

ON A DISTRIBUTION OF LEIPNIK AND PEARCE

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We dedicate this note to the memory and kindness of Professor Roy B. Leipnik.

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Abstract

A recent paper by Leipnik and Pearce introduced the gamma Weibull distribution. One of the main properties given is its characteristic function, which is expressed as an infinite sum. In this note, we provide a simpler representation in terms of the well-known hypergeometric functions in some special cases. We also derive expressions for moments of the distribution.

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1. Introduction

By multiplying the gamma and Weibull densities and renormalizing, Leipnik and Pearce [2] introduced a novel distribution referred to as the *gamma Weibull distribution*. The probability density function (pdf) of this distribution takes the form

$$f(x) = Kx^{\alpha-1} \exp(-\mu x - ax^r) \quad (1.1)$$

for $x > 0$, $r > 0$, $a > 0$, $\mu > 0$ and $\alpha > 0$, where $K = K(r, a, \mu, \alpha)$ is the normalizing constant. Note that the corresponding form of the pdf given in Equation (2.1) of Leipnik and Pearce [2] has five parameters; in fact, only four parameters are needed to define the gamma Weibull distribution.

Leipnik and Pearce derived the characteristic function (chf) and discussed sums of independent random variables having the gamma Weibull distribution. Every property

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derived in the paper is expressed as infinite sums. In particular, the chf is expressed as an infinite sum of gamma functions. In this note, we show that when r is rational a simpler form can be derived in terms of the well-known hypergeometric function defined by

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

where $(f)_k = f(f + 1) \cdots (f + k - 1)$ denotes the ascending factorial. We also derive simple forms for the moments associated with (1.1)—a property not discussed by Leipnik and Pearce.

2. Characteristic function

Let X be a random variable with pdf given by (1.1). The chf of X can be expressed as

$$\begin{aligned} \phi(t) &= E [\exp(itX)] \\ &= K \int_0^{\infty} x^{\alpha-1} \exp(itx - \mu x - ax^r) dx \\ &= K \int_0^{\infty} x^{\alpha-1} \exp(-sx - ax^r) dx \\ &= KI(\alpha, s, a, r), \end{aligned} \tag{2.1}$$

where $s = \mu - it$. By Equation (2.3.1.13) in Prudnikov *et al.* [3, Volume 1], the integral I in (2.1) can be evaluated as

$$I = \begin{cases} \sum_{j=0}^{q-1} \frac{(-a)^j \Gamma(\alpha + rj)}{j! s^{\alpha+rj}} {}_{p+1}F_q(1, \Delta(p, \alpha + rj); \Delta(q, 1 + j); (-1)^q z), & \text{if } 0 < r < 1, \\ \sum_{h=0}^{p-1} \frac{(-1)^h s^h}{r h! a^{(\alpha+h)/r}} \Gamma\left(\frac{\alpha+h}{r}\right) \\ \quad \times {}_{q+1}F_p\left(1, \Delta\left(q, \frac{\alpha+h}{r}\right); \Delta(p, 1+h); \frac{(-1)^p}{z}\right), & \text{if } r > 1, \\ \frac{1}{(a+s)^\alpha}, & \text{if } r = 1, \end{cases} \tag{2.2}$$

provided that $r = p/q$, where $p \geq 1$ and $q \geq 1$ are co-prime integers, where $z = p^p a^q / \{s^p q^q\}$ and $\Delta(k, a) = (a/k, (a + 1)/k, \dots, (a + k - 1)/k)$. Thus, the chf

of X can be expressed as finite sums of the hypergeometric function provided that r is a rational number. The hypergeometric functions are well known and well established (see [1, Section 9.14]). Numerical routines for computing them are widely available (see for example Maple and Mathematica).

3. Moments

Similar expressions can be obtained for the moments. If X is a random variable with pdf given by (1.1) then its n th moment can be expressed as

$$\begin{aligned} E(X^n) &= K \int_0^\infty x^{n+\alpha-1} \exp(-\mu x - ax^r) dx \\ &= KI(n + \alpha, \mu, a, r), \end{aligned}$$

where I is given by (2.2). In particular, the normalizing constant K is given by $K = 1/I(\alpha, \mu, a, r)$.

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