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ON SECOND-ORDER CONVERSE DUALITY FOR A NONDIFFERENTIABLE PROGRAMMING PROBLEM

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Certain shortcomings are described in the second order converse duality results in the recent work of (J. Zhang and B. Mond, Bull. Austral. Math. Soc. 55(1997) 29-44). Appropriate modifications are suggested.

1. Introduction

A second-order dual for a nonlinear programming problem was introduced by Mangasarian ([**1**]). Later, Mond [**2**] proved duality theorems under a condition which is called "second-order convexity". This condition is much simpler than that used by Mangasarian. Later, Mond and Weir [**3**] reformulated the second-order dual.

In [**4**], Mond considered the class of nondifferentiable mathematical programming problems

$$\text{(P)} \qquad \text{minimize} \qquad f(x) + (x^T Bx)^{1/2}$$

$$\text{(1)} \qquad \text{subject to} \qquad g(x) \geqslant 0,$$

where $x \in \mathbb{R}^n$, f and g are twice differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}$ and $\mathbb{R}^m$, respectively, and B is an $n \times n$ positive semi-definite (symmetric) matrix.

Recently, Zhang and Mond in [**5**] formulate a general second-order dual model for nondifferentiable programming problems (P):

$$\text{(GD)} \qquad \text{maximize} \qquad f(u) - \sum_{i \in I_0} y_i g_i(u) + u^T Bw - \frac{1}{2} p^T \Big[\nabla^2 f(u) - \nabla^2 \sum_{i \in I_0} y_i g_i(u) \Big] p,$$

$$\text{(2)} \qquad \text{subject to} \qquad \nabla f(u) - +\nabla\big(y^T g(u)\big) + Bw + \nabla^2 f(u)p - \nabla^2 y^T g(u) p = 0,$$

$$\text{(3)} \qquad \sum_{i \in I_\alpha} y_i g_i(u) - \frac{1}{2} p^T \nabla^2 \sum_{i \in I_\alpha} y_i g_i(u) p \leqslant 0, \alpha = 1, 2, \ldots, r,$$

$$\text{(4)} \qquad w^T Bw \leqslant 1,$$

$$\text{(5)} \qquad y \geqslant 0,$$

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where $u, w, p \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $I_\alpha \subset M = \{1, 2, \ldots, m\}$, $\alpha = 0, 1, 2, \ldots, r$ with $\bigcup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \cap I_\beta = \emptyset$ if $\alpha \neq \beta$.

Zhang and Mond in [**5**] give weak, strong and converse duality theorems for first order and second order nondifferentiable dual models under generalised convexity. In particular, they prove the following second order converse duality Theorem.

Theorem 1. Converse duality (see [**5**, Theorem 6]). *Let (x^*, y^*, w^*, p^*) be an optimal solution of (GD) at which*

(A1) *the $n \times n$ Hessian matrix $\nabla\Big[\nabla^2 f(x^*) - \nabla^2\big({y^*}^T g(x^*)\big)\Big]p^*$ is positive or negative definite,*

(A2) *the vectors*

$$\left\{\Big[\nabla^2 f(x^*) - \nabla^2 \sum_{i \in I_0} {y^*}_i g_i(x^*)\Big]_j,\ \Big[\nabla^2 \sum_{i \in I_\alpha} {y^*}_i g_i(x^*)\Big]_j,\ \alpha = 1, 2, \ldots, r, j = 1, 2, \ldots, n\right\}$$

are linearly independent, where $[\cdot]_j$ denotes the j^{th} row.

If for all feasible (x, u, y, w, p), $f(\cdot) - \sum_{i \in I_0} y_i g_i(\cdot) + (\cdot)^T Bw$ is second order pseudoinvex and $\sum_{i \in I_\alpha} y_i g_i(\cdot)$, $\alpha = 1, 2, \ldots, r$ is second order quasincave with respect to the same η, then x^* is an optimal solution to (P).

We note that the matrix $\nabla\Big[\nabla^2 f(x^*) - \nabla^2\big({y^*}^T g(x^*)\big)\Big]p^*$ is positive or negative definite in the assumption (A_1) of Theorem 1, and the result of Theorem 1 implies $p^* = 0$, see [**5**, proof of Theorem 6]. It is obvious that the assumption and the result are inconsistent. In this note, we shall give appropriate modifications for the deficiency in Theorem 1.

2. Second order converse duality

In the section, we shall present a second order converse duality theorem which corrects Theorem 1.

Theorem 2. (Converse duality.) *Let (x^*, y^*, w^*, p^*) be an optimal solution of* (GD) *at which*

(A1) *for all $\alpha = 1, 2, \ldots, r$, either (a) the $n \times n$ Hessian matrix $\nabla^2 \sum_{i \in I_\alpha} {y^*}_i g_i(x^*)$ is positive definite and ${p^*}^T \nabla \sum_{i \in I_\alpha} {y^*}_i g_i(x^*) \geqslant 0$ or (b) the $n \times n$ Hessian matrix $\nabla^2 \sum_{i \in I_\alpha} {y^*}_i g_i(x^*)$ is negative definite and ${p^*}^T \nabla \sum_{i \in I_\alpha} {y^*}_i g_i(x^*) \leqslant 0$,*

(A2) *the vectors*

$$\left\{\Big[\nabla^2 f(x^*) - \nabla^2 \sum_{i \in I_0} {y^*}_i g_i(x^*)\Big]_j,\ \Big[\nabla^2 \sum_{i \in I_\alpha} {y^*}_i g_i(x^*)\Big]_j,\ \alpha = 1, 2, \ldots, r, j = 1, 2, \ldots, n\right\}$$

are linearly independent, where

(A3) *the vectors* $\left\{\nabla \sum_{i\in I_\alpha} {y^*}_i g_i(x^*),\ \alpha = 1,2,\dots,r\right\}$ *are linearly independent.*

If, for all feasible (x,u,y,w,p), $f(\cdot) - \sum_{i\in I_0} y_i g_i(\cdot) + (\cdot)^T Bw$ *is second order pseudoinvex and* $\sum_{i\in I_\alpha} y_i g_i(\cdot)$, $\alpha = 1,2,\dots,r$ *is second order quasincave with respect to the same* η, *then* x^* *is an optimal solution to (P).*

Proof: Since (x^*, y^*, w^*, p^*) is an optimal solution of (GD), by the generalised Fritz John necessary conditions, there exists, $\tau_0 \in \mathbb{R}$, $v \in \mathbb{R}^n$, $\tau_\alpha \in \mathbb{R}, \alpha = 1,2,\dots,r$, $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R}^m$, such that

$$\begin{aligned}
&\tau_0\Big\{-\nabla f(x^*) + \sum_{i\in I_0} \nabla {y^*}_i g_i(x^*) - Bw^* + \frac{1}{2}p^{*T}\nabla\Big[\nabla^2 f(x^*) - \nabla^2 \sum_{i\in I_0} {y^*}_i g_i(x^*)p^*\Big]\Big\}\\
&\qquad + v^T\{\nabla^2 f(x^*) - \nabla^2 y^{*T} g(x^*) + \nabla\big[\nabla^2 f(x^*)p^* - \nabla^2 y^{*T} g(x^*)p^*\big]\}\\
&\qquad + \sum_{\alpha=1}^{r} \tau_\alpha\Big\{\nabla \sum_{i\in I_\alpha} {y^*}_i g_i(x^*) - \frac{1}{2}p^{*T}\nabla\Big[\nabla^2 \sum_{i\in I_\alpha} {y^*}_i g_i(x^*)p^*\Big]\Big\} = 0, y
\end{aligned} \tag{6}$$

$$\tau_0\Big\{g_i(x^*) - \frac{1}{2}p^{*T}\nabla^2 g_i(x^*)p^*\Big\} - v^T\{g_i(x^*) + \nabla^2 g_i(x^*)p^*\} - \gamma_i = 0,\ i \in I_0, \tag{7}$$

$$\begin{aligned}
&\tau_\alpha\Big\{g_i(x^*) - \frac{1}{2}p^{*T}\nabla^2 g_i(x^*)p^*\Big\}\\
&\qquad - v^T\Big\{\nabla g_i(x^*) + \nabla^2 g_i(x^*)p^*\Big\} - \gamma_i = 0, i \in I_\alpha, \alpha = 1,2,\dots,r,
\end{aligned} \tag{8}$$

$$\tau_0 Bx^* - v^T B - 2\beta^T(Bw^*) = 0, \tag{9}$$

$$\begin{aligned}
&(\tau_0 p^* + v)^T\Big\{\nabla^2 f(x^*) - \nabla^2 \sum_{i\in I_0} {y^*}_i g_i(x^*)\Big\}\\
&\qquad - \sum_{\alpha=1}^{r} (\tau_\alpha p^* + v)T\Big\{\nabla^2 \sum_{i\in I_\alpha} {y^*}_i g_i(x^*)\Big\} = 0,
\end{aligned} \tag{10}$$

$$\tau_\alpha\Big\{\sum_{i\in I_\alpha} {y^*}_i g_i(x^*) - \frac{1}{2}p^{*T}\nabla^2 \sum_{i\in I_\alpha} {y^*}_i g_i(x^*)p^*\Big\} = 0,\ \alpha = 1,2,\dots,r, \tag{11}$$

$$\beta(w^* Bw^* - 1) = 0, \tag{12}$$

$$\gamma^T y^* = 0, \tag{13}$$

$$(\tau_0, \tau_1, \tau_2, \dots, \tau_r, \beta, \gamma) \geqslant 0, \tag{14}$$

$$(\tau_0, \tau_1, \tau_2, \dots, \tau_r, \beta, \gamma, v) \neq 0. \tag{15}$$

Because of Assumption (A2), (10) gives

$$\tau_\alpha p^* + v = 0 \qquad \alpha = 0,1,2,\dots,r. \tag{16}$$

Multiplying (8) by ${y^*}_i, i \in I_\alpha, \alpha = 1,2,\dots,r$ and using (11), we have

$$\tau_\alpha\Big\{y^*{}_i g_i(x^*) - \frac{1}{2}p^{*T}\nabla^2 y^*{}_i g_i(x^*)p^*\Big\} \\ - v^T\big\{\nabla y^*{}_i g(x^*) + \nabla^2 y^*{}_i g(x^*)p^*\big\} = 0, i \in I_\alpha, \alpha = 1, 2, \ldots, r,$$

thus

$$\tau_\alpha\Big\{\sum_{i\in I_\alpha} y^*{}_i g_i(x^*) - \frac{1}{2}p^{*T}\sum_{i\in I_\alpha}\nabla^2 y^*{}_i g_i(x^*)p^*\Big\} \\ - v^T\Big\{\sum_{i\in I_\alpha}\nabla y^*{}_i g(x^*) + \sum_{i\in I_\alpha}\nabla^2 y^*{}_i g(x^*)p^*\Big\} = 0, \alpha = 1, 2, \ldots, r.$$

From (11), it follows that

$$v^T\Big\{\sum_{i\in I_\alpha}\nabla y^*{}_i g(x^*) + \sum_{i\in I_\alpha}\nabla^2 y^*{}_i g(x^*)p^*\Big\} = 0, \alpha = 1, 2, \ldots, r. \tag{17}$$

Using (2) in (6), we have

$$(\tau_\alpha p^* + v)^T\Big\{\nabla^2 f(x^*) - \nabla^2\sum_{i\in I_0} y^*{}_i g_i(x^*) + \nabla\Big[\nabla^2 f(x^*) - \nabla^2\sum_{i\in I_0} y^*{}_i g_i(x^*)\Big]p^*\Big\} \\ - \sum_{\alpha=1}^{r}(\tau_\alpha p^* + v)^T\Big\{\nabla^2\sum_{i\in I_\alpha} y^*{}_i g_i(x^*) + \nabla\Big[\nabla^2\sum_{i\in I_\alpha} y^*{}_i g_i(x^*)\Big]p^*\Big\} \\ -\tau_0\Big\{\nabla\sum_{i\in M\setminus I_0} y^*{}_i g_i(x^*) + \nabla^2\sum_{i\in M\setminus I_0} y^*{}_i g_i(x^*)p^*\Big\} \\ -\frac{1}{2}\tau_0 p^{*T}\Big\{\nabla\Big[\nabla^2 f(x^*) - \nabla^2\sum_{i\in I_0} y^*{}_i g_i(x^*)\Big]p^*\Big\} \\ + \sum_{\alpha=1}^{r}\tau_\alpha\Big\{\nabla\sum_{i\in I_\alpha} y^*{}_i g_i(x^*) + \nabla^2\Big[\sum_{i\in I_\alpha} y^*{}_i g_i(x^*)\Big]p^*\Big\} \\ + \sum_{\alpha=1}^{r}\frac{1}{2}\tau_\alpha p^{*T}\Big\{\nabla\Big[\nabla^2\sum_{i\in I_\alpha} y^*{}_i g_i(x^*)\Big]p^*\Big\} = 0.$$

From (16), it follows that

$$\sum_{\alpha=1}^{r}(\tau_\alpha - \tau_0)\Big\{\nabla\sum_{i\in I_\alpha} y^*{}_i g_i(x^*) + \nabla^2\sum_{i\in I_\alpha} y^*{}_i g_i(x^*)p^*\Big\} \\ + \frac{1}{2}v^T\Big\{\nabla\Big[\nabla^2 f(x^*) - \nabla^2\sum_{i\in I_0} y^*{}_i g_i(x^*)\Big]p^* - \nabla\Big[\nabla^2\sum_{i\in M\setminus I_0} y^*{}_i g_i(x^*)\Big]p^*)\Big\} = 0.$$

That is

$$\sum_{\alpha=1}^{r}(\tau_\alpha - \tau_0)\Big\{\nabla\sum_{i\in I_\alpha} y^*{}_i g_i(x^*) + \nabla^2\sum_{i\in I_\alpha} y^*{}_i g_i(x^*)p^*\Big\} \tag{18}$$

$$+\frac{1}{2}v^T\Big\{\nabla\big[\nabla^2 f(x^*)-\nabla^2 y^{*T}g(x^*)\big]p^*\Big\}=0.$$

If for all $\alpha=0,1,2,\ldots,r$, $\tau_\alpha=0$, then $v=0$ from (16), $\gamma=0$ from (7) and (8), and $\beta=0$ from (9) and (12); that is, $(\tau_0,\tau_1,\tau_2,\ldots,\tau_r,\beta,\gamma,v)=0$, contradicts (15). Thus, there exists an $\overline{\alpha}\in\{0,1,2,\ldots,r\}$, such that $\tau_{\overline{\alpha}}>0$.

We claim that $p^*=0$. Indeed, if $p^*\neq 0$, then (16) gives

$$(\tau_\alpha-\tau_{\overline{\alpha}})p^*=0,\alpha=1,2,\ldots,r,.$$

This implies $\tau_\alpha=\tau_{\overline{\alpha}}>0,\alpha=1,2,\ldots,r,$. So, (17) yields

$$p^{*T}\Big\{\sum_{i\in I_\alpha}\nabla y^*{}_i g(x^*)+\sum_{i\in I_\alpha}\nabla^2 y^*{}_i g(x^*)p^*\Big\}=0,\alpha=1,2,\ldots,r,$$

which contradicts to assumption (A_1). Hence, $p^*=0$. Based on (16) and $p^*=0$, we have $v=0$. In view of (A3), (16), $p^*=0$ and $\tau_{\overline{\alpha}}>0$ for some $\overline{\alpha}\in\{0,1,2,\ldots,r\}$, (18) implies $\tau_\alpha=\tau_{\overline{\alpha}}>0$. Now from (7) and (8), it follows that

(19) $$\tau_0 g_i(x^*)-\gamma_i=0,\ i\in I_0,$$

(20) $$\tau_\alpha g_i(x^*)-\gamma_i=0,\ i\in I_\alpha,\alpha=1,2,\ldots,r,$$

Therefore $g(x^*)\geqslant 0$ since $\gamma\geqslant 0$ and $\tau_\alpha>0,\alpha=0,1,2,\ldots,r$. Thus, x^* is feasible for (P), and the objective functions of (P) and (GD) are equal.

Multiplying (19) by $y^*{}_i, i\in I_0$ and using (13) gives

$$\tau_0 y^*{}_i g_i(x^*)=0,i\in I_0.$$

By $\tau_0>0$, it follows that

(21) $$y^*{}_i g_i(x^*)=0,i\in I_0.$$

Also, $v=0$, $\tau_0>0$ and (9) give

(22) $$Bx^*=(2\beta\tau_0)Bw^*.$$

Hence

(23) $$x^{*T}Bx^*=(x^{*T}Bx^*)^{1/2}(w^{*T}Bw^*)^{1/2}.$$

If $\beta>0$, then (12) gives $w^{*T}Bw^*=1$, and so (23) yields

$$x^{*T}Bw^*=(x^{*T}Bx^*)^{1/2}.$$

If $\beta=0$, then (22) gives $Bx^*=0$. So we still get

$$x^{*T}Bw^*=(x^{*T}Bx^*)^{1/2}.$$

Thus, in either case, we have

$$x^{*T}Bw^* = (x^{*T}Bx^*)^{1/2}. \tag{24}$$

Therefore from (21), (24) and $p^* = 0$, we have

$$f(x^*)+(x^{*T}Bx^*)^{1/2} = f(x^*)-\sum_{i\in I_0} y^*{}_i g_i(x^*)+u^{*T}Bw^*-\frac{1}{2}p^{*T}\left[\nabla^2 f(x^*)-\nabla^2\sum_{i\in I_0} y^*{}_i g_i(x^*)\right]p^*.$$

If, for all feasible (x, u, y, w, p), $f(\cdot) - \sum\limits_{i\in I_0} y_i g_i(\cdot) + (\cdot)^T Bw$ is second order pseudoinvex and $\sum\limits_{i\in I_\alpha} y_i g_i(\cdot)$, $\alpha = 1, 2, \ldots, r$ is second order quasincave with respect to the same η, by [**5**, Theorem 4], then x^* is an optimal solution to (P). ∎

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