$\int_\Omega \Phi(\varepsilon|\nabla(u \vee v)|)\, dx \leq \int_\Omega \Phi(|\nabla(\varepsilon u)|)\, dx + \int_\Omega \Phi(|\nabla(\varepsilon v)|)\, dx < \infty.$$

Hence $|\nabla(u \vee v)| \in L_\Phi$, that is, $\nabla(u \vee v) \in (L_\Phi)^N$. We have shown that $u \vee v \in W^1L_\Phi$. Analogous arguments hold for $u \wedge v$. □

We assume that $\underline{u}_1, \ldots, \underline{u}_k$ are subsolutions of (1.1) (in the sense of Definition 2.1) and put

$$\underline{u} = \max\{\underline{u}_i : 1 \leq i \leq k\} \tag{3.2}$$

and

$$\underline{u}_0 = \min\{\underline{u}_i : 1 \leq i \leq k\}. \tag{3.3}$$

From Lemma 3.1, $\underline{u}$ and $\underline{u}_0$ are in W^1L_Φ. Assume that f has the following growth condition above $\underline{u}_0$:

$$|f(x, u)| \leq a(x) + \Psi'(|u|), \tag{3.4}$$

for a.e. $x \in \Omega$, all $u \in [\underline{u}_0(x), \infty)$, where $a \in L^1(\Omega)$ and Ψ is a Young function such that

$$\Psi \ll \Phi \quad \text{(at infinity).} \tag{3.5}$$

Under these conditions, we have the following existence and comparison results for (1.1).

THEOREM 3.2. *Assume $\underline{u}_1, \ldots, \underline{u}_k$ are subsolutions of (1.1) and that F has the growth condition (3.4). Then there exists a solution u of (1.1) such that $u \geq \underline{u}$.*

In the proof of Theorem 3.2, we need the following estimate.

LEMMA 3.3. *For any $d > 0$, there exists $C > 0$ such that*

$$\int_\Omega \Phi(|\nabla u|)\,dx - d\int_\Omega |u|^2\,dx \geq \frac{1}{2}\int_\Omega \Phi(|\nabla u|)\,dx - C, \quad \forall\, u \in W_0^1 L_\Phi. \tag{3.6}$$

PROOF. From [11, Lemma 5.7], there are positive constants D_1, D_2 such that

$$\int_\Omega \Phi(D_1 u)\,dx \leq D_2 \int_\Omega \Phi(|\nabla u|)\,dx, \quad \forall\, u \in W_0^1 L_\Phi.$$

This implies that

$$\begin{aligned}\int_\Omega \Phi(|\nabla u|)\,dx - d\int_\Omega |u|^2\,dx \\ \geq \frac{1}{2}\int_\Omega \Phi(|\nabla u|)\,dx + \frac{1}{2D_2}\left(\int_\Omega \Phi(D_1 u)\,dx - 2dD_2\int_\Omega |u|^2\,dx\right).\end{aligned}$$

Since $t^2 \ll \Phi(t)$, there is a constant $D_3 > 0$ such that $\Phi(D_1 u) \geq 2dD_2|u|^2 - D_3$, for all $u \in \mathbb{R}$. This shows that

$$\int_\Omega \Phi(|\nabla u|)\,dx - d\int_\Omega |u|^2\,dx \geq \frac{1}{2}\int_\Omega \Phi(|\nabla u|)\,dx - \frac{D_3}{2D_2}|\Omega|, \tag{3.7}$$

implying (3.6). □

We are now ready to prove Theorem 3.2.

PROOF OF THEOREM 3.2. For $x \in \Omega$, $t \in \mathbb{R}$, put

$$b(x,t) = -[\underline{u}(x) - t]^+ = \begin{cases} 0 & \text{if } t \geq \underline{u}(x), \\ t - \underline{u}(x) & \text{if } t < \underline{u}(x). \end{cases}$$

Because of the continuous (in fact, compact) embedding

$$W^1 L_\Phi \hookrightarrow L^\infty(\Omega), \tag{3.8}$$

we have

$$|b(x,t)| \le |t| + \|\underline{u}\|_{L^\infty(\Omega)} \le |t| + C\|\underline{u}\|_{W^1 L_\Phi}, \tag{3.9}$$

for a.e. $x \in \Omega$, all $t \in \mathbb{R}$. Here and in what follows, C denotes a generic positive constant.

This estimate shows that the operator B given by

$$\langle B(u), \phi\rangle = \int_\Omega b(x,u)\phi\, dx, \tag{3.10}$$

is well defined and continuous from $L^\infty(\Omega)$ into its dual and thus from $W^1 L_\Phi$ into its dual. For $u \in L_\Phi$, let us put $T(u) = u \vee \underline{u}$ and $T_j(u) = u \vee \underline{u}_j$ for $j \in \{1, \dots, k\}$. It follows from (3.8) and Lemma 3.1 that

$$T(u),\, T_j(u) \in W^1 L_\Phi (\hookrightarrow L^\infty(\Omega)), \quad \forall\, u \in W^1 L_\Phi.$$

Also, if $u \in L_\Phi$ then $T(u), T_j(u) \in L_\Phi$. It can be easily verified that T and T_j are continuous mappings from $W^1 L_\Phi$ into itself and also from L_Φ into itself. Because $T(u), T_j(u) \ge \underline{u}_0$ a.e. on Ω, we have from (3.4) that

$$|f(\cdot, T(u))| \le a + \Psi'(|T(u)|),$$

for all $u \in L_\Phi$. Therefore, the mapping $u \mapsto f(\cdot, T(u))$ is continuous and bounded from L_Φ into $L_{\bar\Phi} (= (L_\Phi)^*)$ and also from $W^1 L_\Phi$ into $(W^1 L_\Phi)^*$. Similar properties hold for the mapping $u \mapsto f(\cdot, T_j(u))$, $1 \le j \le k$. Let us define

$$\langle \Gamma(u), \phi\rangle = \int_\Omega \left[f(\cdot, T(u)) + \sum_{j=1}^k \left| f(\cdot, T_j(u)) - f(\cdot, T(u)) \right| \right] \phi\, dx, \tag{3.11}$$

for all $u, \phi \in W^1 L_\Phi$. The above arguments show that Γ is bounded and continuous from L_Φ to $L_{\bar\Phi}$. Because the embedding $W^1 L_\Phi \hookrightarrow L_\Phi$ is compact, Γ is completely continuous from $W^1 L_\Phi$ with the weak* topology to $(W^1 L_\Phi)^*$. Similarly, since the embedding $W^1 L_\Phi \hookrightarrow L^\infty(\Omega)$ is compact, the operator B defined in (3.10) is completely continuous from $W^1 L_\Phi$ (again with respect to the weak* topology) into $(W^1 L_\Phi)^*$. Let us consider the variational inequality

$$\begin{cases} J(v) - J(u) + \langle \beta B(u) - \Gamma(u), v - u\rangle \ge 0, & \forall v \in W_0^1 L_\Phi, \\ u \in W_0^1 L_\Phi, \end{cases} \tag{3.12}$$

with some fixed $\beta > 0$. From the above arguments, we see that $\beta B - \Gamma$ is completely continuous from $W^1 L_\Phi$ (with respect to the weak* topology) to $(W^1 L_\Phi)^*$. Let us prove that $J(\cdot) + \langle (\beta B - \Gamma)(\cdot), \cdot\rangle$ is coercive in the following sense:

$$\lim_{\|u\|_{W_0^1 L_\Phi} \to \infty, u \in W_0^1 L_\Phi} \frac{J(u) + \langle \beta B(u) - \Gamma(u), u\rangle}{\|u\|} = \infty, \tag{3.13}$$

(where $\|u\|_{W_0^1 L_\Phi} = \| \, |\nabla u| \, \|_{L_\Phi}$).

In fact, for $j \in \{1, \ldots, k\}$, $u \in W_0^1 L_\Phi$, we have

$$\begin{aligned}
&\int_\Omega |f(\cdot, T_j(u))||u| \, dx \\
&\quad \le \int_{\{x \in \Omega : u(x) \ge \underline{u}_0(x)\}} |f(\cdot, u)||u| \, dx + \int_{\{x \in \Omega : u(x) < \underline{u}(x)\}} |f(\cdot, \underline{u}_j)||u| \, dx \\
&\quad \le \int_\Omega a(x)|u| \, dx + \int_\Omega \Psi'(|u|)|u| \, dx + \int_\Omega |f(\cdot, \underline{u}_j)||u| \, dx \\
&\quad \le \|u\|_{L^\infty(\Omega)} (\|a\|_{L^1(\Omega)} + \|f(\cdot, \underline{u}_j)\|_{L^1(\Omega)}) + \int_\Omega \Psi'(|u|)|u| \, dx.
\end{aligned} \tag{3.14}$$

Note that since Ψ' is nondecreasing and Ψ is even, we have for all $u \in \mathbb{R}$,

$$\Psi(2u) = \Psi(2|u|) = \int_0^{2|u|} \Psi'(s) \, ds \ge \int_{|u|}^{2|u|} \Psi'(s) \, ds \ge \Psi'(|u|)|u| (= \Psi'(u)u).$$

For $\varepsilon, C > 0$, it follows from (3.5) and the convexity of Ψ that there exists $D = D_{\varepsilon C} > 0$ such that

$$\Psi'(|u|)|u| \le \Psi(2u) \le \varepsilon \Phi(Cu) + D_{\varepsilon C}, \quad \forall\, u \in \mathbb{R}. \tag{3.15}$$

Hence

$$\int_\Omega \Psi'(|u|)|u| \, dx \le \varepsilon \int_\Omega \Phi(Cu) \, dx + D_{\varepsilon C} |\Omega|. \tag{3.16}$$

Combining (3.14)–(3.16) with (1.2), one gets, for any $j \in \{1, \ldots, k\}$ and $u \in W_0^1 L_\Phi$,

$$\begin{aligned}
\int_\Omega |f(\cdot, T_j(u))||u| \, dx \le \|u\|_{L^\infty(\Omega)} \big(\|a\|_{L^1(\Omega)} + \|f(\cdot, \underline{u}_j)\|_{L^1(\Omega)} \big) \\
+ \varepsilon D_2 \int_\Omega \Phi(|\nabla u|) \, dx + D_{\varepsilon D_1} |\Omega|.
\end{aligned} \tag{3.17}$$

We have a similar estimate to (3.17) in which $\int_\Omega |f(\cdot, T_j(u))||u| \, dx$ is replaced by

$\int_\Omega |f(\cdot, T(u))||u|\,dx$. Therefore, one obtains the following estimate:

$$\begin{aligned}|\langle \Gamma(u), u\rangle| &\le C\left[(k+1)\|a\|_{L^1(\Omega)} + (k+1)\|f(\cdot,\underline{u})\|_{L^1(\Omega)} + \sum_{j=1}^{k}\|f(\cdot,\underline{u}_j)\|_{L^1(\Omega)}\right]\|u\|_{W_0^1L_\Phi}\\ &\quad + \varepsilon(k+1)D_2\int_\Omega \Phi(|\nabla u|)\,dx + (k+1)D_{\varepsilon D_1}|\Omega|, \ \forall u \in W_0^1L_\Phi. \end{aligned} \tag{3.18}$$

On the other hand, for all $u \in W^1L_\Phi$,

$$|\langle B(u), u\rangle| \le \int_\Omega |\underline{u} - u||u|\,dx \le \frac{3}{2}\int_\Omega |u|^2\,dx + \frac{1}{2}\int_\Omega |\underline{u}|^2\,dx. \tag{3.19}$$

Choosing $\varepsilon > 0$ sufficiently small in (3.18) and using (3.19), one gets (as above, C denotes a generic constant),

$$\begin{aligned} J(u) &+ \langle \beta B(u) - \Gamma(u), u\rangle\\ &\ge \int_\Omega \Phi(|\nabla u|)\,dx - C\|u\|_{W_0^1L_\Phi} - \varepsilon(k+1)D_2\int_\Omega \Phi(|\nabla u|)\,dx\\ &\quad - \frac{3\beta}{2}\int_\Omega |u|^2\,dx - C\\ &\ge \frac{1}{2}\int_\Omega \Phi(|\nabla u|)\,dx - C\left(\|u\|_{W_0^1L_\Phi} + \int_\Omega |u|^2\,dx + 1\right). \end{aligned}\tag{3.20}$$

From (3.6) of Lemma 3.3, (3.7) and (3.20), we obtain for all $u \in W_0^1L_\Phi$,

$$J(u) + \langle \beta B(u) - \Gamma(u), u\rangle \ge \frac{1}{4}\int_\Omega \Phi(|\nabla u|)\,dx - C(\|u\|_{W_0^1L_\Phi} + 1), \tag{3.21}$$

for some $C > 0$ independent of u. Because

$$\lim_{\|u\|_{W_0^1L_\Phi}\to\infty} \frac{1}{\|u\|_{W_0^1L_\Phi}}\int_\Omega \Phi(|\nabla u|)\,dx = \infty, \tag{3.22}$$

(see, for example, [11]), (3.21) immediately implies (3.13).

It follows from the above arguments and classical existence theory for variational inequalities (see, for example, [12] and [19]) that the inequality (3.12) has a solution u. Let us verify that for all $j \in \{1, \ldots, k\}$

$$u \ge \underline{u}_j. \tag{3.23}$$

Let $q \in \{1, \ldots, k\}$. Substituting $v = \underline{u}_q \wedge u \in \underline{u}_q \wedge [W_0^1L_\Phi \cap D(J)]$ in (2.3) (with $\underline{u}_q$ instead of $\underline{u}$) yields

$$J(\underline{u}_q \wedge u) - J(\underline{u}_q) \ge -\int_\Omega f(x, \underline{u}_q)(\underline{u}_q - u)^+\,dx. \tag{3.24}$$

On the other hand, letting $v = \underline{u}_q \vee u$ in (3.12), we obtain

$$J(\underline{u}_q \vee u) - J(u) + \langle \beta B(u) - \Gamma(u), (\underline{u}_q - u)^+ \rangle \geq 0. \tag{3.25}$$

Adding (3.24) to (3.25) and using the fact that for all $v, w \in W^1 L_\Phi$

$$\begin{aligned} J(v \wedge w) + J(v \vee w) &= \int_\Omega \Phi(|\nabla(v \wedge w)|)\, dx + \int_\Omega \Phi(|\nabla(v \vee w)|)\, dx \\ &= \int_\Omega \Phi(|\nabla v|)\, dx + \int_\Omega \Phi(|\nabla w|)\, dx \\ &= J(v) + J(w), \end{aligned}$$

which is a direct consequence of Stampacchia's theorem (see, for example, [12] or [10]), we have

$$\langle \beta B(u) - \Gamma(u), (\underline{u}_q - u)^+ \rangle + \int_\Omega f(x, \underline{u}_q)(\underline{u}_q - u)^+\, dx \geq 0.$$

It follows from (3.11) that

$$\begin{aligned} & -\langle \Gamma(u), (\underline{u}_q - u)^+ \rangle + \int_\Omega f(x, \underline{u}_q)(\underline{u}_q - u)^+\, dx \\ &= \int_\Omega \left\{ [f(\cdot, \underline{u}_q) - f(\cdot, T(u))] - \sum_{j=1}^k |f(\cdot, T_j(u)) - f(\cdot, T(u))| \right\} (\underline{u}_q - u)^+\, dx \\ &= \int_{\{x \in \Omega : \underline{u}_q > u(x)\}} \left\{ [f(\cdot, \underline{u}_q) - f(\cdot, T(u))] - \sum_{j=1}^k |f(\cdot, T_j(u)) - f(\cdot, T(u))| \right\} (\underline{u}_q - u)\, dx \\ &\leq 0. \end{aligned}$$

Hence

$$\begin{aligned} 0 \leq \langle \beta B(u), (\underline{u}_q - u)^+ \rangle &= \beta \int_{\{x \in \Omega : \underline{u}_q(x) > u(x)\}} b(\cdot, u)(\underline{u}_q - u)\, dx \\ &= -\beta \int_{\{x \in \Omega : \underline{u}_q(x) > u(x)\}} (\underline{u} - u)(\underline{u}_q - u)\, dx \leq 0, \end{aligned}$$

and thus

$$0 = \int_{\{x \in \Omega : \underline{u}_q(x) > u(x)\}} (\underline{u} - u)(\underline{u}_q - u)\, dx \geq \int_{\{x \in \Omega : \underline{u}_q(x) > u(x)\}} (\underline{u}_q - u)^2\, dx \geq 0.$$

Consequently, $\underline{u}_q - u = 0$ a.e. in $\{x \in \Omega : \underline{u}_q(x) > u(x)\}$ and this set must have measure 0. We have shown that $u \geq \underline{u}_q$ a.e. in Ω. Since this holds for all $q \in \{1, \ldots, k\}$, one obtains $u \geq \underline{u}$.

From the definitions of B and Γ, we have

$$B(u) = 0 \quad \text{and} \quad \langle \Gamma(u), \phi \rangle = \int_\Omega f(\cdot, u)\phi \, dx.$$

Therefore the variational inequality (3.12) reduces to our original inequality (1.1), that is, u is a solution of (1.1). □

By using similar arguments, one can show the following existence result for solutions lying between the subsolutions and the supersolutions when both exist. In this case, we need only a more relaxed growth condition on the lower term between the sub- and super-solutions. In fact, we have the following existence theorem.

THEOREM 3.4. *Assume* (1.1) *has subsolutions* $\underline{u}_i$, $i = 1, \ldots, k$, *and supersolutions* $\bar{u}_j$, $j = 1, \ldots, m$. *Let* $\underline{u}$ *be as in* (3.2) *and* $\bar{u} = \min\{\bar{u}_j : 1 \le j \le m\}$. *Suppose furthermore that* $\underline{u} \le \bar{u}$ *a.e. in* Ω *and that* f *has the growth condition* (3.4) *for a.e.* $x \in \Omega$, *all* $u \in [\underline{u}_0(x), \bar{u}_0(x)]$, *where* $\bar{u}_0(x) = \max\{\bar{u}_j : 1 \le j \le m\}$.

Then (1.1) *has a solution* u *between* $\underline{u}$ *and* $\bar{u}$.

4. Existence of extremal solutions

In this section, we show a further property of the solution set of the inequality (1.1), namely, we prove that under the assumptions of Theorems 3.2 or 3.4, there exist greatest and/or smallest solutions of (1.1) between the sub- and super-solutions. First, let us show the following result about the existence of greatest solutions above a subsolution.

THEOREM 4.1. *Under the assumptions of Theorem* 3.2, *there exists a greatest solution* u^* *above* $\underline{u}$, *that is,* u^* *is a solution of* (1.1), $u^* \ge \underline{u}$, *and if* u *is any solution of* (1.1) *such that* $u \ge \underline{u}$ *then* $u \le u^*$.

PROOF. Let $\mathcal{S}$ be the set of solutions of (1.1) above $\underline{u}$:

$$\mathcal{S} := \{u \in W_0^1 L_\Phi : u \ \text{ is a solution of (1.1) and } u \ge \underline{u} \text{ a.e. in } \Omega\}. \tag{4.1}$$

In the first step, we show that $\mathcal{S}$ is bounded in $W_0^1 L_\Phi$. In fact, assume $u \in \mathcal{S}$. Letting $v = 0$ in (1.1) yields

$$\begin{aligned}\int_\Omega \Phi(|\nabla u|) \, dx &\le \int_\Omega f(x, u) u \, dx \\ &\le \int_\Omega a|u| \, dx + \int_\Omega \Psi'(|u|)|u| \, dx \\ &\le \|a\|_{L_{\tilde{\Phi}}} \|u\|_{L_\Psi} + \int_\Omega \Psi(2|u|) \, dx,\end{aligned} \tag{4.2}$$

(since Ψ is a Young function, we have $0 \leq \Psi'(|u|)|u| \leq \Psi(2|u|)$, for all $u \in \mathbb{R}$). Again, from [11, Lemma 5.7], there are constants $C, k > 0$ such that

$$\int_\Omega \Phi(|\nabla u|)\,dx \geq \frac{1}{C}\int_\Omega \Phi(k|u|)\,dx, \quad \text{for all } u \in W_0^1 L_\Phi.$$

From (3.5), there exists $M > 0$ such that

$$\frac{\Psi(2|s|)}{\Phi(k|s|)} < \frac{1}{2C}, \quad \text{for all } s \in \mathbb{R}, |s| \geq M.$$

As a consequence, one obtains

$$\begin{aligned}
&\int_\Omega \Phi(|\nabla u|)\,dx \\
&\quad \leq \|a\|_{L_{\bar\Psi}}\|u\|_{L_\Psi} + \int_{\{x\in\Omega:|u(x)|<M\}} \Psi(2M)\,dx + \frac{1}{2C}\int_{\{x\in\Omega:|u(x)|\geq M\}} \Phi(k|u|)\,dx \\
&\quad \leq \|a\|_{L_{\bar\Psi}}\|u\|_{L_\Psi} + |\Omega|\Psi(2M) + \frac{1}{2C}\int_\Omega \Phi(k|u|)\,dx \\
&\quad \leq \|a\|_{L_{\bar\Psi}}\|u\|_{L_\Psi} + |\Omega|\Psi(2M) + \frac{1}{2}\int_\Omega \Phi(|\nabla u|)\,dx.
\end{aligned}$$

Therefore

$$\frac{1}{2}\int_\Omega \Phi(|\nabla u|)\,dx \leq \|a\|_{L_{\bar\Psi}}\|u\|_{L_\Psi} + |\Omega|\Psi(2M). \tag{4.3}$$

On the other hand, because $\bar\Phi$ satisfies a Δ_2 condition, we have from (3.22) a positive number R_0 such that

$$\int_\Omega \Phi(|\nabla u|)\,dx \geq 3\|a\|_{L_{\bar\Psi}}\mu^{-1}\|u\|_{W_0^1 L_\Phi}, \tag{4.4}$$

for all $u \in W_0^1 L_\Phi$, $\|u\|_{W_0^1 L_\Phi} \geq R_0$. Here μ is the best embedding constant for the embedding $W_0^1 L_\Phi \hookrightarrow L_\Psi$, that is,

$$\mu = \inf\{\|u\|_{W_0^1 L_\Phi} : u \in W_0^1 L_\Phi, \|u\|_{L_\Psi} = 1\}.$$

In particular,

$$\mu\|u\|_{L_\Psi} \leq \|u\|_{W_0^1 L_\Phi}, \quad \text{for all } u \in W_0^1 L_\Phi. \tag{4.5}$$

If $\|u\|_{L_\Psi} \geq R_0/\mu$, then from (4.3)–(4.5), it follows that $\frac{1}{2}\|a\|_{L_{\bar\Psi}}\|u\|_{L_\Psi} \leq |\Omega|\Psi(2M)$, that is, $\|u\|_{L_\Psi} \leq 2|\Omega|\Psi(2M)\|a\|_{L_{\bar\Psi}}^{-1}$. We have shown that if $u \in \mathcal{S}$ then

$$\|u\|_{L_\Psi} \leq \max\{R_0\mu^{-1}, 2|\Omega|\Psi(2M)\|a\|_{L_{\bar\Psi}}^{-1}\}.$$

This estimate, together with (4.3) and (4.4), shows that the set $\{\|u\|_{W_0^1 L_\Phi} : u \in \mathcal{S}\}$ is bounded, that is, $\mathcal{S}$ is bounded in $W_0^1 L_\Phi$. From the boundedness of $\mathcal{S}$ in $W_0^1 L_\Phi$, we can choose $M > 0$ such that $\int_\Omega \Phi(u/M)\, dx \le 1$, for all $u \in S$.

Next, we show that $\mathcal{S}$ has a maximal element with respect to the ordering $\le$ in $W_0^1 L_\Phi$, which is a maximal solution of (1.1). In view of Zorn's lemma, we only need to check that every nonempty chain $\mathcal{C}$ in $\mathcal{S}$ has an upper bound. Suppose $\mathcal{C} \neq \emptyset$ is a chain in $\mathcal{S}$. Let $u_0 \in \mathcal{C}$ and put $\mathcal{C}_0 = \{u \in \mathcal{C} : u \ge u_0\}$. To prove that $\mathcal{C}$ has an upper bound in $\mathcal{S}$, one only has to show that $\mathcal{C}_0$ has an upper bound in $\mathcal{S}$. Let

$$\alpha_0 := \sup \left\{ \int_\Omega \Phi\left(\frac{u}{M}\right) dx : u \in \mathcal{C}_0 \right\} (\le 1).$$

By considering $\mathcal{C}_0 - u_0$ instead of $\mathcal{C}_0$, one can assume without loss of generality that $u \ge 0$ a.e. in Ω, for every $u \in \mathcal{C}_0$. There are two cases:

(i) there is a $u \in \mathcal{C}_0$ such that $\int_\Omega \Phi(u/M)\, dx = \alpha_0$, and
(ii) for all $u \in \mathcal{C}_0$, we have $\int_\Omega \Phi(u/M)\, dx < \alpha_0$.

If (i) holds then u is an upper bound of $\mathcal{C}_0$. In fact, for any $v \in \mathcal{C}_0$, either $u \le v$ or $v \le u$. In the first case, we have from the monotonicity of Φ that

$$\alpha_0 = \int_\Omega \Phi\left(\frac{u}{M}\right) dx \le \int_\Omega \Phi\left(\frac{v}{M}\right) dx \le \alpha_0.$$

Thus $\int_\Omega \Phi(u/M)\, dx = \int_\Omega \Phi(v/M)\, dx$. Because $0 \le u \le v$ and Φ is strictly increasing on $[0, \infty)$, this occurs only if $u = v$. Hence $u \ge v$, for all $v \in \mathcal{C}_0$, that is, u is an upper bound of $\mathcal{C}_0$.

Assume now that case (ii) holds. In this case, from the definition of α_0, we can construct inductively a sequence $\{u_n\}$ in $\mathcal{C}_0$ such that

$$\alpha_0 > \int_\Omega \Phi\left(\frac{u_1}{M}\right) dx > \alpha_0 - 1,$$

and

$$\alpha_0 > \int_\Omega \Phi\left(\frac{u_n}{M}\right) dx > \max\left\{ \int_\Omega \Phi\left(\frac{u_{n-1}}{M}\right) dx, \alpha_0 - \frac{1}{n} \right\}, \quad \forall n > 1. \tag{4.6}$$

We note that $u_n \ge u_{n-1}$ (for all $n > 1$). In fact, if this does not hold then $u_{n-1} \ge u_n$ (because $\mathcal{C}$ is a chain) and as above, one must have

$$\int_\Omega \Phi\left(\frac{u_{n-1}}{M}\right) dx \ge \int_\Omega \Phi\left(\frac{u_n}{M}\right) dx,$$

which contradicts (4.6). Consequently, $\{u_n\}$ is an increasing sequence in L_Φ and thus

$$u_n \to u \quad \text{a.e. in } \Omega, \tag{4.7}$$

where $u = \sup\{u_n : n \in \mathbb{N}\}$. On the other hand, from the boundedness of $\{u_n\}$ in $W_0^1 L_\Phi$ and the compact embedding $W_0^1 L_\Phi \hookrightarrow L_\Phi$, by passing to a subsequence if necessary, we can assume that $u_n \rightharpoonup^* \tilde{u}$ in $W_0^1 L_\Phi$, and $u_n \to \tilde{u}$ in L_Φ, and thus in $L^1(\Omega)$. Comparing to (4.7) and by passing again to a subsequence if necessary, we have $u = \tilde{u}$, which implies that $u_n \rightharpoonup^* u$ in $W_0^1 L_\Phi$, and therefore

$$u_n \to u \quad \text{in } \; L_\Phi. \tag{4.8}$$

Next, let us prove that u is an upper bound of $\mathcal{C}_0$. Let $v \in \mathcal{C}_0$. If $v \leq u_n$ for some n, then $v \leq u$. Assume otherwise that $v \not\leq u_n$ for all n. Again, since $\mathcal{C}_0$ is a chain, we must have $u_n \leq v$ for all n. Using again the above arguments, we get

$$\int_\Omega \Phi\left(\frac{u_n}{M}\right) dx \leq \int_\Omega \Phi\left(\frac{v}{M}\right) dx, \quad \text{for all } \; n \in \mathbb{N}. \tag{4.9}$$

Letting $n \to \infty$ in this inequality and using (4.6), one obtains $\alpha_0 \leq \int_\Omega \Phi(v/M)\, dx$. Because $v \in \mathcal{C}_0$, this contradicts our assumption on α_0. Hence u is an upper bound of $\mathcal{C}_0$.

In this last step, let us prove that u belongs to $\mathcal{S}$. Since $u_n \geq \underline{u}$ for all $n \in \mathbb{N}$ due to $\{u_n\} \subset \mathcal{S}$, we have $u \geq \underline{u}$. Because $u_n \in \mathcal{S}$, we have, for any $v \in W_0^1 L_\Phi$,

$$\int_\Omega \Phi(|\nabla v|)\, dx - \int_\Omega \Phi(|\nabla u_n|)\, dx \geq \int_\Omega f(x, u_n)(v - u_n)\, dx. \tag{4.10}$$

It follows from (4.7) and (4.8) that

$$\int_\Omega f(x, u_n)(v - u_n)\, dx \xrightarrow{n\to\infty} \int_\Omega f(x, u)(v - u)\, dx. \tag{4.11}$$

From the lower semicontinuity of J with respect to the weak* topology in $W_0^1 L_\Phi$, we have

$$\int_\Omega \Phi(|\nabla u|)\, dx \leq \liminf_{n\to\infty} \int_\Omega \Phi(|\nabla u_n|)\, dx. \tag{4.12}$$

Combining (4.10)–(4.12), one sees that u is a solution of (1.1). Hence $u \in \mathcal{S}$ and u is therefore an upper bound of $\mathcal{C}$ in $\mathcal{S}$.

We have shown that every nonempty chain in $\mathcal{S}$ has an upper bound. By Zorn's lemma, $\mathcal{S}$ has a maximal element u^*. Let us verify that u^* is in fact the greatest element of $\mathcal{S}$. Assume otherwise that there exists $v \in \mathcal{S}$ such that

$$v \not\leq u^*. \tag{4.13}$$

Because $v \geq \underline{u}$ and f has the growth condition (3.4)–(3.5), v satisfies (2.2) (ii). Also, since v is a solution of (1.1), it clearly satisfies (2.2) (i) and (2.2) (iii). This means that

v is a subsolution of (1.1). Similarly, u^* is a subsolution of (1.1). Let $\tilde{u} = \max\{v, u^*\}$. Note that the growth condition (3.4) also holds for v, u^*, and $\tilde{u}$ in our present case. From Theorem 3.2, (1.1) has a solution w such that $w \geq \tilde{u} (\geq u^* \geq \underline{u})$. Hence $w \in \mathcal{S}$. Because $w \geq u^*$ and u^* is a maximal element of $\mathcal{S}$, one must have $w = u^*$. Thus $u^* = w \geq \tilde{u} \geq v$. This contradicts (4.13) and shows that u^* is in fact the greatest element of $\mathcal{S}$. Our proof is complete. □

By employing analogous arguments, one can show the existence of solutions and extremal solutions of (1.1) between the sub- and the super-solutions. In fact, we have the following result.

THEOREM 4.2. *Under the assumptions of Theorem 3.4, there exist a smallest solution u_* and a greatest solution u^* between $\underline{u}$ and $\bar{u}$, that is, u_*, u^* are solutions of* (1.1) *satisfying $\underline{u} \leq u_* \leq u^* \leq \bar{u}$, and if u is any solution of* (1.1) *such that $\underline{u} \leq u \leq \bar{u}$, then $u_* \leq u \leq u^*$.*

Acknowledgement

The author would like to thank the referee for his(her) careful reading and valuable remarks and suggestions.

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