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EXISTENCE OF POSITIVE SOLUTIONS FOR NONLOCAL AND NONVARIATIONAL ELLIPTIC SYSTEMS

Yujuan Chen and Hongjun Gao

In the paper we prove a result on the existence of positive solutions for a class of nonvariational elliptic system with nonlocal source by Galerkin methods and a fixed point theorem in finite dimensions. We establish another existence result by the super and subsolution method and a monotone iteration.

1. Introduction and main results

In this paper, we consider the positive solutions for a class of elliptic system with nonlocal source:

$$\begin{cases} -\Delta u = f_1(x,u)\|v\|_{\alpha_1}^{p_1}, & x \in \Omega, \\ -\Delta v = f_2(x,v)\|u\|_{\alpha_2}^{p_2}, & x \in \Omega, \\ u > 0,\ v > 0, & x \in \Omega, \\ u = 0,\ v = 0, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N, $N \geqslant 1$, with smooth boundary $\partial\Omega, p_i > 0$, $1 \leqslant \alpha_i \leqslant \infty$ and f_i, $i = 1, 2$, satisfy some assumptions to be stated below.

This system can be used to describe the steady-state solutions of parabolic systems with a nonlocal source:

$$\begin{cases} u_t - \Delta u = f_1(x,u)\|v\|_{\alpha_1}^{p_1}, & x \in \Omega,\ t > 0, \\ v_t - \Delta v = f_2(x,v)\|u\|_{\alpha_2}^{p_2}, & x \in \Omega,\ t > 0, \\ u(x,t) = 0,\ v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

When f_1, f_2 are positive constants, system (1.2) has been discussed by Deng, Li and Xie (see [4]). It was proved that if $p_1 p_2 < 1$ every nonnegative solution is global; whereas if $p_1 p_2 > 1$, and the initial data is sufficiently large, the nonnegative solution of (1.2) blows up in finite time, and if the initial data is sufficiently small, the nonnegative solution of (1.2) is global.

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Over the last few decades, many physical phenomena were formulated into nonlocal mathematical models (see [2, 3, 4, 5, 6],7 or references therein). The corresponding steady-state solutions have been attached great importance. Compared to the existence of positive solutions for elliptic problems, different techniques (variational methods, degree theory, fixed point theory, sub and supersolutions, Galerkin methods, moving hyperplane methods, et cetera) have been used (see for instance [2, 3, 4, 5, 6],7). However, to the author's best knowledge, there is no literture on system (1.1). Since the system (1.1) has no variational structure, it is worthwhile to recall some of related nonvariational methods.

In [1], Alves and de Figureiredo used Galerkin methods and fixed point theorems in finite dimension and proved the existence of solutions of the local semilinear elliptic systems

$$\begin{aligned} -\Delta u &= au^{\alpha} + f(x,u,v), \\ -\Delta v &= bv^{\beta} + g(x,u,v), \end{aligned}$$

in a bounded domain with homogeneous boundary conditions.

In [3] Correa studied the problem

$$-a\left(\int_{\Omega} |u|^{q}\right)\Delta u = H(x)f(u)$$

with homogeneous boundary condition. To prove the existence of positive solutions, the Krasnoselskii fixed point theorem has been used in the unidimensional case and the Schaefer fixed point theorem in the multidimensional case.

In [2] Correa, Silbano Menezes and Ferreira used fixed point theorems and index theory to establish a positive solution for the problem

$$-a\left(\int_{\Omega} |u|\right)\Delta u = f(x,u).$$

Furthermore, they proved another existence result by using sub and supersolutions without monotone iteration, and relying heavily on a comparison principle and the Schaefer fixed point theorem.

The purpose of this paper is to study the existence of solutions of (1.1) in two cases via

(a) Galerkin methods

(b) super and subsolution techniques,

respectively. Our work is motivated by [1, 2, 3],6. From the viewpoint of physics, we are considering the positive weak solutions. We shall use the following definition of the weak solution for (1.1).

DEFINITION 1.1: We say a function $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a weak solution of (1.1) provided

$$\int_\Omega \nabla u \nabla w \, dx = \|v\|_{\alpha_1}^{p_1} \int_\Omega f_1(x,u) w \, dx,$$
$$\int_\Omega \nabla v \nabla w \, dx = \|u\|_{\alpha_2}^{p_2} \int_\Omega f_2(x,v) w \, dx,$$

for all $w \in H_0^1(\Omega)$.

This paper is organised as follows. In the next section, we use Galerkin methods to establish an existence theorem for the solution for system (1.1) when $N \geqslant 3$, $0 < p_i < 1$, $1 < \alpha_i < (2N)/(N-1)$ or $N = 1, 2$, $0 < p_i < 1$, $1 \leqslant \alpha_i < \infty$, $i = 1, 2$. In Section 3, we established another existence result for (1.1) by making use of super and subsolution techniques and where the monotone iteration is used. When f_i $(i = 1, 2)$ is bounded and $p_1 p_2 < 1$, we get a pair of super and subsolution of (1.1) Finally we consider the special case $\alpha_i = p_i = 1$, $i = 1, 2$ and show the only probable solution is the trivial one under some assumptions on f_1, f_2.

Our main results read as follows.

Theorem 1.2. *If either the following* (H1) *or* (H2) *is satisfied, system* (1.1) *has a positive solution.*

(H1) $N \geqslant 3$, $0 < p_i < 1$, $1 < \alpha_i < (2N)/(N-2)$,

$$0 \leqslant D_1(x) \leqslant f_1(x,u) \leqslant A_1(x)|u|^{r_1} + B_1(x),$$
$$0 \leqslant D_2(x) \leqslant f_2(x,v) \leqslant A_2(x)|v|^{r_2} + B_2(x),$$
$$B_i(x) \in L^{(\alpha_j)/(\alpha_j - 1)}, \; A_i(x) \in L^{(\alpha_j)/(\alpha_j - r_i - 1)},$$

$r_i < \min\{1 - p_i, \alpha_j - 1\}$, $D_i(x) \not\equiv 0 (i = 1, 2)$, *where* $ij = 2$ *and* i, j *are positive integers.*

(H2) $N = 1, 2$, *for* $i = 1, 2$, $0 < p_i < 1$, $1 \leqslant \alpha_i < \infty$.

Theorem 1.3. *Suppose that* $f_1(\cdot, u) \in C\big(\overline{\Omega} \times R\big)$, $f_2(\cdot, v) \in C\big(\overline{\Omega} \times R\big)$ *are nondecreasing and Lipschitz continuous in* u *and* v *respectively. Assume there exists a weak supersolution* $\big(\overline{u}(x), \overline{v}(x)\big)$ *and a weak subsolution* $\big(\underline{u}(x), \underline{v}(x)\big)$ *of system* (1.1), *satisfying*

$$\begin{aligned} &\big(\overline{u}(x), \overline{v}(x)\big) \geqslant 0, \big(\underline{u}(x), \underline{v}(x)\big) \leqslant 0 \text{ on } \partial\Omega \text{ in the trace sense}, \\ &\big(\underline{u}(x), \underline{v}(x)\big) \leqslant \big(\overline{u}(x), \overline{v}(x)\big) \text{ almost everywhere in } \Omega. \end{aligned} \tag{1.3}$$

Then there exists a solution (u, v) *of* (1.1)*, such that*

$$\big(\underline{u}(x), \underline{v}(x)\big) \leqslant \big(u(x),\, v(x)\big) \leqslant \big(\overline{u}(x), \overline{v}(x)\big) \text{ almost everywhere in } \Omega. \tag{1.4}$$

Furthermore, if $\big(\underline{u}(x),\, \underline{v}(x)\big) > 0$ *in* Ω*, the solution of system* (1.1) *is positive.*

Theorem 1.4. *In addition to the assumptions on* f_1, f_2 *of Theorem 1.3, we assume further that* $0 < m_1 \leqslant f_1(x,u) \leqslant M_1$, $0 < m_2 \leqslant f_2(x,v) \leqslant M_2$ *in* $\Omega \times R$, $p_1 p_2 < 1$, *then there exists a positive solution of system* (1.1).

2. Galerkin methods

In this section we show the existence of solutions for the system (1.1) via Galerkin methods. The following Fixed Point Theorem is used (see [1] or [6, Chapter 9]).

Proposition 2.1. *Let* $F : R^K \to R^K (K \in \mathbb{N})$ *be a continuous function such that* $\big\langle F(\xi), \xi \big\rangle \geqslant 0$ *on* $|\xi| = r$. *Then, there exists* $z_0 \in \overline{B}_r(0)$ *such that* $F(z_0) = 0$. *Here* $\langle \cdot, \cdot \rangle$ *denotes inner product of two vectors.*

Proof of Theorem 1.2: First, we shall study the existence for the solution of the following systems

$$\begin{aligned} -\Delta u &= f_1(x,u)\|v\|_{\alpha_1}^{p_1} + \lambda\phi, && x \in \Omega, \\ -\Delta v &= f_2(x,v)\|u\|_{\alpha_2}^{p_2} + \lambda\phi, && x \in \Omega, \\ u > 0,\ v &> 0, && x \in \Omega \\ u = 0,\ v &= 0, && x \in \partial\Omega. \end{aligned} \tag{2.1}$$

where $\phi \in C_0^\infty(\Omega)$ is a fixed positive function and λ is a positive parameter.

Let $\sum = \{e_1, \ldots, e_n, \ldots\}$ be an orthogonal basis of the Hilbert space $H_0^1(\Omega)$ and be smooth. For each $m \in N$ define the subspace $V_m = \operatorname{span}\{e_1, \ldots, e_m\}$. It is well known that $\big(V_m, \|\cdot\|\big)$ and $\big(R^m, |\cdot|\big)$ are isometrically isomorphic by the natural linear map $T : V_m \to R^m$ given by

$$v = \sum_{i=1}^{n} \xi_i e_i \mapsto T(v) = \xi = (\xi_1, \xi_2, \ldots, \xi_m).$$

So $\|v\| = \big|T(v)\big| = |\xi|$, where $|\cdot|$ and $\|\cdot\|$ denote the usual norms in R^N and $V_m(\Omega)$, respectively.

Consider the following function $F : R^{2m} \to R^{2m}$ given by

$$F(\xi, \eta) = \big(F_1(\xi,\eta), \ldots, F_m(\xi,\eta), G_1(\xi,\eta), \ldots, G_m(\xi,\eta)\big),$$

where

$$F_i(\xi,\eta) = \int_\Omega \nabla u \nabla e_i \, dx - \|v\|_{\alpha_1}^{p_1} \int_\Omega f_1(x,u) e_i \, dx - \lambda \int_\Omega \phi e_i \, dx,$$

and

$$G_i(\xi,\eta)=\int_\Omega \nabla v\nabla e_i\,dx-\|u\|_{\alpha_2}^{p_2}\int_\Omega f_2(x,v)e_i\,dx-\lambda\int_\Omega \phi e_i\,dx,\ i=1,2,\ldots,m.$$

In the above definitions we are using the identifications

$$\xi\mapsto u=\sum_{i=1}^m \xi_i e_i \text{ and } \eta\mapsto v=\sum_{i=1}^m \eta_i e_i.$$

Note that

$$\begin{aligned}\big\langle F(\xi,\eta),(\xi,\eta)\big\rangle &= \|u\|^2-\|v\|_{\alpha_1}^{p_1}\int_\Omega f_1(x,u)u\,dx\\ &\quad -\lambda\int_\Omega \phi u\,dx+\|v\|^2-\|u\|_{\alpha_2}^{p_2}\int_\Omega f_2(x,v)v\,dx-\lambda\int_\Omega \phi v\,dx,\end{aligned}$$

and when $N\geqslant 3$, we have $H_0^1(\Omega)\subset L^q(\Omega)$, where $1<q<(2N)/(N-2)$. By (H1) and the Hölder inequality, we have

$$\begin{aligned}&\|v\|_{\alpha_1}^{p_1}\int_\Omega f_1(x,u)u\,dx+\|u\|_{\alpha_2}^{p_2}\int_\Omega f_2(x,v)v\,dx\\ &\quad\leqslant \|v\|_{\alpha_1}^{p_1}\int_\Omega \big(A_1(x)|u|^{r_1}+B_1(x)\big)u\,dx+\|u\|_{\alpha_2}^{p_2}\int_\Omega \big(A_2(x)|v|^{r_2}+B_2(x)\big)v\,dx\\ &\quad\leqslant \|v\|_{\alpha_1}^{p_1}\Big[\big\|A_1(x)\big\|_{\alpha_2/(\alpha_2-r_1-1)}\|u\|_{\alpha_2}^{r_1+1}+\big\|B_1(x)\big\|_{\alpha_2/(\alpha_2-1)}\|u\|_{\alpha_2}\Big]\\ &\qquad\qquad+\|u\|_{\alpha_2}^{p_2}\Big[\big\|A_2(x)\big\|_{\alpha_1/(\alpha_1-r_2-1)}\|v\|_{\alpha_1}^{r_2+1}+\big\|B_2(x)\big\|_{\alpha_1/(\alpha_1-1)}\|v\|_{\alpha_1}\Big]\\ &\quad\leqslant C\big(\|u\|^{p_1+r_1+1}+\|v\|^{p_1+r_1+1}+\|u\|^{p_2+r_2+1}+\|v\|^{p_2+r_2+1}\\ &\qquad\qquad+\|u\|^{p_1+1}+\|v\|^{p_1+1}+\|u\|^{p_2+1}+\|v\|^{p_2+1}\big).\end{aligned}$$

Here and elsewhere, we may use the same letter C to indicate (possibly different) positive constants. With $\big\|(u,v)\big\|^2=\|u\|^2+\|v\|^2$, we obtain

$$\begin{aligned}\big\langle F(\xi,\eta),(\xi,\eta)\big\rangle &\geqslant \big\|(u,v)\big\|^2-C\Big[\big\|(u,v)\big\|^{p_1+r_1+1}\\ &\qquad+\big\|(u,v)\big\|^{p_2+r_2+1}+\big\|(u,v)\big\|^{p_1+1}+\big\|(u,v)\big\|^{p_2+1}\Big]-\lambda C\big\|(u,v)\big\|.\end{aligned}$$

As of $0<p_i+r_i<1$ and $0<p_i<1$, there exists a sufficient large ρ such that

$$\big\langle F(\xi,\eta),(\xi,\eta)\big\rangle>0 \text{ on } \big\|(u,v)\big\|=\rho\ (\rho \text{ is independent of } m).$$

It follows from Proposition 2.1 that, for each $m\in N$ there exists $(u_m,v_m)\in V_m\times V_m$ satisfying

$$F(u_m,v_m)=(0,0),\quad \big\|(u_m,v_m)\big\|\leqslant\rho. \tag{2.2}$$

Then (2.2) implies

$$\int_\Omega \nabla u_m \nabla w\,dx = \|v_m\|_{\alpha_1}^{p_1} \int_\Omega f_1(x,u_m)w\,dx + \lambda \int_\Omega \phi w\,dx, \quad \forall w \in V_m,$$
$$\int_\Omega \nabla v_m \nabla w\,dx = \|u_m\|_{\alpha_2}^{p_2} \int_\Omega f_2(x,v_m)w\,dx + \lambda \int_\Omega \phi w\,dx, \quad \forall w \in V_m,$$

with $\|u_m\|, \|v_m\| \leqslant \rho$, $\forall m \in N$. Let $u, v \in H_0^1(\Omega)$ be the weak limit of $\{u_m\}$ and $\{v_m\}$ respectively, choosing subsequences if necessary. So

$(u_m, v_m) \rightharpoonup (u, v)$ weakly in $H_0^1(\Omega)$,
$(u_m, v_m) \to (u, v)$ in $L^q(\Omega)$ for $1 < q < \dfrac{2N}{N-2}$, $\big($since H_0^1 is compact in $L^q(\Omega)\big)$
$(u_m, v_m) \to (u, v)$ almost everywhere in Ω.

Considering $w \in V_k$, $\Psi \in V_k$ and $m \geqslant k$ we have

$$\int_\Omega \nabla u_m \nabla w\,dx = \|v_m\|_{\alpha_1}^{p_1} \int_\Omega f_1(x,u_m)w\,dx + \lambda \int_\Omega \phi w\,dx, \quad \forall w \in V_k,$$
$$\int_\Omega \nabla v_m \nabla \Psi\,dx = \|u_m\|_{\alpha_2}^{p_2} \int_\Omega f_2(x,v_m)\Psi\,dx + \lambda \int_\Omega \phi \Psi\,dx, \quad \forall \Psi \in V_k, \tag{2.3}$$

Then taking the limits as $m \to \infty$, we obtain

$$\int_\Omega \nabla u \nabla w\,dx = \|v\|_{\alpha_1}^{p_1} \int_\Omega f_1(x,u)w\,dx + \lambda \int_\Omega \phi w\,dx, \quad \forall w \in V_k,$$
$$\int_\Omega \nabla v \nabla \Psi\,dx = \|u\|_{\alpha_2}^{p_2} \int_\Omega f_2(x,v)\Psi\,dx + \lambda \int_\Omega \phi \Psi\,dx, \quad \forall \Psi \in V_k, \tag{2.4}$$

The equality (2.4) holds for all functions in H_0^1, as functions of the form w and Ψ are dense in this space. Hence (u, v) is a weak solution of the system (2.1).

Recalling that $\lambda > 0, \phi > 0$, it follows from the maximum principle that $u, v > 0$ in Ω. Then according to (H1), we have

$$-\Delta\big(u/\|v\|_{\alpha_1}^{p_1}\big) \geqslant D_1(x),$$
$$-\Delta\big(v/\|u\|_{\alpha_2}^{p_2}\big) \geqslant D_2(x).$$

Let (z_1, z_2) be the only positive solution of

$$-\Delta z_1 = D_1(x), \quad \text{in } \Omega,$$
$$-\Delta z_2 = D_2(x), \quad \text{in } \Omega,$$
$$z_1 = z_2 = 0, \quad \text{on } \partial\Omega.$$

According to the maximum principle, we have $u/\|v\|_{\alpha_1}^{p_1} \geqslant z_1, v/\|u\|_{\alpha_2}^{p_2} \geqslant z_2$ in Ω, which implies

$$\|u\|_{\alpha_2} \geqslant \left(\|z_1\|_{\alpha_2}\|z_2\|_{\alpha_1}^{p_1}\right)^{1/(1-p_1p_2)} > 0,$$
$$\|v\|_{\alpha_1} \geqslant \left(\|z_1\|_{\alpha_2}^{p_2}\|z_2\|_{\alpha_1}\right)^{1/(1-p_1p_2)} > 0,$$

So $\|u\|_{\alpha_2}, \|v\|_{\alpha_1}$ has a positive lower bound. Moveover, take $\lambda = 1/n, n = 1, 2, \ldots$ and denote the corresponding solution of (2.1) as (u_n, v_n), that is (u_n, v_n) satisfies (2.4) with any w and Ψ in $H_0^1(\Omega)$. Set $w = u_n$, $\Psi = v_n$ in (2.4), we get

$$\left\|(u_n, v_n)\right\|^2 \leqslant C\Big(\left\|(u_n, v_n)\right\|^{p_1+r_1+1} + \left\|(u_n, v_n)\right\|^{p_2+r_2+1} + \left\|(u_n, v_n)\right\|^{p_1+1} + \left\|(u_n, v_n)\right\|^{p_2+1} + \left\|(u_n, v_n)\right\| \cdot \|\phi\|_2\Big) \text{ for all } n \in \mathbb{N}.$$

Since $0 < p_i + r_i < 1$, and $0 < p_i < 1$, $\{(u_n, v_n)\}$ is a bounded sequence. As $n \to \infty$, thanks to the Sobolev embedding and the Lebesgue convergence theorem, a positive solution of (1.1) is obtained.

The case (H2) can be proved similarly, since for $N = 1$,

$$H_0^1(\Omega) \subset C^\alpha(\overline{\Omega}),\ 0 < \alpha \leqslant 1/2,$$

and for $N = 2$,

$$H_0^1(\Omega) \subset L^q(\Omega),\ 1 \leqslant q < \infty.$$

So its proof is omitted. □

3. Super and subsolution method

Over the past two decades, supersolution and subsolution methods have been widely used in the proof of the existence of the solutions of parabolic or elliptic problems. In this section, we shall use this technique to prove the existence of the solution of system (1.1).

First, we shall give the definition of the super and subsolution of system (1.1).

Definition 3.1: A function $(\overline{u}, \overline{v}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is called a weak supersolution of (1.1) if

$$\int_\Omega \nabla\overline{u}\nabla w\,dx \geqslant \|\overline{v}\|_{\alpha_1}^{p_1} \int_\Omega f_1(x, \overline{u})w\,dx, \text{ for each } w \in H_0^1, w > 0 \text{ almost everywhere,} \tag{3.1}$$

$$\int_\Omega \nabla\overline{v}\nabla\Psi\,dx \geqslant \|\overline{u}\|_{\alpha_2}^{p_2} \int_\Omega f_2(x, \overline{v})\Psi\,dx, \text{ for each } \Psi \in H_0^1, \Psi > 0 \text{ almost everywhere.}$$

Similarly, $\big(\underline{u}(x), \underline{v}(x)\big) \in H_0^1(\Omega)$ is called a weak subsolution if it satisfies (3.1) with reversed inequalities.

Proof of Theorem 1.3: The method is standard, but for the reader's convenience, we complete it. Now write $(u_0, v_0) = (\underline{u}, \underline{v})$, we can construct a sequence $\big\{(u_k, v_k)\big\} \in H_0^1(\Omega) \times H_0^1(\Omega)$ from the following iterative process

$$\begin{aligned} \int_\Omega \nabla u_k \nabla w \, dx &= \|v_{k-1}\|_{\alpha_1}^{p_1} \int_\Omega f_1(x, u_{k-1}) w \, dx, \\ \int_\Omega \nabla v_k \nabla \Psi \, dx &= \|u_{k-1}\|_{\alpha_2}^{p_2} \int_\Omega f_2(x, v_{k-1}) \Psi \, dx. \end{aligned} \tag{3.2}$$

We claim

$$(\underline{u}, \underline{v}) \leqslant (u_1, v_1) \leqslant \cdots \leqslant (u_n, v_n) \leqslant \cdots \leqslant (\overline{u}, \overline{v}). \tag{3.3}$$

To confirm this, first note from (3.2) for $k = 1$ that

$$\begin{aligned} \int_\Omega \nabla u_1 \nabla w \, dx &= \|v_0\|_{\alpha_1}^{p_1} \int_\Omega f_1(x, u_0) w \, dx, \\ \int_\Omega \nabla v_1 \nabla \Psi \, dx &= \|u_0\|_{\alpha_2}^{p_2} \int_\Omega f_2(x, v_0) \Psi \, dx \end{aligned} \tag{3.4}$$

for each $w, \Psi \in H_0^1(\Omega)$. Subtract (3.4) from (3.1), recall $(u_0, v_0) = (\underline{u}, \underline{v})$ and set $w = (u_0 - u_1)^+$, $\Psi = (v_0 - v_1)^+$, to get

$$\begin{aligned} \int_\Omega \nabla(u_0 - u_1) \cdot \nabla(u_0 - u_1)^+ \, dx \leqslant 0, \\ \int_\Omega \nabla(v_0 - v_1) \cdot \nabla(v_0 - v_1)^+ \, dx \leqslant 0. \end{aligned} \tag{3.5}$$

But

$$\nabla(u_0 - u_1)^+ = \begin{cases} \nabla(u_0 - u_1), & \text{almost everywhere on } \{u_0 \geqslant u_1\}, \\ 0 & \text{almost everywhere on } \{u_0 \leqslant u_1\}, \end{cases}$$

$$\nabla(v_0 - v_1)^+ = \begin{cases} \nabla(v_0 - v_1), & \text{almost everywhere on } \{v_0 \geqslant v_1\}, \\ 0 & \text{almost everywhere on } \{v_0 \leqslant v_1\}, \end{cases}$$

(see [6, Chapter 5]). Consequently,

$$\begin{aligned} \int_{\{u_0 \geqslant u_1\}} \big|\nabla(u_0 - u_1)\big|^2 \, dx \leqslant 0, \\ \int_{\{v_0 \geqslant v_1\}} \big|\nabla(v_0 - v_1)\big|^2 \, dx \leqslant 0, \end{aligned} \tag{3.6}$$

so that $(u_0, v_0) \leqslant (u_1, v_1)$ almost everywhere in Ω. Now assume inductively

$$(u_{k-1}, v_{k-1}) \leqslant (u_k, v_k) \text{ almost everywhere in } \Omega \tag{3.7}$$

From (3.2) we find similarly

$$\begin{aligned}
&\int_{\{u_k \geqslant u_{k+1}\}} \nabla(u_k - u_{k+1}) \cdot \nabla(u_k - u_{k+1})^+ \, dx \\
&\qquad \leqslant \int_\Omega \big(f_1(x, u_{k-1})\|v_{k-1}\|_{\alpha_1}^{p_1} - f_1(x, u_k)\|v_k\|_{\alpha_1}^{p_1}\big)(u_k - u_{k+1})^+ \, dx \leqslant 0, \\
&\int_{\{v_k \geqslant v_{k+1}\}} \nabla(v_k - v_{k+1}) \cdot \nabla(v_k - v_{k+1})^+ \, dx \\
&\qquad \leqslant \int_\Omega \big(f_2(x, v_{k-1})\|u_{k-1}\|_{\alpha_2}^{p_2} - f_2(x, v_k)\|u_k\|_{\alpha_2}^{p_2}\big)(v_k - v_{k+1})^+ \, dx \leqslant 0.
\end{aligned}$$

The last inequality holds in view of (3.7) and $f_1(\cdot, u), f_2(\cdot, v)$ is nondecreasing in u and v respectively. Therefore $(u_k, v_k) \leqslant (u_{k+1}, v_{k+1})$ almost everywhere in Ω.

$$\begin{aligned}
\int_{\{u_{k+1} \geqslant \overline{u}\}} \big|\nabla(u_{k+1} - \overline{u})\big|^2 \, dx &\leqslant \int_{\{u_{k+1} \geqslant \overline{u}\}} \big(f_1(x, u_k)\|v_k\|_{\alpha_1}^{p_1} \\
&\qquad - f_1(x, \overline{u})\|\overline{v}\|_{\alpha_1}^{p_1}\big)(u_{k+1} - \overline{u})^+ \, dx \leqslant 0, \\
\int_{\{v_{k+1} \geqslant \overline{v}\}} \big|\nabla(v_{k+1} - \overline{v})\big|^2 \, dx &\leqslant \int_{\{v_{k+1} \geqslant \overline{v}\}} \big(f_2(x, v_k)\|u_k\|_{\alpha_2}^{p_2} \\
&\qquad - f_2(x, \overline{v})\|\overline{u}\|_{\alpha_2}^{p_2}\big)(v_{k+1} - \overline{v})^+ \, dx \leqslant 0.
\end{aligned}$$

Next we show that $(u_k, v_k) \leqslant (\overline{u}, \overline{v})$ almost everywhere in Ω. It is valid for $k = 0$ by hypothesis (1.3). Assume now for induction $(u_k, v_k) \leqslant (\overline{u}, \overline{v})$ almost everywhere in Ω. Then we have

Thus $(u_{k+1}, v_{k+1}) \leqslant (\overline{u}, \overline{v})$ almost everywhere in Ω.

Therefore $\big(u(x), v(x)\big) := \lim_{n\to\infty} \big(u_n(x), v_n(x)\big)$ exists for almost all x. Furthermore, by the assumptions on $f_i (i = 1, 2)$, there is a subsequence $\big\{\big(u_{n_j}, v_{n_j}\big)\big\}_{j=1}^{\infty}$ which converges weakly in $H_0^1(\Omega) \times H_0^1(\Omega)$ to (u, v). So, (u, v) is a weak solution of (1.1). The theorem follows. ∎

Theorem 1.4 is an example in which the above result applies.

Proof of Theorem 1.4: Let (w_1, w_2) be the only solution of

$$\begin{aligned}
-\Delta w_1 &= m_1, \quad \text{in } \Omega, \\
-\Delta w_2 &= m_2, \quad \text{in } \Omega, \\
w_1 = w_2 &= 0, \quad \text{on } \partial\Omega.
\end{aligned} \tag{3.8}$$

Set

$$\underline{u} = \left(\|w_1\|_{\alpha_1}^{p_1p_2}\|w_2\|_{\alpha_2}^{p_2}\right)^{1/(1-p_1p_2)}w_1, \quad \underline{v} = \left(\|w_2\|_{\alpha_1}^{p_1p_2}\|w_1\|_{\alpha_2}^{p_2}\right)^{1/(1-p_1p_2)}w_2,$$

then $(\underline{u}, \underline{v})$ is a positive subsolution of system (1.1). Similarly, if we denote by (W_1, W_2) the only positive solution of (3.8) with m_1, m_2 be replaced by M_1, M_2 respectively, we can get the supersolution $(\overline{u}, \overline{v})$ of system (1.1), where

$$\overline{u} = \left(\|W_1\|_{\alpha_1}^{p_1p_2}\|W_2\|_{\alpha_2}^{p_2}\right)^{1/(1-p_1p_2)}W_1, \quad \overline{v} = \left(\|W_2\|_{\alpha_1}^{p_1p_2}\|W_1\|_{\alpha_2}^{p_2}\right)^{1/(1-p_1p_2)}W_2$$

As $m_i \leqslant M_i$, we have $w_i \leqslant W_i, i = 1, 2$. In addition, $0 < p_1p_2 < 1$, so $(\underline{u}, \underline{v}) \leqslant (\overline{u}, \overline{v})$. Theorem 1.3 completes the proof. □

Now, for the the special case $\alpha_i = p_i = 1$, becomes the system

$$-\Delta u = f_1(x, u)\int_\Omega |v|\,dx, \qquad x \in \Omega, \tag{3.9a}$$

$$-\Delta v = f_2(x, v)\int_\Omega |u|\,dx, \qquad x \in \Omega, \tag{3.9b}$$

$$u = v = 0, \qquad x \in \partial\Omega. \tag{3.9c}$$

Denote by $\varphi_0(x)$ the unique positive solution of the linear elliptic problem

$$-\Delta\varphi_0(x) = 1, \quad x \in \Omega; \quad \varphi_0(x) = 0, \quad x \in \partial\Omega.$$

Set $\rho = \int_\Omega \varphi_0(x)\,dx$. Thus, we have the following result.

PROPOSITION 3.2. *In addition to the assumptions on f_1, f_2 of Theorem 1.3, assume further $0 < f_i \leqslant M_i (i = 1, 2)$ in $\Omega \times R$. Then nonegative solutions of (3.9) exist if $\rho^2 \leqslant 1/(M_1M_2)$.*

PROOF: Applying $\rho^2 \leqslant 1/(M_1M_2)$, we see that there exist large positive constants K_1 and K_2 such that

$$M_1\rho \leqslant K_1/K_2 \leqslant 1/(M_2\rho).$$

Let $W(x) = K_1\varphi_0(x)$, $S(x) = K_2\varphi_0(x)$, then (W, S) is a pair of supersolutions of (3.9). Furthermore, $(0, 0)$ is its subsolution, so we can get a nonnegative solution via Theorem 1.3. □

Indeed, we have the following result.

PROPOSITION 3.3. *Assume $0 < m_i < f_i \leqslant M_i (i = 1, 2)$ in $\Omega \times R$, and ρ satisfies*

$$\rho^2 < \frac{1}{M_1M_2} \quad \text{or } \rho^2 > \frac{1}{m_1m_2}. \tag{3.10}$$

Then the system (3.9) only possesses the trivial solution.

PROOF: Suppose (u, v) is a non trivial solution of (3.9). Multiplying (3.9a), (3.9b) through by φ_0 and integrating on Ω respectively, we find

$$\int_\Omega u\,dx = \|v\|_1 \int_\Omega f_1(x,u)\varphi_0\,dx$$
$$\int_\Omega v\,dx = \|u\|_1 \int_\Omega f_2(x,v)\varphi_0\,dx.$$

Since $0 < m_i \leqslant f_i \leqslant M_i (i = 1, 2)$, and φ_0 is a positive function, we have

$$m_1\rho\|v\|_1 \leqslant \|u\|_1 \leqslant M_1\rho\|v\|_1,$$
$$m_2\rho\|u\|_1 \leqslant \|v\|_1 \leqslant M_2\rho\|u\|_1.$$

So

$$m_1m_2\rho^2\|u\|_1\|v\|_1 \leqslant \|u\|_1\|v\|_1 \leqslant M_1M_2\rho^2\|u\|_1\|v\|_1.$$

The conclusion follows since (3.10) holds. □

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