

A NEW PROXIMITY FUNCTION GENERATING THE BEST KNOWN ITERATION BOUNDS FOR BOTH LARGE-UPDATE AND SMALL-UPDATE INTERIOR-POINT METHODS

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Abstract

Interior-Point Methods (IPMs) are not only very effective in practice for solving linear optimization problems but also have polynomial-time complexity. Despite the practical efficiency of large-update algorithms, from a theoretical point of view, these algorithms have a weaker iteration bound with respect to small-update algorithms. In fact, there is a significant gap between theory and practice for large-update algorithms. By introducing self-regular barrier functions, Peng, Roos and Terlaky improved this gap up to a factor of $\log n$. However, checking these self-regular functions is not simple and proofs of theorems involving these functions are very complicated. Roos *et al.* by presenting a new class of barrier functions which are not necessarily self-regular, achieved very good results through some much simpler theorems. In this paper we introduce a new kernel function in this class which yields the best known complexity bound, both for large-update and small-update methods.

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1. Introduction

We deal with the standard linear optimization problem

$$\min \{c^\top x : Ax = b, x \geq 0\} \quad (1.1)$$

where $A \in R^{m \times n}$ is a real $m \times n$ matrix with $\text{rank}(A) = m$, $b \in R^m$ and $c \in R^n$. The dual problem of (1.1) is given by

$$\max \{b^\top y : A^\top y + s = c, s \geq 0\}. \quad (1.2)$$

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Kernel functions play an important role in defining new search directions in primal-dual interior point algorithms for solving linear optimization problems. A kernel function is a univariate strictly convex function which is defined for all positive reals t and is minimal at $t = 1$ where the minimal value equals zero. In other words $\psi(t) : D \rightarrow R_+$ with $R_{++} \subseteq D$, is a kernel function when it satisfies

$$\psi'(1) = \psi(1) = 0, \quad \psi''(t) > 0.$$

Moreover, $\psi(t)$ has the barrier property, that is, $\psi(t)$ goes to infinity if either $t \rightarrow 0$ or $t \rightarrow \infty$. Note that the above properties imply that $\psi(t)$ is completely determined by its second derivative

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi.$$

This kernel function may be extended to a positive n -dimensional vector μ by

$$\Psi(v) = \sum_{i=1}^n \psi(v_i), \quad (1.3)$$

yielding the scaled barrier function $\Psi(v)$. Note that the barrier function $\Psi(v)$ is nonnegative, and zero if and only if v is a vector of ones. Therefore, the value of the barrier function can be considered as a measure for the closeness of x and (y, s) to the μ -centres of (1.1) and (1.2). Hence some authors also call $\Psi(v)$ a proximity function. In the next section we briefly describe how any such barrier function defines a primal-dual interior-point method. The iteration bound for so-called large-update methods is obtained by showing that each iteration decreases $\Psi(v)$ by a sufficient amount. Table 1 gives some examples of kernel functions that have been analyzed so far, and the complexity results for the corresponding algorithms.

Note that all kernel functions in this table depend on a parameter and the mentioned iteration bounds in the table occur by choosing this parameter as indicated in the third column of the table. The first kernel function, the so-called self-regular function, was introduced and analyzed by Peng *et al.* Also the second kernel function is self regular and is the special case of the self regular function $\gamma_{p,q}(t)$, for $p = 1$. The third function is not self regular and has been proposed and analyzed by Bai *et al.* A surprising feature of this kernel function is that it is finite for $t = 0$, a property which separates it from self-regular functions, because self-regular functions become unbounded when t approaches zero. Also Bai *et al.* introduced the following kernel function in [2]:

$$\frac{t^2 - 1}{2} + \frac{e^{1/t} - e}{e}.$$

TABLE 1. Kernel functions with the best known iteration bound for large-update methods. The iteration bound for all kernel functions is $O(\sqrt{n} \log n) \log(n/\epsilon)$.

Kernel function	Parameter	References
$\frac{t^2 - 1}{2} + \frac{t^{1-q} - 1}{q - 1}, q > 1$	$q = \frac{1}{2} \log n$	[5, 6]
$\frac{t^2 - 1}{2} + \frac{t^{1-q} - 1}{q(q - 1)} - \frac{q - 1}{q}(t - 1), q > 1$		
$\frac{t^2 - 1}{2} + \frac{1}{\sigma} (e^{\sigma(1-t)} - 1), \sigma > 0$	$\sigma = O(\log n)$	[1]
$\frac{t^2 - 1}{2} + \int_1^t e^{q(1/\xi - 1)} d\xi, q \geq 1$	$q = O(\log(1 + n))$	[3]

They showed that the iteration bound for the corresponding algorithm is $O(\sqrt{n}(\log n)^2 \log(n/\epsilon))$. This bound is a factor $\log n$ worse than the bound in Table 1. In this paper we introduce a new kernel function as follows:

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{e^{q(1/t-1)} - 1}{q}, \quad q \geq 1. \quad (1.4)$$

We show that this kernel function yields the best-known iteration bound for large-update methods, that is, $O(\sqrt{n} \log n) \log(n/\epsilon)$. Figure 1 depicts the graph of $\psi(t)$.

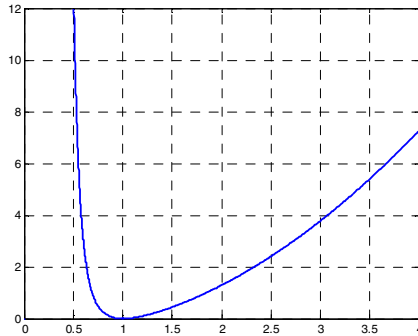


FIGURE 1. The graph of ψ for $q = 4$.

The paper is organized as follows. In Section 2, we briefly recall how a given kernel function defines a primal-dual interior-point algorithm. In Section 3, we describe the

simple conditions on the kernel function that define a general class of kernel functions introduced by Bai *et al.* in [2]. Then we show that the new kernel function introduced in (1.4) satisfies these conditions. In what follows, we use the general scheme for analyzing the generic algorithm, as presented in [2]. We obtain the iteration bounds for both large-update and small-update methods based on a new kernel function in Section 4 and finally, Section 5 contains some concluding remarks.

2. The generic primal-dual interior-point algorithm

It is well known that finding an optimal solution of (1.1) and (1.2) is equivalent to solving the following system:

$$\begin{cases} Ax = b, & x \geq 0, \\ A^\top y + s = c, & s \geq 0, \\ xs = 0. \end{cases}$$

The basic idea of the primal-dual IPMs is to replace the third equation, the so-called *complementarity condition*, by the parameterized equation $xs = \mu e$, with $\mu > 0$. Thus we consider the following system:

$$\begin{cases} Ax = b, & x \geq 0, \\ A^\top y + s = c, & s \geq 0, \\ xs = \mu e. \end{cases}$$

We assume that the primal and dual problem in (1.1) and (1.2) satisfies the interior-point condition (IPC), that is, there exists (x^0, s^0, y^0) such that

$$Ax^0 = b, \quad x^0 > 0, \quad A^\top y^0 + s^0 = c, \quad s^0 > 0.$$

It is well known that the IPC can be assumed without loss of generality. By using the self-dual embedding model, we will assume that $x^0 = s^0 = e$ [4, 7]. If $\text{rank}(A) = m$ and the IPC holds, then for each $\mu > 0$ the above parameterized system has a unique solution. We denote this solution as $(x(\mu), y(\mu), s(\mu))$ and call $x(\mu)$ the μ -centre of (1.1) and $(y(\mu), s(\mu))$ the μ -centre of (1.2). The set of μ -centres (with μ running through all positive real numbers) gives a homotopy path, which is called the central path of (1.1) and (1.2). If $\mu \rightarrow 0$ then the limit of the central path exists and since the limit point satisfies the complementarity condition, the limit yields to optimal solutions for (1.1) and (1.2). If $(x(\mu), y(\mu), s(\mu))$ is known for some positive μ , then we decrease μ to $\mu := (1 - \theta)\mu$, for some fixed $\theta \in (0, 1)$ and solve the following

system:

$$\begin{cases} A\Delta x = 0, \\ A^\top \Delta y + \Delta s = 0, \\ s\Delta x + x\Delta s = \mu e - xs. \end{cases} \quad (2.1)$$

This system uniquely defines a search direction $(\Delta x, \Delta s, \Delta y)$. This direction approximates the next μ -centre. Hence all IPMs follow the central path approximately. Now we define

$$v = \sqrt{\frac{xs}{\mu}}.$$

One can easily check that System (2.1) which defines the search direction can be rewritten as follows:

$$\begin{cases} \bar{A}d_x = 0, \\ \bar{A}^\top \Delta y + d_s = 0, \\ d_x + d_s = v^{-1} - v, \end{cases} \quad (2.2)$$

where $d_x = v\Delta x/x$, $d_s = v\Delta s/s$ and $\bar{A} = AV^{-1}X$, with $V = \text{diag}(v)$ and $X = \text{diag}(x)$. The third equation in (2.2) is called the scaled centring equation. The right-hand side in the scaled centring equation equals minus the gradient of the basic logarithmic barrier function, that is,

$$\Psi_c(v) = \sum_{i=1}^n \left(\frac{v_i^2 - 1}{2} - \log v_i \right).$$

The basic idea in IPMs is to replace the scaled barrier function $\Psi_c(v)$ by an arbitrary strictly convex function $\Psi(v)$, such that $\Psi(v)$ is minimal at $v = e$ with $\Psi(e) = 0$. Thus System (2.2) converts to

$$\begin{cases} \bar{A}d_x = 0, \\ \bar{A}^\top \Delta y + d_s = 0, \\ d_x + d_s = -\Delta\Psi(v). \end{cases}$$

Since the vectors d_x and d_s belong to the null and row spaces of the matrix \bar{A} , these vectors are orthogonal. By taking a suitable step size $\alpha \in (0, 1)$, these search directions construct a new triple (x_+, y_+, s_+) with

$$x_+ = x + \alpha\Delta x, \quad y_+ = y + \alpha\Delta y, \quad s_+ = s + \alpha\Delta s.$$

We repeat the procedure until we find an iterate in a certain neighbourhood of $(x(\mu), y(\mu), s(\mu))$. Then μ is again reduced by the factor $1 - \theta$ and we apply Newton's method targeting the new μ -centres, and so on. This process is repeated until μ is small enough and at this stage we have found an ϵ -solution of the problems (1.1) and (1.2). The generic form of the algorithm is shown in Figure 2.

Input:

A threshold parameter $\tau > 0$;

An accuracy parameter $\epsilon > 0$;

A fixed barrier update parameter $\theta, 0 < \theta < 1$;

begin

$x := e; s := e; \mu := 1$;

while $n\mu > \epsilon$ **do**

begin

$\mu := (1 - \theta)\mu$;

while $\Psi(v) > \tau$ **do**

begin

$x_+ := x + \alpha \Delta x$;

$y_+ := y + \alpha \Delta y$;

$v := \sqrt{\frac{xs}{\mu}}$;

end

end

end

FIGURE 2. A generic primal-dual algorithm for Lo

3. A general class of the kernel functions

In [2] Bai, El Ghami and Roos introduced a general class of kernel functions by using the following conditions:

$$t\psi''(t) + \psi'(t) > 0, \quad t < 1, \quad (3.1)$$

$$t\psi''(t) - \psi'(t) > 0, \quad t > 1, \quad (3.2)$$

$$\psi'''(t) < 0, \quad t > 0, \quad (3.3)$$

$$2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0, \quad t < 1, \quad (3.4)$$

$$\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi'''(\beta t) > 0, \quad t > 1, \beta > 1. \quad (3.5)$$

The kernel function $\psi(t)$ is called eligible if it satisfies (3.1) and (3.3) to (3.5). It was shown in [2] that (3.2) and (3.3) imply (3.5). So a kernel function $\psi(t)$ is

eligible if it satisfies conditions (3.1) to (3.4). First we show that the kernel function introduced in (1.4) satisfies these conditions. The first three derivatives of $\psi(t)$ are shown in the following table:

TABLE 2. ψ and its derivatives.

$\psi(t)$	$\frac{t^2 - 1}{2} + \frac{e^{q/t} - e^q}{qe^q}$
$\psi'(t)$	$t - \frac{e^{q(1/t-1)}}{t^2}$
$\psi''(t)$	$1 + \frac{q + 2t}{t^4} e^{q(1/t-1)}$
$\psi'''(t)$	$-\frac{q^2 + 6qt + 6t^2}{t^6} e^{q(1/t-1)}$

Since for $0 < t < 1$ and $q \geq 1$

$$t\psi''(t) + \psi'(t) = 2t + \left[\frac{q + 2t}{t^3} - \frac{1}{t^2} \right] e^{q(1/t-1)} = 2t + \left(\frac{q + t}{t^3} \right) e^{q(1/t-1)} > 0,$$

Condition (3.1) is satisfied. Furthermore, for $t > 1$

$$t\psi''(t) - \psi'(t) = \frac{q + 3t}{t^3} e^{q(1/t-1)} > 0.$$

So (3.2) is satisfied as well. Also from Table 2 it is obvious that (3.3) is satisfied. Finally for satisfying (3.4), it is seen that

$$\begin{aligned} & 2\psi''(t)^2 - \psi'(t)\psi'''(t) \\ &= \frac{2[t^4 + (q + 2t)e^{q(1/t-1)}]^2 + e^{q(1/t-1)}(q^2 + 6qt + 6t^2)(t^3 - e^{q(1/t-1)})}{t^8}. \end{aligned}$$

If we set $A = e^{q(1/t-1)}$, after some elementary reduction we have

$$2\psi''(t)^2 - \psi'(t)\psi'''(t) = \frac{A^2 [2t^2 + 2qt + q^2] + A [14t^2 + 10qt + q^2t^3] + 2t^8}{t^8}.$$

On the other hand, because $0 < t < 1$, it follows that $1/t - 1 > 0$ and for $q \geq 1$ we have

$$A = e^{q(1/t-1)} > 1.$$

So we have $2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0$ and hence condition (3.4) holds. Therefore $\psi(t)$ lies in the general class introduced in [2]. Having such a kernel function, we can now construct the barrier function $\Psi(v)$ by (1.3). The function $\Psi(v)$ not only serves to define a search direction, but also as a measure of closeness of the current iterates to the μ -centre. We use the norm-based proximity measure $\delta(v)$ defined by

$$\delta(v) = \frac{1}{2} \|\nabla\Psi(v)\| = \frac{1}{2} \sqrt{\sum_{i=1}^n (\psi'(v_i))}.$$

Since $\Psi(v)$ is strictly convex and minimal at $v = e$ we have

$$\Psi(v) = 0 \iff \delta(v) = 0 \iff v = e.$$

In other words, the proximity measure $\delta(v)$ is zero if and only if the current iterates are the μ -centres. For any kernel function in the mentioned class, iteration bounds for both small-update and large-update methods can be obtained by using the following scheme:

Step 1. Solve the equation $-\psi'(t)/2 = s$ to get $\rho(s)$, the inverse function of $-\psi'(t)/2, t \in (0, 1]$. If the equation is hard to solve, derive a lower bound for $\rho(s)$.

Step 2. Calculate the decrease of $\Psi(v)$ in terms of δ for the default step size $\tilde{\alpha} = 1/\psi''(\rho(2\delta))$ from $f(\tilde{\alpha}) = \delta^2/\psi''(\rho(2\delta))$.

Step 3. Solve the equation $\psi(t) = s$ to get $Q(s)$, the inverse function of $\psi(t), t \geq 1$. If the equation is hard to solve, derive lower and upper bound for $Q(s)$.

Step 4. Derive a lower bound for δ in term of $\Psi(v)$ by using $\delta(v) \geq \psi'(Q(\Psi(v)))/2$.

Step 5. Using the result of Steps 3 and 4 find positive constants κ and γ , with $\gamma \in (0, 1]$, such that $f(\tilde{\alpha}) \leq -\kappa\Psi(v)^{1-\gamma}$.

Step 6. Calculate the upper bound for Ψ_0 from

$$\Psi_0 \leq L_\psi(n, \theta, \tau) = n\psi\left(\frac{Q(\tau/n)}{\sqrt{1-\theta}}\right).$$

Step 7. Derive an upper bound for the total number of iterations from

$$\frac{\Psi_0^\gamma}{\kappa\gamma\theta} \log \frac{n}{\epsilon}.$$

Step 8. To calculate a complexity bound for large-update algorithms set $\tau = O(n)$ and $\theta = \Theta(1)$ and for small-update method algorithms set $\tau = O(1)$ and $\theta = \Theta(1/\sqrt{n})$.

In the next section, we apply this scheme to obtain an iteration bound for the algorithm generated by the kernel function introduced in (1.4).

4. Iteration bounds of the algorithm

Since we are unable to get explicit expressions for the inverse functions ρ and Q in the above steps, we recall two lemmas from [2].

LEMMA 4.1. *Let $Q : [0, \infty] \rightarrow [1, \infty]$ be the inverse function of $\psi(t)$ for $t \geq 1$. Then we have*

$$\sqrt{1 + 2s} \leq Q(s) \leq 1 + \sqrt{2s}, \quad s \geq 0.$$

LEMMA 4.2. *Let $\psi_b(t)$ be the barrier term of $\psi(t)$ ($\psi(t) = (t^2 - 1)/2 + \psi_b(t)$) and let $\underline{\rho} : [0, \infty) \rightarrow (0, 1]$ be the inverse function of the restriction of $-\psi'_b(t)$ to the interval $(0, 1]$. Then one has*

$$\rho(s) \geq \underline{\rho}(1 + 2s).$$

Now by using these two lemmas, we derive some bounds for ρ and Q .

Step 1. From the equation $-\psi'_b(t) = s$ we have

$$\begin{aligned} -\psi'_b(t) &= \frac{e^{q(1/t-1)}}{t^2} = s, \\ e^{q(1/t-1)} = st^2 &\iff q \left(\frac{1}{t} - 1 \right) = \log s + 2 \log t. \end{aligned}$$

From $0 < t \leq 1$ we find that $\log t$ and as a result

$$q \left(\frac{1}{t} - 1 \right) \leq \log s \iff t = \underline{\rho}(s) \geq \frac{1}{1 + q^{-1} \log s}.$$

Now by using Lemma 4.2, since $\rho(s) \geq \underline{\rho}(1 + 2s)$, we derive a lower bound for $\rho(s)$ as follows:

$$\rho(s) \geq \frac{1}{1 + q^{-1} \log(1 + 2s)}, \quad s \geq 0. \tag{4.1}$$

Step 2. The function $\psi''(t)$ is monotonically decreasing, hence

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi''(\rho(2\delta))} \leq -\frac{\delta^2}{\psi''(\underline{\rho}(1 + 4\delta))}.$$

Putting $t = \underline{\rho}(1 + 4\delta)$, we have $t \leq 1$ and can write

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi''(t)} \leq -\frac{\delta^2}{1 + (q + 2t)t^{-4}e^{q(1/t-1)}}. \tag{4.2}$$

Note that

$$t = \underline{\rho}(1 + 4\delta) \iff (1 + 4\delta) = -\psi'_b(t) = \frac{e^{q(1/t-1)}}{t^2}.$$

Substituting this equation in (4.2), we have

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{1 + (q + 2t)t^{-2}(1 + 4\delta)} \leq -\frac{\delta^2}{1 + 3qt^{-2}(1 + 4\delta)}.$$

On the other hand,

$$\frac{1}{t^2} = \frac{1}{\underline{\rho}(1 + 4\delta)^2} \leq \frac{1}{\rho(2\delta)^2} \leq (1 + q^{-1} \log(1 + 4\delta))^2.$$

So

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{1 + 3q(1 + 4\delta)(1 + q^{-1} \log(1 + 4\delta))^2}. \tag{4.3}$$

Step 3. By Lemma 4.1 the inverse function of $\psi(t)$ for $t \in [1, \infty)$ satisfies

$$\sqrt{1 + 2\psi(t)} \leq Q(\psi(t)) \leq 1 + \sqrt{2\psi(t)}. \tag{4.4}$$

Step 4. Using $\delta(v) \geq \psi'(Q(\Psi(v)))/2$, we have

$$\delta \geq \frac{\psi'(Q(\Psi(v)))}{2} \geq \frac{1}{2} \left(\sqrt{1 + 2\Psi} - \frac{e^{q(1/\sqrt{1+2\Psi}-1)}}{(1 + 2\Psi)^2} \right). \tag{4.5}$$

Note that $\Psi \geq 1$, so $(1 + 2\Psi)^{3/2} \geq e^{q(1/\sqrt{1+2\Psi}-1)}$ and $\sqrt{1 + 2\Psi} \leq \sqrt{3\Psi}$. So we can write

$$\delta \geq \frac{1}{2} \left(\sqrt{1 + 2\Psi} - \frac{1}{\sqrt{1 + 2\Psi}} \right) = \frac{\Psi}{\sqrt{1 + 2\Psi}} \geq \frac{\Psi}{\sqrt{3\Psi}} \geq \sqrt{\frac{\Psi}{3}}. \tag{4.6}$$

Step 5. Let $\Psi_0 \geq \Psi \geq \tau \geq 3$. We deduced that $\delta \geq 1$ and $\sqrt{\Psi} \leq \sqrt{3}\delta \leq 2\delta$. Now by using (4.3) we have

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{16q\delta(1 + q^{-1} \log(1 + 4\delta))^2} \leq -\frac{\delta}{16q(1 + q^{-1} \log(1 + 4\delta))^2},$$

that is,

$$f(\tilde{\alpha}) \leq -\frac{\sqrt{\Psi}}{48q(1 + q^{-1} \log(1 + \sqrt{\Psi_0}))^2}. \tag{4.7}$$

Thus it follows that

$$\Psi_{k+1} \leq \Psi_k - \kappa(\Psi_k)^{1-\gamma}, \quad k = 0, 1, \dots, K - 1, \tag{4.8}$$

where $\kappa = \frac{1}{48q(1 + q^{-1} \log(1 + \sqrt{\Psi_0}))^2}$, $\gamma = \frac{1}{2}$

and K denotes the number of inner iterations.

Step 6. From Lemma 4.1 we have $Q(\tau/n) \leq 1 + \sqrt{2\tau/n}$. As a consequence

$$\Psi_0 \leq L_\psi(n, \theta, \tau) = n\psi\left(\frac{Q(\tau/n)}{\sqrt{1-\theta}}\right) \leq n\psi\left(\frac{1 + \sqrt{2\tau/n}}{\sqrt{1-\theta}}\right).$$

Since $\psi(t) \leq (t^2 - 1)/2$ for $t \geq 1$,

$$\Psi_0 \leq \frac{n^2 \sqrt{2\tau/n} + 2\tau/n}{2(1-\theta)} = \frac{\tau + \sqrt{2n\tau}}{1-\theta}. \quad (4.9)$$

Step 7. By inequality (4.8), the number of inner iterations is bounded by

$$K \leq \frac{\Psi_0^\gamma}{\kappa\gamma} = 96q \left(1 + q^{-1} \log\left(1 + \sqrt{\Psi_0}\right)\right)^2 \Psi_0^{1/2}.$$

Substituting (4.9) in this inequality gives

$$K \leq 96q \left(1 + q^{-1} \log\left(1 + \sqrt{\frac{\tau + \sqrt{2n\tau}}{1-\theta}}\right)\right)^2 \left(\frac{\tau + \sqrt{2n\tau}}{1-\theta}\right)^{1/2}.$$

Thus the total number of iterations is bounded above by

$$\frac{K}{\theta} \log \frac{n}{\epsilon} \leq 96q \left(1 + q^{-1} \log\left(1 + \sqrt{\frac{\tau + \sqrt{2n\tau}}{1-\theta}}\right)\right)^2 \left(\frac{\tau + \sqrt{2n\tau}}{1-\theta}\right)^{1/2} \frac{1}{\theta} \log \frac{n}{\epsilon}. \quad (4.10)$$

Step 8. For large-update methods set $\tau = O(n)$ and $\theta = \Theta(1)$. As a consequence,

$$\Psi_0 \leq \frac{\tau + \sqrt{2n\tau}}{1-\theta} = O(n).$$

By choosing

$$q = \log\left(1 + \sqrt{\frac{\tau + \sqrt{2n\tau}}{1-\theta}}\right) = O(\log n)$$

the total iteration bound in (4.10) becomes

$$O(\sqrt{n}(\log n)) \log \frac{n}{\epsilon}.$$

Setting $\tau = O(1)$ and $\theta = \Theta(1/\sqrt{n})$ for small-update methods, we can obtain the best known bound as follows.

By Lemma 2.4 in [2], we have $\psi(t) < \psi''(1)(t-1)^2/2$, $t > 1$. Since $\psi''(1) = q + 3$, we achieve an upper bound for Ψ_0 of

$$\begin{aligned} \Psi_0 &= \frac{n\psi''(1)}{2} \left(\frac{1 + \sqrt{2\tau/n}}{\sqrt{1-\theta}} - 1\right)^2 \leq \frac{n\psi''(1)}{2} \left(\frac{\theta + \sqrt{2\tau/n}}{\sqrt{1-\theta}}\right)^2 \\ &= \frac{q+3}{2} \frac{(\sqrt{2\tau} + \theta\sqrt{n})^2}{1-\theta}, \end{aligned}$$

where we also use that $1 - \sqrt{1 - \theta} \leq \theta$. Now $\Psi_0 = O(q)$ and the iteration bound becomes

$$O(q\sqrt{qn}) \log \frac{n}{\epsilon}.$$

By choosing the parameter q as a constant which is independent of n , this is the best bound for small-update methods.

5. Concluding remarks

In this paper, we introduced a new kernel function and showed that it generates the best possible iteration bounds, both for small-update and for large-update methods. This paper was inspired as a result of the work on primal-dual interior-point methods (IPMs) for linear optimization based on kernel functions and the scheme for analyzing such methods. The new kernel function is a parameterized version of a kernel function introduced in [2].

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