

GENERALIZATIONS OF A WEIGHTED TRAPEZOIDAL INEQUALITY FOR MONOTONIC FUNCTIONS AND APPLICATIONS

KUEI-LIN TSENG¹, GOU-SHENG YANG² and SEVER S. DRAGOMIR³

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Abstract

In this paper we establish some generalizations of a weighted trapezoidal inequality for monotonic functions and give several applications for the r -moments, the expectation of a continuous random variable and the Beta and Gamma functions.

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1. Introduction

The *trapezoidal inequality* states that if f'' exists and is bounded on (a, b) , then

$$\left| \int_a^b f(x) dx - \frac{b-a}{2} [f(a) + f(b)] \right| \leq \frac{(b-a)^3}{12} \|f''\|_\infty, \quad (1.1)$$

where $\|f''\|_\infty := \sup_{x \in (a,b)} |f''(x)| < \infty$.

Now if we assume that $I_n : a = x_0 < x_1 < \dots < x_n = b$ is a partition of the interval $[a, b]$ and f is as above, then we can approximate the integral $\int_a^b f(x) dx$ by the *trapezoidal quadrature rule* $A_T(f, I_n)$, having an error denoted by $R_T(f, I_n)$, where

$$A_T(f, I_n) := \frac{1}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] l_i,$$

¹Department of Mathematics, Aletheia University, Tamsui 25103, Taiwan; e-mail: klseng@email.au.edu.tw.

²Department of Mathematics, Tamkang University, Tamsui 25137, Taiwan.

³School of Computer Science and Mathematics, Victoria University, PO Box 14428, MCMC 8001, Victoria, Australia; e-mail: sever.dragomir@vu.edu.au.

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and the remainder $R_T(f, I_n)$ satisfies the estimation

$$|R_T(f, I_n)| \leq \frac{1}{12} \|f''\|_\infty \sum_{i=0}^{n-1} l_i^3,$$

with $l_i := x_{i+1} - x_i$ for $i = 0, 1, \dots, n - 1$.

For some recent results which generalize, improve and extend this classic inequality (1.1), see the papers [2, 3, 4, 5, 6, 7, 8].

Recently, Cerone and Dragomir [3] proved the following two trapezoidal-type inequalities for monotonic functions:

THEOREM A. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a monotonic nondecreasing function on $[a, b]$, then*

$$\begin{aligned} & \left| \int_a^b f(t) dt - [(x - a)f(a) + (b - x)f(b)] \right| \\ & \leq (b - x)f(b) - (x - a)f(a) + \int_a^b \operatorname{sgn}(x - t)f(t) dt \\ & \leq (x - a)[f(x) - f(a)] + (b - x)[f(b) - f(x)] \\ & \leq \left[\frac{1}{2}(b - a) + \left| x - \frac{a + b}{2} \right| \right] [f(b) - f(a)], \end{aligned} \tag{1.2}$$

for all $x \in [a, b]$. The above inequalities are sharp.

Let I_n, l_i ($i = 0, 1, \dots, n - 1$) be as above and let $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, 1, \dots, n - 1$) be intermediate points. Define the *generalized trapezoidal quadrature rule* by

$$T_P(f, I_n, \xi) := \sum_{i=0}^{n-1} [(\xi_i - x_i)f(x_i) + (x_{i+1} - \xi_i)f(x_{i+1})].$$

We have the following result for the approximation of $\int_a^b f(x) dx$ in terms of T_P .

THEOREM B. *Let f be defined as in Theorem A, then we have*

$$\int_a^b f(x) dx = T_P(f, I_n, \xi) + R_P(f, I_n, \xi).$$

The remainder term $R_P(f, I_n, \xi)$ satisfies the inequalities

$$\begin{aligned} |R_P(f, I_n, \xi)| & \leq \sum_{i=0}^{n-1} [(x_{i+1} - \xi_i)f(x_{i+1}) - (\xi_i - x_i)f(x_i)] + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}(\xi_i - t)f(t) dt \\ & \leq \sum_{i=0}^{n-1} (\xi_i - x_i)[f(\xi_i) - f(x_i)] + \sum_{i=0}^{n-1} (x_{i+1} - \xi_i)[f(x_{i+1}) - f(\xi_i)] \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=0}^{n-1} \left[\frac{1}{2} l_i + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] [f(x_{i+1}) - f(x_i)] \\
&\leq \left[\frac{1}{2} \nu(l) + \max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] [f(b) - f(a)] \\
&\leq \nu(l) [f(b) - f(a)],
\end{aligned} \tag{1.3}$$

where $\nu(l) := \max \{l_i \mid i = 0, 1, \dots, n-1\}$.

In this paper we establish weighted generalizations of Theorems **A** and **B** and give several applications for the r -moments, the expectation of a continuous random variable and the Beta and Gamma functions.

2. Some integral inequalities

The following result shows a generalization of the weighted trapezoidal inequality.

THEOREM 1. *Let $g : [a, b] \rightarrow \mathbb{R}$ be nonnegative and continuous with $g(t) > 0$ on (a, b) and let $h : [a, b] \rightarrow \mathbb{R}$ be differentiable such that $h'(t) = g(t)$ on $[a, b]$.*

(a) *If $f : [a, b] \rightarrow \mathbb{R}$ is a monotonic nondecreasing function on $[a, b]$, then*

$$\begin{aligned}
&\left| \int_a^b f(t)g(t) dt - [(x - h(a)) f(a) + (h(b) - x) f(b)] \right| \\
&\leq (h(b) - x) f(b) - (x - h(a)) f(a) + \int_a^b \operatorname{sgn}(h^{-1}(x) - t) f(t)g(t) dt \\
&\leq (x - h(a)) [f(h^{-1}(x)) - f(a)] + (h(b) - x) [f(b) - f(h^{-1}(x))] \\
&\leq \left[\frac{1}{2} \int_a^b g(t) dt + \left| x - \frac{h(a) + h(b)}{2} \right| \right] [f(b) - f(a)]
\end{aligned} \tag{2.1}$$

for all $x \in [h(a), h(b)]$.

(b) *If $f : [a, b] \rightarrow \mathbb{R}$ is a monotonic nonincreasing function on $[a, b]$, then*

$$\begin{aligned}
&\left| \int_a^b f(t)g(t) dt - [(x - h(a)) f(a) + (h(b) - x) f(b)] \right| \\
&\leq (x - h(a)) f(a) - (h(b) - x) f(b) + \int_a^b \operatorname{sgn}(t - h^{-1}(x)) f(t)g(t) dt \\
&\leq (x - h(a)) [f(a) - f(h^{-1}(x))] + (h(b) - x) [f(h^{-1}(x)) - f(b)] \\
&\leq \left[\frac{1}{2} \int_a^b g(t) dt + \left| x - \frac{h(a) + h(b)}{2} \right| \right] [f(a) - f(b)]
\end{aligned} \tag{2.2}$$

for all $x \in [h(a), h(b)]$.

The above inequalities are sharp.

PROOF. (a) Let $x \in [h(a), h(b)]$. Using the integration by parts formula for the Riemann-Stieltjes integral, we have the following identity:

$$\begin{aligned} \int_a^b (x - h(t)) df(t) &= (x - h(t))f(t) \Big|_a^b + \int_a^b f(t)g(t) dt \\ &= \int_a^b f(t)g(t) dt - [(x - h(a))f(a) + (h(b) - x)f(b)]. \end{aligned} \quad (2.3)$$

It is well known [1, page 813] that if $\mu, \nu : [a, b] \rightarrow \mathbb{R}$ are such that μ is continuous on $[a, b]$ and ν is monotonic nondecreasing on $[a, b]$, then

$$\left| \int_a^b \mu(t) d\nu(t) \right| \leq \int_a^b |\mu(t)| d\nu(t). \quad (2.4)$$

Now, using identity (2.3) and inequality (2.4), we have

$$\begin{aligned} &\left| \int_a^b f(t)g(t) dt - [(x - h(a))f(a) + (h(b) - x)f(b)] \right| \\ &\leq \int_a^b |x - h(t)| df(t) \\ &= \int_a^{h^{-1}(x)} (x - h(t))df(t) + \int_{h^{-1}(x)}^b (h(t) - x)df(t) \\ &= (x - h(t))f(t) \Big|_a^{h^{-1}(x)} + \int_a^{h^{-1}(x)} f(t)g(t) dt \\ &\quad + (h(t) - x) \Big|_{h^{-1}(x)}^b - \int_{h^{-1}(x)}^b f(t)g(t) dt \\ &= (h(b) - x)f(b) - (x - h(a))f(a) + \int_a^b \operatorname{sgn}(h^{-1}(x) - t) f(t)g(t) dt \end{aligned} \quad (2.5)$$

and the first inequalities in (2.1) are proved.

As f is monotonic nondecreasing on $[a, b]$, we obtain

$$\int_a^{h^{-1}(x)} f(t)g(t) dt \leq f(h^{-1}(x)) \int_a^{h^{-1}(x)} g(t) dt = (x - h(a))f(h^{-1}(x))$$

and

$$\int_{h^{-1}(x)}^b f(t)g(t) dt \geq f(h^{-1}(x)) \int_{h^{-1}(x)}^b g(t) dt = (h(b) - x)f(h^{-1}(x)),$$

then

$$\int_a^b \operatorname{sgn}(h^{-1}(x) - t) f(t)g(t) dt \leq (x - h(a))f(h^{-1}(x)) + (x - h(b))f(h^{-1}(x)).$$

Therefore

$$\begin{aligned}
 & (h(b) - x)f(b) - (x - h(a))f(a) + \int_a^b \operatorname{sgn}(h^{-1}(x) - t) f(t)g(t) dt \\
 & \leq (h(b) - x)f(b) - (x - h(a))f(a) \\
 & \quad + (x - h(a))f(h^{-1}(x)) + (x - h(b))f(h^{-1}(x)) \\
 & = (x - h(a)) [f(h^{-1}(x)) - f(a)] + (h(b) - x) [f(b) - f(h^{-1}(x))], \quad (2.6)
 \end{aligned}$$

which proves the second inequality in (2.1).

As f is monotonic nondecreasing on $[a, b]$, we have $f(a) \leq f(h^{-1}(x)) \leq f(b)$ and

$$\begin{aligned}
 & (x - h(a)) [f(h^{-1}(x)) - f(a)] + (h(b) - x) [f(b) - f(h^{-1}(x))] \\
 & \leq \max\{x - h(a), h(b) - x\} [f(h^{-1}(x)) - f(a) + f(b) - f(h^{-1}(x))] \\
 & = \left[\frac{h(b) - h(a)}{2} + \left| x - \frac{h(a) + h(b)}{2} \right| \right] [f(b) - f(a)] \\
 & = \left[\frac{1}{2} \int_a^b g(t) dt + \left| x - \frac{h(a) + h(b)}{2} \right| \right] [f(b) - f(a)]. \quad (2.7)
 \end{aligned}$$

Thus, by (2.5)–(2.7), we obtain (2.1).

Let $g(t) \equiv 1, h(t) = t, t \in [a, b]$;

$$f(t) = \begin{cases} 0, & t \in [a, b), \\ 1, & t = b; \end{cases}$$

and $x = (a + b)/2$. Then a simple calculation reveals that

$$\begin{aligned}
 & \left| \int_a^b f(t)g(t) dt - [(x - h(a))f(a) + (h(b) - x)f(b)] \right| \\
 & = (h(b) - x)f(b) - (x - h(a))f(a) + \int_a^b \operatorname{sgn}(h^{-1}(x) - t) f(t)g(t) dt \\
 & = (x - h(a)) [f(h^{-1}(x)) - f(a)] + (h(b) - x) [f(b) - f(h^{-1}(x))] \\
 & = \left[\frac{1}{2} \int_a^b g(t) dt + \left| x - \frac{h(a) + h(b)}{2} \right| \right] [f(b) - f(a)] \\
 & = \frac{b - a}{2},
 \end{aligned}$$

which proves that the inequalities (2.1) are sharp.

(b) If f is replaced by $-f$ in (a), then (2.2) is obtained from (2.1). This completes the proof. \square

REMARK 1. If we choose $g(t) \equiv 1$ and $h(t) = t$ on $[a, b]$, then inequalities (2.1) reduce to (1.2).

COROLLARY 1. If we choose $x = (h(a) + h(b))/2$, then we get

$$\begin{aligned} & \left| \int_a^b f(t)g(t) dt - \frac{f(a) + f(b)}{2} \int_a^b g(t) dt \right| \\ & \leq \frac{1}{2}[f(b) - f(a)] \int_a^b g(t) dt \\ & \quad + \int_a^b \operatorname{sgn} \left(h^{-1} \left(\frac{h(a) + h(b)}{2} \right) - t \right) f(t)g(t) dt \\ & \leq \frac{1}{2}[f(b) - f(a)] \int_a^b g(t) dt, \end{aligned} \quad (2.8)$$

where f and g are defined as in (a) of Theorem 1, and

$$\begin{aligned} & \left| \int_a^b f(t)g(t) dt - \frac{f(a) + f(b)}{2} \int_a^b g(t) dt \right| \\ & \leq \frac{1}{2}[f(a) - f(b)] \int_a^b g(t) dt \\ & \quad + \int_a^b \operatorname{sgn} \left(t - h^{-1} \left(\frac{h(a) + h(b)}{2} \right) \right) f(t)g(t) dt \\ & \leq \frac{1}{2}[f(a) - f(b)] \int_a^b g(t) dt, \end{aligned} \quad (2.9)$$

where f and g are defined as in (b) of Theorem 1.

The inequalities (2.8) and (2.9) are the “weighted trapezoid” inequalities.

Note that the trapezoidal inequalities (2.8) and (2.9) are, in a sense, the best possible inequalities we can obtain from (2.1) and (2.2). Moreover, the constant $1/2$ is the best possible for both inequalities in (2.8) and (2.9), respectively.

REMARK 2. The following inequality is well known in the literature as the Fejér inequality (see for example [9]):

$$f \left(\frac{a+b}{2} \right) \int_a^b g(t) dt \leq \int_a^b f(t)g(t) dt \leq \frac{f(a) + f(b)}{2} \int_a^b g(t) dt, \quad (2.10)$$

where $f : [a, b] \rightarrow \mathbb{R}$ is convex and $g : [a, b] \rightarrow \mathbb{R}$ is positive, integrable and symmetric to $(a+b)/2$.

Using the above results and (2.8)–(2.9), we obtain the following error bound of the second inequality in (2.10):

$$\begin{aligned}
 0 &\leq \frac{f(a) + f(b)}{2} \int_a^b g(t) dt - \int_a^b f(t)g(t) dt \\
 &\leq \frac{1}{2}[f(b) - f(a)] \int_a^b g(t) dt \\
 &\quad + \int_a^b \operatorname{sgn} \left(h^{-1} \left(\frac{h(a) + h(b)}{2} \right) - t \right) f(t)g(t) dt \\
 &\leq \frac{1}{2}[f(b) - f(a)] \int_a^b g(t) dt,
 \end{aligned} \tag{2.11}$$

provided that f is monotonic nondecreasing on $[a, b]$.

Also,

$$\begin{aligned}
 0 &\leq \frac{f(a) + f(b)}{2} \int_a^b g(t) dt - \int_a^b f(t)g(t) dt \\
 &\leq \frac{1}{2}[f(a) - f(b)] \int_a^b g(t) dt + \int_a^b \operatorname{sgn} \left(t - h^{-1} \left(\frac{h(a) + h(b)}{2} \right) \right) f(t)g(t) dt \\
 &\leq \frac{1}{2}[f(a) - f(b)] \int_a^b g(t) dt,
 \end{aligned} \tag{2.12}$$

provided that f is monotonic nonincreasing on $[a, b]$.

3. Applications for the quadrature rules

Throughout this section, let g and h be defined as in Theorem 1.

Let $f : [a, b] \rightarrow \mathbb{R}$, and let $I_n : a = x_0 < x_1 < \dots < x_n = b$ be a partition of $[a, b]$ and $\xi_i \in [h(x_i), h(x_{i+1})]$ ($i = 0, 1, \dots, n - 1$) be intermediate points. Put $l_i := h(x_{i+1}) - h(x_i) = \int_{x_i}^{x_{i+1}} g(t) dt$ and define the quadrature rule

$$T_P(f, g, h, I_n, \xi) := \sum_{i=0}^{n-1} [(\xi_i - h(x_i)) f(x_i) + (h(x_{i+1}) - \xi_i) f(x_{i+1})].$$

We have the following result concerning the approximation of $\int_a^b f(t)g(t) dt$ in terms of T_P .

THEOREM 2. Let $\nu(l) := \max \{l_i \mid i = 0, 1, \dots, n - 1\}$, f be defined as in Theorem 1 and let

$$\int_a^b f(t)g(t) dt = T_P(f, g, h, I_n, \xi) + R_P(f, g, h, I_n, \xi). \tag{3.1}$$

(a) If f is monotonic nondecreasing on $[a, b]$, then

$$\begin{aligned}
 & |R_P(f, g, h, I_n, \xi)| \\
 & \leq \sum_{i=0}^{n-1} [(h(x_{i+1}) - \xi_i) f(x_{i+1}) - (\xi_i - h(x_i)) f(x_i)] \\
 & \quad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}(h^{-1}(\xi_i) - t) f(t)g(t) dt \\
 & \leq \sum_{i=0}^{n-1} (\xi_i - h(x_i)) [f(h^{-1}(\xi_i)) - f(x_i)] \\
 & \quad + \sum_{i=0}^{n-1} (h(x_{i+1}) - \xi_i) [f(x_{i+1}) - f(h^{-1}(\xi_i))] \\
 & \leq \sum_{i=0}^{n-1} \left[\frac{1}{2}l_i + \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] [f(x_{i+1}) - f(x_i)] \\
 & \leq \left[\frac{1}{2}v(l) + \max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] [f(b) - f(a)] \\
 & \leq v(l)[f(b) - f(a)]. \tag{3.2}
 \end{aligned}$$

(b) If f is monotonic nonincreasing on $[a, b]$, then

$$\begin{aligned}
 & |R_P(f, g, h, I_n, \xi)| \\
 & \leq \sum_{i=0}^{n-1} [(\xi_i - h(x_i)) f(x_i) - (h(x_{i+1}) - \xi_i) f(x_{i+1})] \\
 & \quad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}(t - h^{-1}(\xi_i)) f(t)g(t) dt \\
 & \leq \sum_{i=0}^{n-1} (\xi_i - h(x_i)) [f(x_i) - f(h^{-1}(\xi_i))] \\
 & \quad + \sum_{i=0}^{n-1} (h(x_{i+1}) - \xi_i) [f(h^{-1}(\xi_i)) - f(x_{i+1})] \\
 & \leq \sum_{i=0}^{n-1} \left[\frac{1}{2}l_i + \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] [f(x_i) - f(x_{i+1})] \\
 & \leq \left[\frac{1}{2}v(l) + \max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] [f(a) - f(b)] \\
 & \leq v(l)[f(a) - f(b)].
 \end{aligned}$$

PROOF. (a) Apply Theorem 1 on the intervals $[x_i, x_{i+1}]$ ($i = 0, 1, \dots, n-1$) to get

$$\begin{aligned}
 & \left| \int_{x_i}^{x_{i+1}} f(t)g(t) dt - [(\xi_i - h(x_i)) f(x_i) + (h(x_{i+1}) - \xi_i) f(x_{i+1})] \right| \\
 & \leq (h(x_{i+1}) - \xi_i) f(x_{i+1}) - (\xi_i - h(x_i)) f(x_i) \\
 & \quad + \int_{x_i}^{x_{i+1}} \operatorname{sgn}(h^{-1}(\xi_i) - t) f(t)g(t) dt \\
 & \leq (\xi_i - h(x_i)) [f(h^{-1}(\xi_i)) - f(x_i)] \\
 & \quad + (h(x_{i+1}) - \xi_i) [f(x_{i+1}) - f(h^{-1}(\xi_i))] \\
 & \leq \left[\frac{1}{2}l_i + \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] [f(x_{i+1}) - f(x_i)]
 \end{aligned}$$

for all $i \in \{0, 1, \dots, n-1\}$.

Using this and the generalized triangle inequality, we have

$$\begin{aligned}
 & |R_p(f, g, h, I_n, \xi)| \\
 & \leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(t)g(t) dt - [(\xi_i - h(x_i)) f(x_i) + (h(x_{i+1}) - \xi_i) f(x_{i+1})] \right| \\
 & \leq \sum_{i=0}^{n-1} [(h(x_{i+1}) - \xi_i) f(x_{i+1}) - (\xi_i - h(x_i)) f(x_i)] \\
 & \quad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn}(h^{-1}(\xi_i) - t) f(t)g(t) dt \\
 & \leq \sum_{i=0}^{n-1} (\xi_i - h(x_i)) [f(h^{-1}(\xi_i)) - f(x_i)] \\
 & \quad + \sum_{i=0}^{n-1} (h(x_{i+1}) - \xi_i) [f(x_{i+1}) - f(h^{-1}(\xi_i))] \\
 & \leq \sum_{i=0}^{n-1} \left[\frac{1}{2}l_i + \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] [f(x_{i+1}) - f(x_i)] \\
 & \leq \left[\frac{1}{2}v(l) + \max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \right] [f(b) - f(a)]. \tag{3.3}
 \end{aligned}$$

Next, observe that

$$\left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \leq \frac{1}{2}l_i \quad (i = 0, 1, \dots, n-1); \tag{3.4}$$

and then

$$\max_{i=0,1,\dots,n-1} \left| \xi_i - \frac{h(x_i) + h(x_{i+1})}{2} \right| \leq \frac{1}{2}v(l). \tag{3.5}$$

Thus, by (3.3)–(3.5), we obtain (3.2).

(b) The proof is similar to that for (a) and we omit the details. □

REMARK 3. If we choose $g(t) \equiv 1, h(t) = t$ on $[a, b]$, then the inequalities (3.2) reduce to (1.3).

Now, let $\xi_i = (h(x_i) + h(x_{i+1}))/2$ ($i = 0, 1, \dots, n - 1$) and let $T_{PW}(f, g, h, I_n)$ and $R_P(f, g, h, I_n)$ be defined as

$$T_{PW}(f, g, h, I_n) = T_P(f, g, h, I_n, \xi) = \frac{1}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] \int_{x_i}^{x_{i+1}} g(t) dt$$

and

$$\begin{aligned} R_{PW}(f, g, h, I_n) &= R_P(f, g, h, I_n, \xi) \\ &= \int_a^b f(t)g(t) dt - \frac{1}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] \int_{x_i}^{x_{i+1}} g(t) dt. \end{aligned}$$

If we consider the *weighted trapezoidal formula* $T_{PW}(f, g, h, I_n)$, then we have the following corollary.

COROLLARY 2. Let f, g, h be defined as in Theorem 2 and $\xi_i = (h(x_i) + h(x_{i+1}))/2$ ($i = 0, 1, \dots, n - 1$). Then

$$\int_a^b f(t)g(t) dt = T_{PW}(f, g, h, I_n) + R_{PW}(f, g, h, I_n)$$

where the remainder satisfies the following estimates:

(a) If f is monotonic nondecreasing on $[a, b]$, then

$$\begin{aligned} |R_{PW}(f, g, h, I_n)| &\leq \frac{1}{2} \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} g(t) dt \right) [f(x_{i+1}) - f(x_i)] \\ &\quad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn} \left(h^{-1} \left(\frac{h(x_i) + h(x_{i+1})}{2} \right) - t \right) f(t)g(t) dt \\ &\leq \frac{1}{2} \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} g(t) dt \right) [f(x_{i+1}) - f(x_i)] \\ &\leq \frac{v(l)}{2} [f(b) - f(a)]. \end{aligned} \tag{3.6}$$

(b) If f is monotonic nonincreasing on $[a, b]$, then

$$\begin{aligned}
 |R_{PW}(f, g, h, I_n)| &\leq \frac{1}{2} \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} g(t) dt \right) [f(x_i) - f(x_{i+1})] \\
 &\quad + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \operatorname{sgn} \left(t - h^{-1} \left(\frac{h(x_i) + h(x_{i+1})}{2} \right) \right) f(t) g(t) dt \\
 &\leq \frac{1}{2} \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} g(t) dt \right) [f(x_i) - f(x_{i+1})] \\
 &\leq \frac{v(l)}{2} [f(a) - f(b)]. \tag{3.7}
 \end{aligned}$$

REMARK 4. In Corollary 2, suppose f is monotonic on $[a, b]$,

$$x_i = h^{-1} \left[h(a) + \frac{i(h(b) - h(a))}{n} \right] \quad (i = 0, 1, \dots, n),$$

and

$$l_i := h(x_{i+1}) - h(x_i) = \frac{h(b) - h(a)}{n} = \frac{1}{n} \int_a^b g(t) dt. \quad (i = 0, 1, \dots, n - 1).$$

If we want to approximate the integral $\int_a^b f(t)g(t) dt$ by $T_{PW}(f, g, h, I_n)$ with an accuracy greater than $\varepsilon > 0$, we need at least $n_\varepsilon \in \mathbb{N}$ points for the partition I_n , where

$$n_\varepsilon := \left\lceil \frac{1}{2\varepsilon} \int_a^b g(t) dt \cdot |f(b) - f(a)| \right\rceil + 1$$

and $[r]$ denotes the Gaussian integer of r ($r \in \mathbb{R}$).

4. Some inequalities for random variables

Throughout this section, let $0 < a < b$, $r \in \mathbb{R}$, and let X be a continuous random variable having the continuous probability density function $g : [a, b] \rightarrow \mathbb{R}$ with $g(t) > 0$ on (a, b) , $h : [a, b] \rightarrow \mathbb{R}$ with $h'(t) = g(t)$ for $t \in (a, b)$ and the r -moment $E_r(X) := \int_a^b t^r g(t) dt$, which is assumed to be finite.

THEOREM 3. We have the inequalities

$$\begin{aligned}
 \left| E_r(X) - \frac{a^r + b^r}{2} \right| &\leq \frac{1}{2} (b^r - a^r) + \int_a^b \operatorname{sgn} \left(h^{-1} \left(\frac{1}{2} \right) - t \right) t^r g(t) dt \\
 &\leq \frac{1}{2} (b^r - a^r) \quad \text{as } r \geq 0
 \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \left| E_r(X) - \frac{a^r + b^r}{2} \right| &\leq \frac{1}{2}(a^r - b^r) + \int_a^b \operatorname{sgn} \left(t - h^{-1} \left(\frac{1}{2} \right) \right) t^r g(t) dt \\ &\leq \frac{1}{2}(a^r - b^r) \quad \text{as } r < 0, \end{aligned} \tag{4.2}$$

respectively.

PROOF. If we put $f(t) = t^r$ ($t \in [a, b]$), $h(t) = \int_a^t g(x) dx$ ($t \in [a, b]$) and $x = (h(a) + h(b))/2 = 1/2$ in Corollary 1, then we obtain (4.1) and (4.2). This completes the proof. \square

The following corollary is a special case of Theorem 3.

COROLLARY 3. *The inequalities*

$$\left| E(X) - \frac{a + b}{2} \right| \leq \frac{b - a}{2} + \int_a^b \operatorname{sgn} \left(h^{-1} \left(\frac{1}{2} \right) - t \right) t g(t) dt \leq \frac{b - a}{2}$$

hold, where $E(X)$ is the expectation of the random variable X .

5. Inequalities for the Beta and Gamma functions

The following two functions are well known in the literature as the *Beta function*

$$B(x, y) := \int_0^1 t^{x-1} (1 - t)^{y-1} dt, \quad x > 0, y > 0$$

and the *Gamma function*,

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0,$$

respectively.

The following inequality is an application of Theorem 1 for the Beta function.

THEOREM 4. *Let $p, q > 0$. Then we have the inequality*

$$\begin{aligned} |B(p + 1, q + 1) - x| &\leq x + \int_a^b \operatorname{sgn} [t - ((p + 1)x)^{1/(p+1)}] t^p (1 - t)^q dt \\ &\leq x + \left(\frac{1}{p + 1} - 2x \right) [1 - ((p + 1)x)^{1/(p+1)}]^q \\ &\leq \frac{1}{2(p + 1)} + \left| x - \frac{1}{2(p + 1)} \right| \end{aligned} \tag{5.1}$$

for all $x \in [0, 1/(p + 1)]$.

PROOF. If we put $a = 0$, $b = 1$, $f(t) = (1-t)^q$, $g(t) = t^p$ and $h(t) = t^{p+1}/(p+1)$ ($t \in [0, 1]$) in Theorem 1, we obtain inequality (5.1) for all $x \in [0, 1/(p+1)]$. \square

The following remark is an application of Theorem 4 for the Gamma function.

REMARK 5. As $B(p+1, q+1) = \Gamma(p+1)\Gamma(q+1)/\Gamma(p+q+2)$, inequality (5.1) is equivalent to

$$\begin{aligned} \left| \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)} - x \right| &\leq x + \int_a^b \operatorname{sgn} [t - ((p+1)x)^{1/(p+1)}] t^p (1-t)^q dt \\ &\leq x + \left(\frac{1}{p+1} - 2x \right) [1 - ((p+1)x)^{1/(p+1)}]^q \\ &\leq \frac{1}{2(p+1)} + \left| x - \frac{1}{2(p+1)} \right|, \end{aligned}$$

that is,

$$\begin{aligned} &|(p+1)\Gamma(p+1)\Gamma(q+1) - x(p+1)\Gamma(p+q+2)| \\ &\leq \left[x + \int_a^b \operatorname{sgn} [t - ((p+1)x)^{1/(p+1)}] t^p (1-t)^q dt \right] (p+1)\Gamma(p+q+2) \\ &\leq \left[x + \left(\frac{1}{p+1} - 2x \right) [1 - ((p+1)x)^{1/(p+1)}]^q \right] (p+1)\Gamma(p+q+2) \\ &\leq \left[\frac{1}{2} + \left| x(p+1) - \frac{1}{2} \right| \right] \Gamma(p+q+2) \end{aligned}$$

and as $(p+1)\Gamma(p+1) = \Gamma(p+2)$, we get

$$\begin{aligned} &|\Gamma(p+2)\Gamma(q+1) - x(p+1)\Gamma(p+q+2)| \\ &\leq \left[x + \int_a^b \operatorname{sgn} [t - ((p+1)x)^{1/(p+1)}] t^p (1-t)^q dt \right] (p+1)\Gamma(p+q+2) \\ &\leq \left[x + \left(\frac{1}{p+1} - 2x \right) [1 - ((p+1)x)^{1/(p+1)}]^q \right] (p+1)\Gamma(p+q+2) \\ &\leq \left[\frac{1}{2} + \left| x(p+1) - \frac{1}{2} \right| \right] \Gamma(p+q+2) \end{aligned}$$

for any $x \in [0, 1/(p+1)]$.

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