THE AUSTRALIAN MATHEMATICAL SOCIETY

GAZETTE

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THE AUSTRALIAN MATHEMATICAL SOCIETY

Gazette

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The *Gazette* follows policy established in Vol. 1, No. 1 and reiterated in Vol. 14, No. 1.

The *Gazette* seeks to publish items of the following types.

- Mathematical articles of general interest, particularly historical and survey articles
- Reviews of books, particularly by Australian authors, or books of wide interest
- Classroom notes on presenting mathematics in an elegant way
- Items relevant to mathematics education
- Letters on relevant topical issues
- Information on conferences, particularly those held in Australasia and the region
- Information on recent major mathematical achievements
- Reports on the business and activities of the Society
- Staff changes and visitors in mathematics departments
- News of members of the Australian Mathematical Society

Local correspondents are asked to submit news items and act as local Society representatives. Material for publication and editorial correspondence should be submitted to the editors.

Notes for contributors

The editors seek contributions for the *Gazette*. These can be sent through Local Correspondents or directly to the editors. Articles should be fairly short, easily read and of interest to many of our readers. Technical articles are refereed.

Authors are encouraged to typeset technical articles in LaTeX 2ε, AMS-LaTeX or variants. A special *Gazette* classfile `gazette.cls` can be downloaded from http://www.austms.org.au/Gazette or may be obtained by email from the editors. In exceptional cases other formats may be accepted. Other contributions should preferably also be typeset in LaTeX 2ε or variants, but may also be submitted in other editable electronic formats such as plain text or Word documents.

Editorial

What do you write as your ‘usual occupation’ on the entry card when returning to Australia from overseas? ‘Mathematician’, or the more generic ‘Academic’? Is Mathematics a profession and how is it looked upon by the outside world? In this issue’s **Math matters** Cheryl Praeger makes a strong case for all of us to increase the visibility of the mathematical profession in everyday life. As part of this she proposes to include undergraduates as members of the AustMS, free of charge. Similar feelings are expressed in a **non-mathematician’s apology** by Ian Enting, responding to Tony Dooley’s Math matters column in issue two of the *Gazette*.

Views on the future of our profession and government involvement are offered in Tony Guttmann’s final **President’s column** and the last of the **Brain drain** articles by gainee and Federation Fellow Richard Brent.

At the time of writing the federal election campaign is in full swing with our politicians travelling the country begging for our attention, and ultimately, our votes. That voting does not just come down to attracting the majority of votes is something well understood by all sides of politics. But whether John Howard, Mark Latham and other political leaders know all about Condorcet’s voting paradox and the Gibbard-Satterthwaite Theorem is doubtful. To see if a proper understanding of the mathematics behind voting would have changed your vote in the federal election read Norman Do’s article. Norman — a graduate student at The University of Melbourne — is our new **Mathellaneous** columnist, replacing Daniel Mathews.

At this time of year it is probably not just voting that is on our minds, but also the rapidly approaching holiday season. If sand, beach and surf is your thing then Neville (Iron Man) de Mestre’s article on the history and mathematics of bodysurfing is a must.

It has come to our attention that some of our readers are under the impression that contributions to the *Gazette* are by invitation only. Although this happens to be true for the President’s column and Math matters, we very much welcome submissions from all of our readers.

Deadlines for submissions to the *Gazette*

Volume	Number	Deadline
31	5	12 October 2004
32	1	15 February 2005
32	2	19 April 2005

President's column by Tony Guttmann

This is my last column as President, so I thought I would indulge myself by musing on both the positive and the negative developments in the mathematical sciences over the last two years. Unfortunately, the negative developments far outweigh the positives.

It has been an exciting time for researchers, as I reported in the President's column in the previous issue, with breakthroughs in the search for solutions of a number of long-standing and celebrated problems. The positives also include the growing recognition by both State and Federal Governments of the importance of the mathematical sciences. This has been manifested by the support for AMSI, MASCOS and ICE-EM.

Unfortunately, the State and Federal Governments' generosity in establishing institutes has not been backed up by much needed expenditure across the board. Peter Hall, in his trenchant article in the March 2004 issue of the *Gazette*, listed many of the problems facing the profession, and highlighted the particular difficulties facing the discipline of statistics.

Currently, direct Federal Government funding typically accounts for less than 40% of a university's operating budget. It is the full-fee paying cohort of students, both overseas and local, that is propping up our universities, thereby making universities incredibly vulnerable to fluctuations in the level of this resource. Indeed, with the appreciation of the Australian dollar, we are already seeing a decline in overseas student enrolments. I have heard this process evocatively described as "strip mining" the educational sector. Rather than using income generated by full-fee paying students to create superb universities, the Federal Government is using it as an opportunity to allocate them ever-decreasing levels of funding. For the mathematical sciences, the effect of this is exacerbated by the current Relative Funding Model, which leaves us at a disadvantage relative to other disciplines.

As a result, while there are a reasonable number of fixed-term opportunities at universities for post-doctoral job seekers, there are only limited opportunities for them to obtain tenure track positions. As a result we are losing many of our best and brightest young mathematicians overseas.

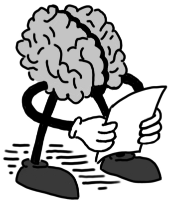
The Federation Fellowship scheme has been essentially useless in addressing this issue. Not one young mathematical scientist has been attracted back to Australia by it, and only by using a broad definition of the mathematical sciences can we say that any mathematicians have been awarded Federation Fellowships at all. Indeed, the ARC seems to have lessened its regard for the mathematical sciences, as evidenced by the number of mathematicians on the relevant panel. As Peter Hall points out, the Canadian Research Chairs system is a far more effective (and less expensive) way of bolstering the university sector.

Nor have government budgets addressed the teacher crisis in schools. There are no new funds for professional development, no mechanism to address the problem of teachers teaching out of field, and little sense of concern for, or even awareness of, a diminishing population of trained mathematics teachers, many of whom are approaching retiring age.

Support for R&D is also too low if Australia is to have a future as a scientifically advanced nation. Admittedly, government expenditure on R&D is above the OECD average, but little seems to be being done, by any political party, to promote policies to raise R&D expenditure by industry. The demand for mathematical scientists created by initiatives in bio-informatics, stem cell research, neuro-informatics and nanotechnology, to name but a few of the areas in which Australia seeks to have a major impact, is huge. Yet the supply of mathematical scientists is not there. The "brain drain", carefully quantified by our Executive Officer, Jan Thomas, shows no signs of improving.

The establishments of the three institutes mentioned above may buy more time, but cannot work miracles, nor compensate for billions of dollars of lost funding. Unfortunately, there seems little sense of urgency in the policies of any political party to address these issues.

It has been a privilege to have been President of the Society for the past two years, and I have been ably assisted—perhaps "propped up" is a more accurate description—by many members of the Executive. I am grateful to all of them, particularly our Secretary, Liz Billington, Treasurer Algy Howe and Executive Officer Jan Thomas. I congratulate and welcome Michael Cowling as the incoming President, and wish him every success in addressing some of the issues raised above.



Letter to the editors

The editors through their new series of articles are highlighting some of the issues that Mathematics is facing in Australia. Yet I cannot help but feel that the issues that are being raised are to some extent ancient history. We have moved on from those dark days and in fact Mathematics has done quite well recently, as has been documented in the excellent article by Tony Dooley. Why not have more emphasis on the positive and on our collective achievements. Doom and gloom is all well and good but it can be self fulfilling and can give quite the wrong impression, especially with the recent Federal Fundings in Mathematics.

Finally, I suppose it is up to the Editors and to the readers of the *Gazette* to form their own views as to the appropriateness of the article by Philip Broadbridge and the comments on page 91 about colleagues knifing people in the back. The ARC does its best in difficult circumstances and I cannot understand why the *Gazette* would allow such comments. Informed opinion is fine but not vitriol.

I urge the *Gazette* to think about what they are trying to achieve with such articles.

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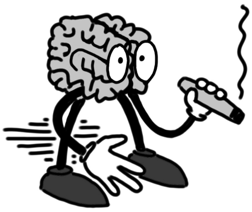
Our aim with the series of articles on the brain drain has been to complement the wealth of statistical data on this important issue with personal stories of drainees. As editors we have chosen to give free reign to our contributors and to not shy away from the sometimes confronting comments being made. Indeed, although the Gazette *should not (just) become a forum for disgruntled mathematicians, the Australian mathematical community should hopefully be strong and mature enough to allow for a free and open debate. We would rather have the* Gazette *be sometimes controversial than be uninteresting and irrelevant.*

As to whether the brain drain is something of the past, only time will tell. It will be clear from Kevin Burrage's letter, the President's column and Richard Brent's 'brain gain' contribution that views on this issue are not at all clear cut.
The editors



Math matters

The Profession of Mathematics

Cheryl E. Praeger

You who glance at this column may care about Mathematics as passionately as I do. You may feel that the importance of Mathematics education is self-evident, for both individuals and society. On the other hand you may also have experienced conversations such as one I had with a friend this week. On telling her that I intended to write about the Profession of Mathematics, she responded that she didn't think that Mathematics was a profession. To be a profession, she said, there had to be a range of careers available, and there were none listed for Mathematics in careers material that her high school-aged son had brought home to show his parents. As a student, she continued, she knew that she had strong problem-solving skills, and good logical and analytical thinking. However, based on her (negative) experience in studying calculus, she had decided to build on her strengths and *not* proceed with Mathematics.

The skills my friend chose in describing her strengths are close to those I would nominate as generic skills to be obtained from a good Mathematics education. However my friend had a very different view from me on the purpose and outcome of a Mathematics education. My view, already on record, is that "the most important outcome from a mathematics education [is] an automatic expectation by students that mathematical thinking will play a key role in their understanding, and problem-solving in every part of their lives"[1].

Why did my perceptive and well-educated friend have such a different understanding from me of the role of Mathematics? *Is there a Mathematics Profession*? If so, what is it like? If not, and if we want there to be one, what must we do to achieve this?

Let us assume for the moment that there is a Mathematics Profession and that we are members of it. What are our perceptions of the Profession? Who are the members and where do they work? What do they need professionally, and in particular, what do they need from a professional association?

Purpose of the Profession

The two hallmarks of Mathematics are its power and its beauty. "The high technology that is so celebrated today is essentially mathematical technology"[2]. Moreover mathematical literacy is critical for an individual to function effectively in modern society. Politicians, industrial leaders, and educators all say they recognise this. To summarise, let's say that the role of the Mathematics Profession is:

[1]In *The Essential Elements of Mathematics*, a paper I wrote in March 2004 in response to an invitation from the Victorian Curriculum and Assessment Authority with respect to its work on developing *Framework of Essential Learning*.

[2]E.E. David, President of Exxon Research and Engineering, see http://www.maths.uwa.edu.au/students/prospective/first_year_general.php

- To strengthen Mathematics education in schools and tertiary institutions, in order to fit young people to function effectively in society;
- To enhance the impact of Mathematics research for the health of our own and other disciplines, and ultimately for the public good; and
- To promote effective applications of Mathematical methods and analysis in commerce and industry, for the economic benefit of our community and nation.

Inevitably most of us, as members of the profession, will focus on some aspects more than others. *Indeed it is the major challenge for the Mathematics Profession to harness the energy and commitment of all its members to work together towards fulfilling this role.*

Many mathematicians are striving towards this. One celebrated successful initiative is the long-running Mathematics in Industry Study Group that seeks to provide annually a forum where mathematicians and other professionals meet to bring mathematical thinking and expertise to bear on a range of problems arising in industry.

Moreover, the mission of the recently established Australian Mathematical Sciences Institute[3] is aligned precisely with this role, as discussed by Garth Gaudry in the third of these columns. In his role as Director of AMSI, Garth found "the level of appreciation of our discipline and its extraordinary impact [to be] extremely high". He called on us in the profession to "broaden our horizons" and "demonstrate our willingness to cooperate, not only among ourselves but with people from the many other endeavours in which the mathematical sciences play a significant role"[4].

Similarly, in the second of these columns, Tony Dooley[5] argued cogently that, in the area of mathematics research, the Mathematics Profession must take "greater control of the mysterious process between theory and applications" and develop "better structures for sharing ideas and projects across the whole spectrum from the purest to the most highly applied research". Taking control of the connection between theory and application is critical, and must be taken seriously. The reason why the process may look "mysterious" is that many of us have not done it – it is challenging and sometimes very difficult as a real application is rarely as clear-cut as theory. However, the mathematical mind is a good one for solving these problems because of the ability to think clearly, recognise what is a proof (or more commonly, what is not) and to simplify complex systems.

Membership of the Profession

The Mathematics Profession deserves a depth of membership that embraces undergraduate "trainees", mathematics teachers at all levels, mathematics researchers, and commercial and industrial mathematicians. Universities certify as graduate mathematicians those who have a major in a mathematical science. Usually this means graduates with three year degrees. At the least, all of these graduates are part of our profession.

However the profession is broader than this. For example, the PhD program in any of the mathematical sciences offers a rigorous training in research, and this is one of the possible routes into a mathematical career. Some graduates in disciplines other than the mathematical sciences become members of the Mathematics Profession through such a program.

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[3] See http://www.amsi.org.au/about.html. The AMSI mission is to become a nationally and internationally recognised centre for the mathematical sciences, providing service to its member institutions, improving the international competitiveness of Australian industry and organisations and enhancing the national level of school mathematics, by the provision and support of mathematical and statistical expertise.

[4] AustMS Gazette **31** (2004), 145–146.

[5] AustMS Gazette **31** (2004), 76–78.

Alternative routes into the Mathematics Profession for those without a complete mathematics or statistics major are of equal validity: for example, the “in-house” and “on-the-job” training and experience that produce effective commercial or industrial mathematicians and statisticians; or the professional development and further study that enable those without a mathematics major to become competent mathematics teachers in schools. *Mathematical ability and commitment should determine membership of the Mathematics Profession, not formal qualifications* – certainly not the holding of a PhD degree. And let us not forget our undergraduate student members.

When it comes to defining the profession, it is important to use the broadest possible umbrella. It is especially relevant to embrace those diverse users of Mathematics and Statistics (in fields such as Computer Science and Bioinformatics) who may not describe their work as Mathematics. Much of what they do is Mathematics by any reasonable definition. The Mathematics Profession should be taking credit for it and welcoming those who practise it into our fold.

Professional associations

There are many mathematical associations in Australia, of which we may like to think of the Australian Mathematical Society as one of the major ones. The membership of each covers only a “slice” of the profession’s membership. Most members of the Australian Mathematical Society (including ANZIAM) are mathematicians or statisticians in universities. A minority are research or commercial mathematicians and statisticians from government or private enterprise, and some are mathematics teachers in schools.

Several other professional mathematical associations include teachers of mathematics in primary and secondary schools, statisticians from all sectors, and mathematical scientists from special sub-disciplines such as Operations Research. There is no single organisation to which all professional mathematicians can logically belong. Moreover, none caters very well for undergraduate mathematics students as members.

By contrast Engineers Australia[6] offers free student membership to all undergraduate engineering students, entailing a monthly student newsletter, and access to careers services, discussion forums and professional advice. From this early stage undergraduate engineering students are welcomed into the Engineering Profession. In addition Engineers Australia has active programs run by its branches, and offers structured professional development programs for individual members and teams.

Efforts to provide opportunities for undergraduate mathematics and statistics students led to the inaugural AMSI Summer School in Melbourne in February 2003, whose success was praised by the Federal Minister, Dr Brendan Nelson[7]. In addition, the Statistical Society of Australia runs a Young Statisticians Section[8] for statistics students and new graduates. Both the Statistical Society of Australia and the Australian Association of Mathematics Teachers have strong state branches that run their own programs independently of the central organisation.

The Accredited Mathematical Scientist

Several mathematical associations have tried to raise public awareness of Mathematics and Statistics, and the quality of members of the Mathematics Profession, by introducing accreditation of their members

[6] http://www.ieaust.org.au

[7] Garth Gaudry, *Math Matters*, AustMS Gazette **31** (2004), 145–146.

[8] http://www.statsoc.org.au/Sections/YoungStatisticians.htm

or of university courses. While accreditation may benefit an individual by providing recognition of their qualifications and experience, *the most valuable purpose of accreditation is to assure those outside the profession that the accredited person can help them mathematically.*

When the Australian Mathematical Society introduced its accreditation scheme in 1994 during my term as President, it was a controversial decision. The scheme was conservative, measuring worthiness for accreditation against performance levels of university academic staff. Three levels of accreditation were offered, and the Society website currently lists 108 persons who have been accredited as Fellows (the highest level). However, no lists of accredited members, or accredited graduate members are given. We have, it seems, failed to attract young mathematicians to accredited graduate membership, which is available to those who are graduates with a major in a mathematical science.

The Statistical Society of Australia (SSAI) introduced its accreditation scheme in 1996, and more recently decided to offer graduate accreditation status to those with a three year degree with a major in statistics. Unlike the Society, the SSAI provides a list[9] of all Accredited Statisticians and Graduate Statisticians, together with their contact details and professional areas of interest, thus helping to achieve the major purpose of accreditation. I question whether the accreditation scheme of the Society is sufficiently outwardly focused.

How is the Profession perceived?

As well as internal strength, the Mathematics Profession needs external recognition to ensure that:

- Young people see the relevance of a mathematical training for developing strong problem-solving skills and critical thinking, and the possibility of a variety of satisfying mathematical careers;
- Companies expect and obtain maximum, and cost-effective, benefits from incorporating mathematicians on their staff, or as consultants, to enable them to achieve their competitive edge; and
- Government comprehends the value of investing in mathematics education for its citizens.

How are we as a profession faring in terms of recognition? Some data indicate that Mathematics is facing a crisis, with decreased resources, splintering of the discipline, and dissipation of mathematics content in courses at all levels. In the first of these columns Peter Hall[10] analysed the negative impact on Mathematics and Statistics in Australian universities of government policies on research and higher education, principally the "penalising of highly performing Australian mathematical scientists" and lack of provision of "adequate career paths for younger Australian mathematical scientists". Documenting the decline in resources is important, but does not necessarily shed light on the causes.

If we, as members of the Mathematics Profession, assume that the only relevant issues are government decisions on support for Mathematics, then the cause of the decline is the government. But this might divert the Mathematics Profession from facing the possibility that much of the problem may lay within itself.

The most recent university mathematics enrolment data I have seen indicate that, in the Mathematical Sciences during the period 1995–99[11], the ratio of the number of graduates with majors in a mathematical

[9] http://www.statsoc.org.au/Accreditations/AccreditedMembers.htm

[10] AustMS Gazette **31** (2004), 6–11. Peter called for increased government funding for university mathematics teaching, structured research/teaching fellowships, and fundamental changes to government policy on the measurement of research performance.

[11] Information collected from Heads of Mathematics Departments in 2001.

science from Australian universities to the number who graduate with honours is more than 5:1. In past years the mathematical associations have largely ignored the former, and they account for 80% of the Profession. Unless the government sees a vigorous Mathematics Profession that acknowledges and engages *all* its members, we cannot expect it to regard Mathematics as important politically.

A way forward

All undergraduate students studying a mathematics subject should be of interest to the Mathematics Profession, not only because they are its clients, but also because they are students with a good mathematics education and we want to ensure that they understand the "unreasonable effectiveness"[12] of mathematics in solving real-world problems. Two simple initiatives the Society might take are:

- *To extend the offer of free student membership of the Australian Mathematical Society for the duration of a student's undergraduate career*[13]; and
- *To establish a Young Mathematicians section of the Society.*

However, from a student's perspective (and indeed the perspective of most members of the community), the "angles of separation" between academic mathematicians and statisticians, industrial and commercial mathematicians, and school mathematics teachers, are very small indeed. Students should be welcomed into the Mathematics Profession early in their undergraduate careers – before most of them distinguish (let alone choose) between the various mathematical career paths, not to mention alternative career paths such as the computing or physical sciences, or engineering.

An appropriate initiative to achieve this would involve:

- *A joint initiative by several mathematical bodies*[14] *to welcome and engage undergraduate mathematics students in the Mathematics Profession.*

Fortunately for this country, the Australian Mathematics Trust engages many thousands of Australian students in mathematics challenges and enrichment activities while they are in primary and secondary school. Equally fortunately, the Australian Mathematical Sciences Institute has a focus on school and undergraduate training in the mathematical sciences, and the professional development of mathematics teachers. The Trust's activities could become the first part of a seamless program of mathematics enrichment, promotion and information offered to young people by the Mathematics Profession.

Australia is a talented country. It is also a small country, too small to waste its mathematical resources. Success in building the Mathematics Profession demands the goodwill and commitment of all Australian Mathematicians (in the broadest sense of the word).

Acknowledgements

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[12] E. Wigner, *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*, Communications in Pure and Applied Mathematics **1** (1960).

[13] Currently one year's free membership is offered.

[14] Involving at least the Australian Mathematical Society, the Statistical Society of Australia, the Australian Association of Mathematics Teachers, the Australian Mathematics Trust, and the Australian Mathematical Sciences Institute.

Mathellaneous by Norman Do

The mathematics of voting

1 Introduction

In the animal kingdom, leaders are chosen by instinct, tradition, and occasionally the bumping of heads. We humans, on the other hand, have moved away from such primitive and barbaric behaviour. Democracy, from the same people who brought us the Olympic Games, necessitated a method to elect leaders in a way that would accurately reflect the will of the people. Thus, voting was born.

Having been practised for thousands upon thousands of years, voting is now rife in our society, whether it be used to opt between holiday destinations, choose a new member for a committee or elect a head of state. And what could be simpler? All it takes is for someone to count the votes and declare whichever candidate has the most to be the winner. Indeed, the subject of voting did not catch the eye of the mathematical world until the late eighteenth century when some clever people noticed that there were strange and paradoxical anomalies lurking behind this seemingly elementary process. Such simple observations gave rise to the marriage of mathematics with the field now known as social choice theory, which analyzes how collective decisions are made by a set of voters.

2 Two Alternatives

Voting between two alternatives is so simple that it can be seen being practised by young children in their playground politics. If more children want to play hide-and-seek than chasey, then it is generally understood that that is the game that they should play. This system, known for obvious reasons as *majority vote*, seems like the optimal way to choose a winner in order to please the most people and displease the fewest. But let us ask ourselves the following question...

> Are there any other ways, besides majority vote, to elect a winner from two alternatives?

What we are on the hunt for is some method of producing from an input of many preferences an output consisting of only one preference. This is an example of what is known in the literature as a *social choice function.* More precisely, a social choice function

- accepts as input a sequence of preference lists which strictly rank the elements of some set, and
- produces as output a winner or a list of tied winners from the set. The input sequence of preference lists is known as a *profile* and the output list of winners is called the *social choice.*

Of course, whether we are voting between hide-and-seek and chasey or between the lesser of two evil candidates for the leader of a nation is totally irrelevant. Also, in this simple case of two alternatives, ranking all of the alternatives is equivalent to choosing a preferred one. So, for convenience, we can give the two alternatives abstract names, such as $+1$ and -1, and a voter's preference list can simply be described as one of these two numbers. Furthermore, a profile can be given by an m-tuple of numbers which are either $+1$ or -1, where m corresponds to the number of voters. As long as there are no ties involved, our problem now translates into finding a function $f : \{-1, +1\}^m \to \{-1, +1\}$. But the number of such functions — which happens to be 2^{2^m} — is phenomenally large, even for reasonably small values of m. The following are just three simple examples of social choice functions for two alternatives.

- Dictatorship: Let one of the voters be the dictator and then let the social choice simply coincide with their preference.
- Constant: No matter how people vote, let the social choice always be $+1$.
- Parity: Let the social choice be $+1$ if an even number of people vote for $+1$ and let it be -1 otherwise.

It should be clear that all of these, although legitimate examples of the abstract notion of a social choice function, are preposterous attempts to accurately reflect the will of the people. The dictatorship does not treat all voters equally while the constant function does not treat both alternatives equally. The parity function allows a voter who changes his or her mind from -1 to $+1$ to make the social choice change in the reverse direction. So we need to put some conditions on our function to avoid these pathological examples. The following are three conditions which seem reasonable along with their translations into the more abstract language of mathematics.

- Anonymous: The social choice function should treat all voters equally.
  If $(x_1, x_2, \ldots, x_m)$ is a permutation of $(y_1, y_2, \ldots, y_m)$, then $f(x_1, x_2, \ldots, x_m) = f(y_1, y_2, \ldots, y_m)$.
- Neutral: The social choice function should treat both alternatives equally.
  For all $(x_1, x_2, \ldots, x_m)$, $f(-x_1, -x_2, \ldots, -x_m) = -f(x_1, x_2, \ldots, x_m)$.
- Monotone: Voting for someone cannot hurt their chances.
  If $x_k \geq y_k$ for all k, then $f(x_1, x_2, \ldots, x_m) \geq f(y_1, y_2, \ldots, y_m)$.

The natural question to ask now is ...

> Besides majority vote, are there any social choice functions for two alternatives which are anonymous, neutral and monotone?

If we restrict our interest to the case where there is an odd number of voters and no ties, then the answer is given in the following...

May's Theorem (1952): Suppose that we have a social choice function for two alternatives which

- has an odd number of voters;
- does not allow ties; and
- is anonymous, neutral and monotone.

Then the social choice function is a majority vote.

Problem: Prove May's Theorem by finding all functions $f : \{-1, +1\}^m \to \{-1, +1\}$ for odd m which are anonymous, neutral and monotone.

3 Three or More Alternatives

May's Theorem tells us that voting between two alternatives offers no surprises, so let us turn our attention to the more interesting case of three or more alternatives. Of course, one might be tempted to think that we are just about ready to conquer the world of voting, as long as we can generalize May's theorem to a larger number of alternatives. The most obvious way to do this is to simply take as the social choice the alternative whom most people think is the best. This method is known as *plurality voting* and is widely adopted, most notably in the United States presidential elections. Despite this fact, it is generally accepted in the social choice theory community that plurality voting is flawed. The following profile of voter preferences from a hypothetical election will give just one reason of many as to why this is so.

Suppose that there are fifteen voters and three alternatives. It may be the case that six of the voters prefer A to B to C, while five prefer C to B to A and the rest prefer B to C to A. This information can be conveniently captured in the following table.

6 voters	5 voters	4 voters
A	C	B
B	B	C
C	A	A

If this election were to use plurality voting as the social choice function, then it is clear that alternative A would be the winner. However, notice that A is the last choice for nine of the fifteen voters... does choosing A accurately reflect the will of the people? Most people would be inclined to think not and would perhaps even believe that A is the worst possible social choice. One might be tempted to think that such an anomaly is particular to this case and a handful of other concocted examples. But this is far from the truth and the flaws of plurality voting have been witnessed in many situations, most notably in the controversial US presidential race of 2000.

To avoid this flaw in plurality voting, it seems sensible to take advantage of the full preference lists of the voters, and voting systems which use this information are known as *preferential voting systems*. However, with this extra information at our disposal, there are far more options for us in determining a winner. Social choice functions abound and have been constructed by such voting conscious mathematicians as Charles Lutwidge Dodgson[15] and Edward John Nanson[16]. The following are three simple examples of social choice functions.

Borda Count: In order to overcome the deficiencies of plurality voting, Jean-Charles de Borda introduced in the late eighteenth century a voting system which would take advantage of each individual's intensity of preference for each alternative. He proposed assigning a number of points to each alternative, equal to the distance from the bottom of each voter's preference list. Thus, an alternative would receive 0 points for each last place vote, 1 point for each next-to-last place vote, all the way up to $n-1$ points for each first place vote, where n is the number of alternatives. The winner is, of course, the alternative that has been awarded the most points. You may be wondering what is so special about the numbers $0, 1, 2, 3, 4, \ldots$ that they should be the number of points assigned to the alternatives. Why

[15]Charles Lutwidge Dodgson (1832–1898) is better known by his pseudonym Lewis Carroll, under which he wrote the popular book "Alice's Adventures in Wonderland". Less well known is the fact that he was a mathematician and particularly interested in the mathematics of voting. In 1876, his exploration into voting led him to design a social choice function, but since then mathematicians have shown that the problem of calculating a winner by his method is NP-hard.

[16]Edward John Nanson (1850–1936) was a professor at The University Of Melbourne for 48 years. He was a strong advocate of preferential voting in the years leading up to the federation of Australia.

not use a sequence like $1, 2, 3, 5, 8, \ldots$ or $1, 2, 5, 14, 42, \ldots$? Such social choice functions which generalize the Borda count are known as *positional* voting systems.

The Hare System: Also known as single transferable vote, this voting system was introduced by Thomas Hare in 1861. The utilitarian philosopher John Stuart Mill described it as "among the greatest improvements yet made in the theory and practice of government" and it is currently in use to elect officials in Australia, Malta and Ireland. The idea behind the Hare system is that the winner should be voted in by a majority of the voters. But this is a rare occurrence when there are many alternatives to choose between, so what can we do? Simple — just delete some of the alternatives! More precisely, if any alternative is at the top of a majority of preference lists, then they are declared the winner. If not, then the alternative which appears at the top of the fewest preference lists is eliminated and the process is repeated. Of course, it may be the case that there is more than one alternative tied with the fewest number of votes and in that case, we can delete all of them. The process terminates when all remaining alternatives are at the top of the same number of preference lists. These remaining alternatives are the winners.

Dictatorship: This social choice function is one of the simplest to implement! One of the voters is assumed to be a dictator and the social choice is simply whoever is at the top of the dictator's preference list.

The following problem highlights the distinction between these different social choice functions. In particular, notice that in many elections, the person who wins can be heavily dependent on the type of voting system in use.

Problem: Consider the following simple profile with seven voters and four alternatives. Check that A wins using plurality, B wins using the Borda count, C wins using the Hare system and D wins using a dictatorship by voter 7. Who would you choose as the winner?

Voter 1	Voter 2	Voter 3	Voter 4	Voter 5	Voter 6	Voter 7
A	A	A	C	C	B	D
B	B	B	B	B	C	C
C	D	D	D	A	A	B
D	C	C	A	D	D	A

4 Condorcet's Voting Paradox

One of the first surprise results in voting was noticed by the Marquis de Condorcet, a contemporary of Borda, who considered the possibility of determining a social choice by using pairwise elections. He proposed that the alternative which would beat all others in a one-on-one majority vote should be the social choice. This alternative is known in the literature as the *Condorcet winner.* This voting system offers a natural generalization to majority vote which overcomes the flaws of plurality voting by using each individual's full list of preferences.

However, such a voting system is not without its flaws, as can be demonstrated by considering the following voting profile.

23 voters	17 voters	2 voters	10 voters	8 voters
A	B	B	C	C
B	C	A	A	B
C	A	C	B	A

Suppose now that candidate C dropped out of the running, leaving A and B in a head-to-head contest. Then using majority vote, as May's Theorem suggests we should, shows us that A would beat B by 33 votes to 27. In other words, a majority of voters prefer A to B, and we can encapsulate this statement in the convenient notation $A \succ B$. Now by comparing B and C, we find that $B \succ C$, winning by 42 votes to 18. It seems clear from this analysis that society prefers A to B and B to C, thereby making A the logical social choice.

But wait a minute... why did we neglect to compare alternative C with alternative A? Indeed, it turns out that in a direct comparison, C would have beaten A by 35 votes to 25. So we have the relations

$$A \succ B, \qquad B \succ C, \qquad C \succ A,$$

and it turns out that for this particular profile that there is no Condorcet winner. Common sense dictates that if a person prefers an apple to a banana and a banana to a cherry, then they should prefer an apple to a cherry. However, Condorcet's voting paradox tells us that society as a whole does not obey such common sense. If society prefers A to B and B to C, it may also occur that society prefers C to A. Of course, this argument relies on the assumption that society prefers one option to another if a majority of voters do. But what other ways are there in which to decide, given that we only have everyone's preference listing to deal with?

Condorcet himself realized the possibility for this type of intransitive behaviour to occur in the following much simpler example, which is now known as the Condorcet profile.

Voter 1	Voter 2	Voter 3
A	B	C
B	C	A
C	A	B

Even on this small scale, we have the relations $A \succ B$, $B \succ C$ and $C \succ A$, which are reminiscent of the famous children's game known as rock-paper-scissors.

Condorcet's Voting Paradox: There are particular profiles in which a social choice function must choose a particular alternative X as the winner, even though a majority of people prefer some other alternative Y.

This result from the late eighteenth century gave mathematicians a preview of the paradoxes that were in store for future social choice theorists. The upshot of it is that even though voting between a pair of alternatives is easy, pairwise voting falls apart when three or more alternatives are involved.

Problem: Show that for three voters and three alternatives, the probability that a Condorcet winner exists is $\frac{17}{18}$.

5 Arrow's Theorem

Thus far, our exploration of voting has only dealt with social choice functions, voting systems which turn a profile into a set of tied winners. In this section, we will be interested in more general voting systems which turn a profile into a ranking of the alternatives. More precisely, a *social welfare function*

- accepts as input a sequence of individual preference lists of some set, and
- produces as output a listing (perhaps with ties) of the set. This list is called the *social preference list*.

Social welfare functions are easy to find, and it turns out that we already have a few examples of them. For example, the Borda count can easily be converted from picking a winner to determining a social preference list by ordering the alternatives from highest to lowest depending on the number of points that they have received. The Hare system can also be converted into a social welfare function. To determine the set of people tied for first place is easy — just take the set of winners. To determine the set of people tied for second place, just delete the alternatives tied for first and repeat the Hare system. Iterating the procedure yields an ordered social preference list, perhaps with ties, as desired. This construction is not specific to the Hare system, but actually shows that every social choice function gives rise to a social welfare function. Of course, the reverse is also true, by obvious reasons. Note also that, in terms of social welfare functions, a dictatorship assumes that there is a voter who is a dictator and the social preference list coincides exactly with the dictator's preference list.

Now that we have the definition and some examples of a social welfare function, it seems natural for us to ask the question...

Can we find a *reasonable* social welfare function?

Of course, the answer to this question will depend on what exactly we mean by the imprecise and subjective term "reasonable". It is not overly difficult to find properties that almost everyone will agree are desirable in a social welfare function. One such property is illustrated in the following dialogue, which takes place in a dimly-lit fancy restaurant.

Waiter:	Good evening, madam. Where would you and your family like to sit — in the smoking section or the non-smoking section?
Madam:	I'd prefer non-smoking, but let me ask my family first... [turning to family] Where would you all like to sit?
Father:	I'd rather non-smoking.
Son:	Yeah, I hate the smoking section.
Daughter:	Me too!
Waiter:	Well, in that case, let me put you all in the smoking section!

Anyone in their right mind would find the waiter's decision to be incomprehensibly absurd. And it is surely reasonable for us to exclude from our consideration all social welfare functions which are incomprehensibly absurd. We do this by requiring the following property to be satisfied.

Unanimity: If every voter has the same preference list, then the social preference list should agree with it.

Another desirable property that we might impose on our social welfare function is illustrated in the following dialogue, which takes place in the very same dimly-lit fancy restaurant only minutes later.

Waiter:	Good evening, madam. Can I get you a drink to start off with?
Madam:	Yes, please. What kind of juices do you have?
Waiter:	We have apple juice and orange juice.
Madam:	OK, well I'll have the orange juice, thanks.
Waiter:	Excellent choice, madam! Oh, I just remembered... we also have cranberry juice available.
Madam:	You also have cranberry? Well in that case, I'll have the apple juice!

This time, it is the waiter's turn to be surprised by the lady's absolute irrationality. Why on earth would the presence of a third alternative alter which one of the first two is preferred? Again, it is surely reasonable for us to exclude from our consideration all social welfare functions which are absolutely irrational. We do this by requiring the following property to be satisfied.

> Independence of Irrelevant Alternatives: The relative positions of X and Y in the social choice should depend only on the relative positions of X and Y in the preference list of each voter. In other words, if every voter changes their preference list but decides to keep the relative positions of X and Y the same, then the social choice should keep the relative positions of X and Y the same.

Now we seem to have two desirable properties that any reasonable social welfare function should possess. Of course, we could go searching for more, but then we might be here all day without knowing when to stop. Anyway, it might pay to not be so greedy for the moment. So let us now ask the question...

> Can we find a social welfare function which satisfies unanimity and independence of irrelevant alternatives?

The quick-witted reader might already have noticed that there is an obvious, although undesirable, candidate for such a social welfare function — namely, a dictatorship. So for those quick-witted readers, let us rephrase our question just slightly.

> Can we find a social welfare function which satisfies unanimity and independence of irrelevant alternatives without being a dictatorship?

This question was answered by Kenneth Arrow in 1950 with a surprising and resounding, "No!"

Arrow's Theorem (1950): Suppose that we have a social welfare function which

- has at least three alternatives;
- satisfies unanimity; and
- satisfies independence of irrelevant alternatives.

Then the social welfare function is a dictatorship.

From our earlier discussion, we were led to the fact that any reasonable social welfare function should satisfy unanimity and independence of irrelevant alternatives, and probably a host of other desirable properties as well. But Arrow's Theorem tells us that even with these two obvious conditions, a dictatorship is unavoidable. In 1972, Kenneth Arrow was awarded the Nobel Prize in Economics for "pioneering contributions to general economic equilibrium theory and welfare theory".

Paul Samuelson, himself a Nobel laureate in Economics, put it this way...

> "The search of the great minds of recorded history for the perfect democracy, it turns out, is the search for a chimera, for logical self-contradiction. New scholars all over the world — in mathematics, politics, philosophy, and economics — are trying to salvage what can be salvaged from Arrow's devastating discovery that is to mathematical politics what Kurt Gödel's 1931 impossibility-of-proving-consistency theorem is to mathematical logic."

Problem: Suppose that we have a social welfare function which has at least three alternatives, satisfies unanimity and satisfies independence of irrelevant alternatives. Show that the social welfare function will never produce ties in the output (without using Arrow's Theorem, of course).

In light of the this fact, Arrow's Theorem boils down to the following problem of purely mathematical content. It is interesting to note that this problem and its solution may never have been uncovered by mathematicians were it not for democracy. The truly adventurous reader may like to try their hand at proving it before reading the next section.

Problem: Let $f : S_n^m \to S_n$ be a function which takes m-tuples of permutations to permutations and which satisfies the following two properties[a].

- For all $p \in S_n$, the function satisfies $f(p, p, \ldots, p) = p$.
- If $(p_1, p_2, \ldots, p_m)$ and $(q_1, q_2, \ldots, q_m)$ are m-tuples of permutations and the integers a and b satisfy

  $$\text{sign}[p_i(a) - p_i(b)] = \text{sign}[q_i(a) - q_i(b)]$$

  for all i, then

  $$\text{sign}[P(a) - P(b)] = \text{sign}[Q(a) - Q(b)]$$

  where $P = f(p_1, p_2, \ldots, p_m)$ and $Q = f(q_1, q_2, \ldots, q_m)$.

Then there exists an integer k such that for every m-tuple of permutations $(x_1, x_2, \ldots, x_m)$

$$f(x_1, x_2, \ldots, x_m) = x_k.$$

[a] As usual, S_n denotes the symmetric group on n elements and we will consider an element of S_n to be a bijection $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

6 A proof of Arrow's Theorem in five bite-sized pieces

We have now come to the meaty part of the exposition, into which much of the mathematical argument has been condensed. For easier digestion, the proof of Arrow's Theorem has been divided into five bite-sized pieces of steadily increasing size.

Bite-sized piece 1: If every voter ranks X over Y, then society ranks X over Y.

Consider the profile where every voter has the same preference list with X at the top and Y second from the top. By unanimity, society must also have X at the top and Y second from the top. But independence of irrelevant alternatives tells us that whether society ranks X over Y or not depends only on each voter's relative ranking of X and Y. In particular, if everyone were to rearrange their list while keeping X over Y, then the social choice should not change. So we conclude that for any profile where every voter ranks X over Y, society must also rank X over Y.

Bite-sized piece 2: If every voter ranks A at the top or bottom of their preference list, then society ranks A at the top or bottom.

Let us assume on the contrary that there is a profile where every voter ranks A at the top or bottom but society does not. Then there must be three distinct alternatives A, B and C such that society ranks B at least as high as A and A at least as high as C.

But consider now what happens if every voter moves their ranking of C to be just over B. Since A occupies an extremal position in every preference list, this does not disturb any voter's relative ranking between A and B. Nor does it disturb any voter's relative ranking between A and C. So society must continue to rank B at least as high as A and A at least as high as C. Hence, society must rank B at least as high as C. But as we have shown above, if everyone ranks C over B, society must also rank C over B. Since society cannot simultaneously rank B at least as high as C and C over B, we have the desired contradiction.

Bite-sized piece 3: There exists a profile at which A is at the bottom of the social ranking and a voter who can move A to the top of the social ranking by changing his or her preference list.

Consider a profile where every voter has the same preference list with A placed at the bottom. By unanimity, society must also place A at the bottom. Now consider what happens to the social ranking when each voter in turn moves A from the bottom of their ranking to the top. After this process has finished, A is at the top of every voter's ranking and hence, must be at the top of the social ranking as well. So there must have been some voter $V(A)$ whose change caused A to move from the bottom of the social ranking. Let profile I denote the profile just prior to $V(A)$ moving A from the bottom to the top and let profile II denote the profile just after.

1	2	$\cdots$	$V-1$	V	$V+1$	$\cdots$	N
A	A	$\cdots$	A	$\star$	$\star$	$\cdots$	$\star$
$\star$	$\star$	$\cdots$	$\star$	$\star$	$\star$	$\cdots$	$\star$
$\star$	$\star$	$\cdots$	$\star$	$\star$	$\star$	$\cdots$	$\star$
$\star$	$\star$	$\cdots$	$\star$	$\star$	$\star$	$\cdots$	$\star$
$\star$	$\star$	$\cdots$	$\star$	A	A	$\cdots$	A

Profile I

1	2	$\cdots$	$V-1$	V	$V+1$	$\cdots$	N
A	A	$\cdots$	A	A	$\star$	$\cdots$	$\star$
$\star$	$\star$	$\cdots$	$\star$	$\star$	$\star$	$\cdots$	$\star$
$\star$	$\star$	$\cdots$	$\star$	$\star$	$\star$	$\cdots$	$\star$
$\star$	$\star$	$\cdots$	$\star$	$\star$	$\star$	$\cdots$	$\star$
$\star$	$\star$	$\cdots$	$\star$	$\star$	A	$\cdots$	A

Profile II

Bite-sized piece 4: The voter $V(A)$ is a dictator over any pair B and C not including A. In other words, if $V(A)$ ranks B over C, then society ranks B over C, and if $V(A)$ ranks C over B, then society ranks C over B.

Let us consider profile III which is constructed from profile II in the following way.

- Let $V(A)$ move B and C to be just above and below A, so that his or her first three preferences are B, A and C.
- Let all other voters keep A in the same extremal position as in profile II but let them change the relative order of B and C however they please.

1	2	$\cdots$	$V-1$	V	$V+1$	$\cdots$	N
A	A	$\cdots$	A	B	C	$\cdots$	B
C	B	$\cdots$	C	A	B	$\cdots$	C
B	C	$\cdots$	B	C	$\star$	$\cdots$	$\star$
$\star$	$\star$	$\cdots$	$\star$	$\star$	$\star$	$\cdots$	$\star$
$\star$	$\star$	$\cdots$	$\star$	$\star$	$\star$	$\cdots$	$\star$
$\star$	$\star$	$\cdots$	$\star$	$\star$	A	$\cdots$	A

Profile III

Note that in profiles I and III every voter has the same relative ranking for A and B. So it must be the case that they yield the same relative ranking in their output, due to independence of irrelevant alternatives. And since the social ranking for profile I places B over A, the social ranking for profile III must also place B over A.

Also note that in profiles II and III, every voter has the same relative ranking for A and C. So it must be the case that they yield the same relative ranking in their output, due to independence of irrelevant alternatives. And since the social ranking for profile II places A over C, the social ranking for profile III must also place A over C.

So the social ranking for profile III places B over A and A over C, and hence, B over C. But remember that we let all voters other than $V(A)$ arrange their relative ranking of B and C arbitrarily. But by independence of irrelevant alternatives, it follows that whenever $V(A)$ places B over C, society must as well. And by a totally analogous argument, if $V(A)$ places C over B, society must as well. So $V(A)$ is a dictator over any pair B and C not including A.

Bite-sized piece 5: The voter $V(A)$ is a dictator over any pair A and B. In other words, if $V(A)$ ranks A over B, then society ranks A over B, and if $V(A)$ ranks B over A, then society ranks B over A.

Now one thing you might be wondering is what is so special about alternative A. Indeed, it is contemptible to play favourites with letters, so it may serve us well to consider what might have happened had we begun our construction with alternative B instead. Supposing we had done this, we would have found out that there exists some voter $V(B)$ who is a dictator over every pair which does not involve B. In particular, $V(B)$ is a dictator over the pair A and C which means that society's relative ranking of A and C must always agree with $V(B)$'s relative ranking of A and C.

But in profiles I and II we noted that $V(A)$ had the power to alter the relative rankings of A and C, while everyone else kept their preference lists the same. So it must be the case that $V(A)$ and $V(B)$ are one and the same person! Of course, there is nothing from stopping us running through the whole process with alternative C and consequently, we would have found out not only that $V(C)$ is a dictator over every pair which does not involve C, but also that $V(A)$, $V(B)$ and $V(C)$ are all one and the same person. It turns out that this particular one person is a dictator over every pair of alternatives and hence, the social welfare function in question must be a dictatorship.

7 Strategy-Proofness and the Gibbard-Satterthwaite Theorem

Consider an election where there are four voters wishing to choose a winner from a set of four alternatives. Suppose that their preferences for the alternatives are as shown in the profile below and that the social choice function to be used is the Borda count.

Voter 1	Voter 2	Voter 3	Voter 4
A	C	C	B
B	D	B	C
C	B	A	D
D	A	D	A

A quick tally of the votes reveals that C wins the election with a count of nine, closely followed by B with eight, then A with four and finally D with a measly three points. If you were voter 1, you probably couldn't help but feel chagrined by such an election outcome. So suppose now that you were crafty and devilish and instead of submitting your true preferences $ABCD$, you instead submitted the insincere preference list $BADC$. Then the resulting profile would have looked a little more like this.

Voter 1	Voter 2	Voter 3	Voter 4
B	C	C	B
A	D	B	C
D	B	A	D
C	A	D	A

And running through the Borda count for this profile shows that B would have won with nine points, followed closely by C with eight points, then D with four and finally A with a measly three points. This quick calculation highlights an important flaw of the Borda count: that a crafty and devilish voter can sometimes obtain a better outcome by voting insincerely!

A social choice function is said to be *strategy-proof* if there is no profile in which one of the voters can vote insincerely in order to obtain a better outcome. Unfortunately, this definition is somewhat incomplete, since it is difficult to define what exactly a better outcome for a voter is. Suppose, for example, that your true preference list is $ABCD$ and that voting sincerely produces A and D as tied winners, while voting insincerely might produce B and C as tied winners. Which of these two choices is a better outcome for you? If, for example, ties were to be broken by the flip of a coin, it might depend on whether you are a pessimist or an optimist. But instead of worrying about whether the glass is half-empty or half-full, let us just sweep the problem under the rug and focus exclusively on social choice functions in which there are no ties. In such cases, it is obvious which one of two outcomes is better for a particular voter.

Now there are several reasons why it is desirable to have a social choice function which is strategy-proof, such as the following.

(1) Insincere voting introduces an element of randomness into collective decisions.
(2) Unequal strategic skills amongst voters means that they will not be treated equally.
(3) Voters will waste resources in making strategic calculations.
(4) Voters are encouraged to conceal their preferences, reducing a flow of information that might aid in collective decision making.

So the obvious question to ask now is...

> Can we find a social choice function which has no ties and is strategy-proof?

An example of a strategy-proof social choice function is the constant one in which one particular alternative is always declared the winner, no matter how people vote. Of course, this is terribly unfair to the other candidates, so to avoid this pathological example, we might look for social choice functions which allow all alternatives to win. So the question we are trying to answer is now the following...

> Can we find a social choice function which has no ties, is strategy-proof and allows all alternatives to win?

A little bit of thought might suggest that a dictatorship satisfies all of these conditions, so let us refine our question just one more time...

> Can we find a social choice function which has no ties, is strategy-proof, allows all alternatives to win, but is not a dictatorship?

This question was answered independently by Allan Gibbard and Mark Satterthwaite with a surprising and resounding, "No!" Thus did mathematics deal another devastating blow to democracy.

Gibbard-Satterthwaite Theorem (1973): Suppose that we have a social choice function which

- has at least three alternatives;
- does not allow ties;
- allows all alternatives to win; and
- is strategy-proof.

Then the social choice function is a dictatorship.

8 The Future of Voting and Social Choice Theory

The mathematics of voting has been inspired by a trilogy of theorems that we have presented here — Condorcet's Voting Paradox, Arrow's Theorem and the Gibbard-Satterthwaite Theorem. It is surprising, though somewhat disheartening, that social choice theory seems to be based on a bunch of negative results. Does the mathematics suggest that we should give up on democracy? Should we appoint a dictator? Should we give up on society altogether, become hermits and eke out the rest of our lives as turnip farmers? Unless you happen to have a particular penchant for dictators or turnips, then such drastic measures are probably not required. Instead, these results should be taken as a warning that voting is not as simple as it seems and as a mathematical challenge to design and analyze more democratic voting procedures. They also highlight the power of mathematics to uncover beautiful structures and surprising paradoxes in areas where intuition tells us there should be none.

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Brain drain

There is growing concern about Australia's brain drain. In this last issue of the series of personal essays by mathematicians who went overseas, Federation Fellow Richard Brent talks about returning to Australia and reversing the brain drain.

A more positive note

Richard P. Brent

The first three essays in this series were by Australians who contributed to the 'brain drain' by moving overseas and who, for reasons explained in their essays, are unlikely to return soon, if at all. Here I will strike a more positive note by explaining why, after six years in Oxford, I am planning to return to Australia. Thus, although I may be counted in the statistics for the brain drains from both Australia and the UK, the net drain (or gain) will be zero. What follows is my personal story, and does not necessarily have any relevance to others.

To start at the beginning, my childhood was spent in a small country town in Gippsland. When I was eight my family moved to Melbourne. After completing secondary school there I enrolled for a BSc degree at Monash University. In those days (1964–1967) Monash was a small and new university, with many young and enthusiastic academic staff, some of whom had contributed to Australia's 'brain gain' by migrating from UK/Europe a few years earlier. I considered Melbourne University, but it seemed to have too much of a 19th century feel. That might not bother me now, since I am writing in Oxford, where the 19th century seems only yesterday. However, Monash appealed to me and turned out to be a good choice at the time. That was well before the period of cutbacks described in the first essay in this series.

At Monash I discovered that I had more interest in mathematics than physics or chemistry. Computer science was not an option then – the first professor (Chris Wallace) did not arrive until 1968. My interest in astronomy led to a vacation scholarship at Mt Stromlo Observatory, where I first learned something about computing on an IBM 1620 (an interesting machine, but that is another story). My computing skills proved useful when I returned to Monash, since I was able to perform some rudimentary computational group theory for Prof Janko's PhD students on a Ferranti Sirius computer. The results had to be written out by hand before Janko saw them, since he did not trust computers!

After graduating from Monash I decided to continue my studies overseas. I do not regret that decision, as my career would otherwise have been quite different and probably much less interesting. Thanks to the excellent teaching at Monash, I did well enough on the GRE that Stanford offered me a place in their Computer Science PhD program. I had applied to Computer Science rather than Mathematics because at that time computer science was a new and exciting field, and one in which I could use my mathematical ability. In fact, the Computer Science Department at Stanford was founded by a mathematician (George Forsythe) and computer science students were able to take several mathematics courses in their first year.

At that time CSIRO offered 'overseas studentships' that would pay Australian students to study overseas. Unfortunately, such studentships no longer exist, so it is more difficult for students nowadays. Thanks to CSIRO I was able to study full-time at Stanford and did not have to earn my living as a teaching assistant. Some of the Stanford professors who I particularly remember were Gene Golub and George Forsythe (my thesis advisors), George Pólya (then in his eighties, but ably assisted by Bob Tarjan who was in his twenties), Bob Floyd (whose take-home exam question inspired one of my first papers), and visitors such as Peter Henrici and Jim Wilkinson. Don Knuth arrived in Stanford in the same year (1968) that I did. Fortunately I became friendly with his secretary, Phyllis Winkler, who typed my thesis when she was not typing Don's books and papers. This was in the days before TEX, and a good mathematical typist was a precious commodity. (Following Wilkinson's excellent advice, my wife never admitted that she could type.)

I completed my PhD at Stanford rather too quickly – looking back, it might have been better to take advantage of the opportunities there for a few quarters longer. The reason for hurrying was that I had an offer of a lectureship in Computer Science at Monash. However, IBM's recruitment team was very persuasive, and paid for my wife and me to visit the IBM Research Center in Yorktown Heights, New York, to meet people there and see the beautiful location. Thus, at the last minute I decided to turn down the Monash offer and to take a post-doctoral position (officially 'practical training' since it was done on a student visa) in the Mathematical Sciences Department at IBM Research. It was a good decision, for it enabled me to get some industrial experience, to meet some of the 'East Coast' mathematics and computer science community (Goldstine, Rabin, Winograd, ...), and to revise my thesis and publish it as a book.

In 1972, after 18 months at Yorktown Heights, I decided that it was time to return to Australia. Bob Anderssen and Mike Osborne persuaded me to take up a Research Fellowship in the Computer Centre at the Australian National University. In those days the ANU made it easy for new staff from overseas by offering adequate removal expenses and excellent temporary housing.

My intention was to stay at ANU for three years, but as it turned out I stayed for 26 years. In that period my position (and office) changed many times. In 1978, when the Computer Centre was abolished in an administrative shuffle, I moved to the small Computer Science Department in the Faculties (then SGS, the part of ANU that did undergraduate teaching) to become the Foundation Professor of Computer Science.

In 1983–1985 I was on secondment to Neil Trudinger's 'Centre of Excellence', the *Centre for Mathematical Analysis*. That was great while it lasted, but eventually the money ran out. The government at the time apparently thought that such a Centre could become self-supporting; sadly that was not the case. Not wanting to revert to the role of Head of an undergraduate teaching department, and seeing the writing on the wall, I moved to the IAS (the other part of ANU) as its first Professor of Computer Sciences. This was initially in the Department of Engineering Physics under

Prof Kaneff (a pioneer of solar energy who was ahead of his time), and then in a separate Department, called the Computer Sciences Laboratory to distinguish it from the undergraduate teaching Computer Science Department.

I was never a member of the IAS Mathematics Department, but I came close. Kurt Mahler encouraged me to write some multiple-precision software in order to compute interesting transcendental numbers such as $\exp(\pi\sqrt{163}) \approx 262537412640768743.99999999999925$. For a while I occupied the office that had previously belonged to Bernhard Neumann and then John Coates, before John contributed to the brain drain by moving overseas and the IAS Mathematics Department moved to the other side of campus. My period under the influence of the ghosts of former occupants lasted only a few years: after another reorganisation I also moved to the other side of the campus, to the new Research School of Information Science and Engineering. This might have caused an identity crisis – was I a mathematician, computer scientist, or engineer? However, such distinctions did not bother me. It can be useful to have different hats for different occasions.

The first time that I contemplated joining the brain drain was in the late eighties, when John Dawkins, the Minister for Education at the time, was embarking on his 'reforms' of the Australian higher education system. Funding became tight and universities started to be run more by accountants and politicians than by academics. However, for personal reasons (two children at school, elderly relatives, etc.) it was difficult to move. It was only in 1997, after the children had left home, that an unexpected phone call inviting me to apply for a chair in Oxford made me realise that the time for a move was ripe.

Early in 1998 I took up the chair of Computing Science at the University of Oxford. Even though the move was unexpectedly difficult and it took some time for my wife and me to settle into our new life in Oxford, we now enjoy living in the UK, and especially enjoy the opportunity to explore Europe. Some pleasant things that I noticed when I arrived in Oxford were the better ratio of support staff to academic staff, and the lack of pressure to perform 'stunts' to get publicity and obtain funding.

Academically, Oxford is a stimulating place. The undergraduate students are excellent. There are distinguished colleagues in the department, both in the Programming Research Group, where my chair is officially located, and in the Numerical Analysis group (sometimes I wear an NA hat, since my thesis and some early publications were in that area). There are often interesting visitors passing through and giving seminars. The Computing Laboratory (Oxford's name for its Computer Science Department) is close to Physics, where there is a strong group working in quantum computing (a subject that I am interested in, if only because I do not believe in the 'hype' associated with it), and to the Mathematical Institute, where I have interests in common with number theorists such as Roger Heath-Brown and Bryan Birch. Thus, why would I want to leave Oxford? There are of course a few practical problems related to living in Oxford, such as high house prices (comparable to Sydney; but fortunately we were able to buy a house when we first arrived), and the climate (but it is not really that bad – a hot Canberra summer can be much worse than a wet Oxford winter). The complicated and devolved University and College system at Oxford makes it very difficult to change anything, so the undergraduate courses are often out of date, and the examination system is arcane, but perhaps these minor flaws add to Oxford's charm.

The UK, while not the same as Australia, has many ties to Australia, and living in the UK I feel much more 'at home' than I would in the USA. On the other hand, North American universities are, in my experience, more welcoming to newcomers. In the UK, and especially at Oxbridge, class distinctions still persist, and foreigners find it difficult to make friends amongst the natives. Thus, I understand why the first three authors of essays in this series decided to move to the USA rather than the UK, although I made a different choice, and I am not tempted to try the USA at present.

The University of Oxford is theoretically independent of government control, but in practice it is dependent on government funding, just like all major Australian universities. Thus, Oxford has to put up with various bureaucratic inconveniences imposed from above. A particularly irksome one is the Research Assessment Exercise (RAE), which rates the research done in departments and indirectly determines their level of funding. This is widely seen as divisive, biased against interdisciplinary or novel research, discouraging scholarship and teaching as they compete for time with research, and encouraging department heads to worry more about the ever-changing rules of the RAE than about encouraging genuine research. Certainly the RAE is time-consuming, expensive, and has capricious outcomes. Unfortunately, Australia has the habit of adopting fashions from overseas even as they are being recognised as failures where they originated. Thus, there are moves to introduce something like the RAE in Australia. I hope that this does not happen, because at present the lack of an RAE is one of Australia's advantages over the UK.

Although living happily in the UK, I feel some bond with the country of my birth, and would like to contribute to it by, for example, training some of the younger generation of Australian computer scientists and mathematicians. Also, of course, as one grows older it is best to live close to one's children. Thus, whenever someone suggested applying for a Federation Fellowship, as happened several times after I moved to Oxford, I would reply "yes, it's a good idea, but not just yet, as I would like to stay a few more years in Oxford". However, by 2003, I realised that it was 'now or never'. I would soon be too old and would either have to stay in Oxford until retiring age, or return to a less attractive position in Australia. Thus, I applied for a Federation Fellowship, and was lucky enough to be offered one. Once the formalities are completed (at the time of writing the formal contract remains to be signed), I expect to return to Australia for at least five years.

The Federation Fellowship will give me the opportunity to make a contribution to Australia by training graduate students and building up a research group that will, hopefully, continue to flourish after I retire. Of course, I also hope to do some research, insofar as someone of my age is capable of it. Failing that, I shall follow Hardy's advice and write some books. Returning to Australia for a Federation Fellowship is a much more attractive proposition than returning to a position as a Head of Department or other administrative position.

What can we conclude from this personal history? In my case, the Federation Fellowship scheme will (most likely) succeed in its aim of bringing Australians back home. However, for a younger person, such as the author of the previous essay in this series, applying for a Federation Fellowship might not be so attractive. There is the question of what happens at the end of the five-year Fellowship. It is not yet clear what ex-Federation Fellows will do – we may hope that the majority of them will stay in Australia, but quite likely many of them will start contributing to the brain drain. Another concern is that so few Federation Fellowships have been awarded to mathematicians, statisticians, or computer scientists. I do not know the reasons for this. However, I hope that my success will encourage others to apply in the future.

One problem with the Federation Fellowship scheme is that, by the time someone is well enough known and has a good enough track record to be offered such an attractive

Fellowship, he (or, in rare cases, she) will be old enough to have established strong ties to his/her present location, e.g. a spouse who can not easily change jobs, children at high school, etc. To bring back early- and mid-career academics it is necessary to improve overall working conditions in Australian Universities, and to improve morale in academic departments. This is not the place, and I am not the best person, to say how to achieve such aims, but a good start might be to take a hard look at the 'reforms' of the past two decades and decide which of them were ultimately harmful and should be reversed, if possible.

To conclude, I will continue to advise good Australian students and postdocs to go overseas for a few years, but remind them not to stay there too long, lest they find it impossible to return and regret that in their old age. Those in positions of influence in the Australian higher education system should aim to make it as attractive as possible for academics to return to Australia. This means help with relocation, housing, child care, the 'two body problem', travel funds, and generally improving conditions and morale in our universities. The aim should be to make our intellectual 'trade balance' positive in the long run. Inevitably some talented Australians will settle overseas and never return, but at least an equal number of talented immigrants should be attracted to take their place. Otherwise, Australia's intellectual capital will decline, and we will all be the poorer for it.

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Beyond disciplines? – A non-mathematician's apology

Ian Enting

The pronoun 'we' has caused me problems for at least a decade. Right now I am getting used to the fact that, for the first time in 23 years, 'we' no longer means CSIRO. At present I am working in MASCOS [1] with a brief that includes doing the sort of linking to end-users that Tony Dooley [2] regards as essential for the survival of the discipline of mathematics (research?). This article aims to expand on Tony's "if you want to live you will do this", drawing on my own experience in CSIRO and from watching the physics community grapple with similar issues.

In professional terms, I am not sure who should constitute 'we'. My preferred self-description is 'mathematical scientist' [3] as being more comprehensible (and more comprehensive) than 'biogeochemical modeller', not to mention 'ex-physicist'. Some details of my various roles are scattered through the endnotes.

One comment is that engagement is hard [4] – Tony Dooley commented on the 'massive investment of time needed to understand the insights and methods' [of each other]. It is essential to avoid what Maurice Kendall identified as:

Hiawatha, who at college
majored in applied statistics
consequently felt entitled
to instruct his fellow man on
any subject whatsoever [5].

The best model that I have seen for real engagement was the CSIRO Division of Mathematics and Statistics under Joe Gani, with mathematicians seconded part-time into CSIRO divisions which were oriented to disciplines and/or application areas. The aim (presumably) was to be 'time-effective' in that the mathematicians/statisticians would gain some familiarity with a field so that each new problem did not involve a major new learning experience [6]. DMS in this form was a casualty of the 1987 McKinsey review of CSIRO. For much of my subsequent CSIRO career, the CSIRO mathematics that I might wish to engage with were split into two groups. I was forbidden to talk to group A; group B were forbidden to talk to me [7] – in each case the issue was intra-CSIRO budget allocation [8]. This is, of course, the CSIRO equivalent of the territorial take-overs in universities that have mathematics service courses increasingly taught by 'user' departments, not by mathematics departments [9]. In some areas – e.g. theoretical physics and dynamical meteorology, autonomous departments exist whose mathematical research is at sufficiently high level that researchers move between such departments and mathematics departments. This is, I believe, a 'good thing' [10] in its way, but it pre-supposes a large critical mass of research, and so has limited applicability as a model for engagement across the whole spectrum of areas of application of mathematics.

If I select four points on the mathematical research continuum, described by Tony Dooley as "the mysterious process between theory and applications", I can talk about four roles: (1) proving theorems (2) devising ways of calculating things (3) calculating things about the real world (4) 'marketing' the results of such calculations [11]. The 'research' component of role (4) is going to be mainly researching the real and perceived needs of the end-users [12].

My own career has spanned (2) to (4) [13]. I hang around with people who prove theorems [14] – in fact I hang around such people enough to have acquired an Erdös number of 2, but my only two papers published in mathematics journals were in 1979 and 1992 [15], hence my reluctance to describe myself as a mathematician. This raises the question of whether – putting aside such special circumstances as MASCOS [16] – could/would/should your mathematics department employ me? [17]. Probably not, for a bunch of very good reasons, starting with the small number of courses that I would be equipped to teach, but this 'gap' illustrates one of the barriers to engagement.

Most of my working career has involved studying the carbon cycle. Apart from its considerable importance, the fun aspect of carbon cycle studies is that it is an excuse for getting involved in almost every field of science (with a need to forge cross-disciplinary partnerships, avoiding the Hiawatha syndrome). Mathematics is just one of the areas where I need to learn new things from time to time, and, to use Tony Dooley's example, often getting what I need from a 30-year-old textbook is easier than the phone (or e-mail or the internet).

Is this a bad thing? The nature of mathematics means that mathematics that was valid 30 years ago is valid today, computer algorithms being an important exception. Do mathematicians want their role to be as support for people who are too lazy to walk to the library?

An important professional support for cross-disciplinary carbon cycle study comes through being a member of the American Geophysical Union (AGU). The AGU includes sections on Atmospheric Sciences, Biogeosciences, Geodesy, Geomagnetism & Paleomagnetism, Hydrology, Ocean Sciences, Planetary Sciences, Seismology [18], Space Physics, Tectonophysics, and finally, Vulcanology, Geochemistry & Petrology. However, the AGU itself, along with the The American Physics Society, The Optical Society of America, The Acoustical Society of America, American Association of Physics Teachers, America Crystallographic Society, American Astronomical Society, American Association of Physicists in Medicine and AVS comes under the umbrella of the American Institute of Physics. What I lack is correspondingly comprehensive links across the mathematical sciences community.

Philip Broadbridge's 'brain drain' article [19] notes officers of Australian professional associations who: 'recommend old mates for medals [20]' and 'denigrate any activity of mathematical science that is more than two journal pages way from their own beloved paradigms'. This is a rather harsh description, but in comparison to the American situation, Australian professional societies really do seem excessively parochial and fragmented [21]. In contrast, a more plausible analysis of how one might operate effectively in a relatively small (and geographically-isolated) nation would suggest a need for less rather than more fragmentation.

Part of this fragmentation reflects a situation where professional scientific societies reflect the divisions enshrined in university departments and, are of less direct relevance to those whose science cuts across traditional discipline boundaries [22]. This is, of course, a self-sustaining condition. In this regard, a recent article in Physics Today [23] (of which more later) sees engagement with all their graduates as being essential for the long-term health of the (US) physics discipline.

Using Douglas Adams' "SEP" recipe for invisibility, much of this is 'Somebody Else's Problem' [24]. A better-linked arrangement of associations of natural scientists in Australia (a) won't happen any time soon; (b) won't do much for the corresponding problems for mathematics.

There is, however, one very important 'fragmentation' issue for mathematics: Probably the most commonly-applied field of mathematics is statistics [25]. (It is also, almost certainly, the most commonly mis-applied, but that is another story). In their names, both MASCOS and my department at The University of Melbourne (and formerly CSIRO) treat 'Mathematics' and 'Statistics' as two, presumably different, things [26]. From my own work [27], and a wide range of studies that I encountered through the CSIRO complex systems science initiative, I can report that, for much of the real world, this nice division between (applied) mathematics vs. statistics is not particularly helpful.

Returning to the Rigden and Stith *Physics Today* article [23], the vast majority of their words about physics could apply as well, or better, to mathematics. Some of the issues to which they attribute a decline in student numbers is:

- the invisibility of physicists [read mathematicians] in the workplace, since their high-level problem-solving skills take them into jobs [e.g. biogeochemical modeller] with non-physicist titles.
- the academic attitude that the only real physicists [again read mathematicians] are those who leave the department with a Ph.D.

Rather than inadequately summarise their proposed solutions, my main take-home message is to read the article if your care about these issues.

Two provocative recommendations:

- Australia needs an ANZIAM++, i.e. a group that covers the full range of applicable mathematics, not just what currently goes under the name of applied mathematics.
- Membership of ANZIAM++ should be available as a low-cost add-on for members of **any** other appropriate professional society in Australia (and New Zealand?), not just the Australian Mathematical Society.

Acknowledgement and identification of potential conflict of interest

Fifty percent of my salary at MASCOS is paid by CSIRO [16].

Endnotes

(1) MASCOS is the ARC Centre of Excellence for Mathematics and Statistics of Complex Systems, http://www.complex.org.au.

(2) Tony Dooley, *Math matters*, AustMS Gazette **31** (2004), 76.

(3) In Physics Today **57(5)** (2004), p. 10, http://www.aip.org/pt/vol-57/iss-5/p10.html, David Mermin describes the difference between mathematical physics and theoretical physics as: Theoretical physics is done by physicists who lack the necessary skills to do real experiments; mathematical physics is done by mathematicians who lack the necessary skills to do real mathematics. All I can do is plead guilty on both counts.

(4) Remember that this has to happen along with what Philip Broadbridge [19] identifies as the roles of researchers in the present climate: entrepreneur, administrator, PR expert, project manager, performer, stenographer and innovative teacher. He appears to have left out editor, graphic artist and IT manager. Of course in the mathematical sciences we have it easy – our list doesn't normally include the OHS responsibility for laboratories and/or field work – in the normal course of our work, no-one risks death if we foul up.

(5) This is really a minor aside in Kendall's parody of Longfellow's poem which seems to be about bias vs. efficiency. It is more relevant to issues of 'engagement' if one interprets it more generally as being about elegance vs. practical applicability. The parody, *Hiawatha designs an experiment*, can be found at a number of locations including http://www.ed.uiuc.edu/csg/documents/hiawatha.html.

(6) The direct consequence for me was M.L. Thompson, I.G. Enting, G.I. Pearman and P. Hyson, *Interannual variation of atmospheric CO_2 concentration*, J. Atmos.

Chem. **4** (1985), 125–155. The indirect consequences shaped much of my subsequent career in atmospheric science [27].

(7) Group A was the re-directed DMS who were to interact with the 'wealth-producing' half of CSIRO and could only interact with the other CSIRO divisions if they paid money. Group B were the Biometrics Units who were created to interact with the rest of CSIRO, apart from those divisions (including my own) whose chiefs refused to relinquish a position to create the units. Fortunately, CSIRO people like Bob Anderssen often had joint university roles and we were allowed to talk while they were wearing their non-CSIRO hats.

(8) I will have to leave it to others, and the passage of time, to assess whether the current 'One-CSIRO' slogan reflects a real improvement. Although I am probably biased on this matter, the CSIRO complex systems initiative (http://www.dar.csiro.au/css) shows great promise if it can survive amidst the prevailing micro-accountability.

(9) It has also been put to me, from the 'client side', that another reason for this loss of service teaching was that mathematics departments assigned such courses to the newest, most junior, least experienced faculty, or even to less qualified sessional teaching assistants. For mathematics, I have only hearsay – for physics, I've been there, done that.

(10) My view is that what matters most is the survival of mathematical research, not the survival of the name 'Department of Mathematics'.

(11) The people who devise techniques for calculations will, of course, draw on theorems to prove that their techniques will work. The other links along the spectrum are even more obvious.

(12) One of my most direct end-use involvements was the study for the inelegantly-named Subsidiary Body for Scientific and Technical Advice (SBSTA) for the Framework Convention on Climate Change (FCCC). This involved looking at a proposal, put forward by Brazil in the negotiations leading to the Kyoto Protocol, that emission reduction targets should be set on the basis of nations' relative blame for the greenhouse effect (see http://ms.unimelb.edu.au/~enting/brazil.html). However the suspicion remains that referring the issue to a scientific panel was the diplomats' alternative to doing something undiplomatic like telling the Brazilians to piss off. It was also presumably cost effective – scientists are much more likely than diplomats to be expected to fly economy class.

(13) One indication of serious engagement with the real world is when people try to suppress your work. For me that happened when representatives of US/Middle East oil interests tried to prevent the IPCC Radiative Forcing Report from referencing the CSIRO Atmospheric Research Technical Paper 31 on carbon cycle modelling results (republished at http://www.dar.csiro.au/publications/enting_2001a0.htm).

(14) My engagement was facilitated by having some powerful computational techniques that were of considerable interest to the Statistical Mechanics group at Melbourne – for a review see I. Enting *Series expansions from the finite lattice method* Nucl. Phys. B (proc. suppl.) **47** (1996), 180–197. This was a special circumstance which limits the applicability of my own experience as a role model for engagement. From my side, the motivations for engagement were (a) it's fun (b) the CSIRO constraints noted above [7]. CSIRO tolerated this statistical physics activity for many years, praised it highly when complex systems became trendy and used it as a basis for getting me out of the organisation when the budget got squeezed.

(15) D. Kim and I. G. Enting, *The limit of chromatic polynomials.* J. Combinatorial Theory B., **26** (1979), 327–336, *and* I. Enting, A. Guttmann, L. Richmond and N. Wormald, *Enumeration of almost-convex polygons*, Random Structures and Algorithms **3** (1992), 445–461.
(16) For which I am exceedingly grateful.
(17) My full publication list can be found at http://ms.unimelb.edu.au/~enting/ige_pubs.pdf.
(18) The paper by Rayner, Enting and Trudinger: *Optimizing the CO_2 observing network for constraining sources and sinks*, Tellus, **48B** (1996), 433–444 (and follow-on papers and contract work for CSIRO) would not have happened if I hadn't seen the poster version of the Hardt and Scherbaum paper (Geophys. J. Int., **117** (1994), 716) at an AGU meeting.
(19) Philip Broadbridge, *Brain drain: looking back from across the big pond*, AustMS Gazette **31** (2004), 89.
(20) Of course mutual medal-awarding is alive and well in the USA. See N.D. Mermin, *What's wrong with these prizes?*, Physics Today (1989), 9, reprinted in N.D. Mermin, *Boojums all the way through: Communicating science in a prosaic age* (CUP 1990).
(21) One minor, but pernicious, consequence of this fragmentation is the local persistence of claims of the form 'the science of the greenhouse effect is invalid because atmospheric scientists have not listened to geologists'. In North America or Europe, away from Australian parochialism, such claims would be more commonly treated with derision, as being contrary to the way science operates, without having to revisit specific examples starting with Högbom's input to Arhennius' 1896 calculations of global warming. The greenhouse position statement of the American **Geophysical** Union can be found at http://www.agu.org/sci_soc/policy/climate_change_position.html.
(22) As well as the AGU, I am a member of the Australian Mathematical Society (and ANZIAM) and the Australian Meteorological and Oceanographic Society and the TeX Users Group but have resigned from the Australian Computer Society, the Australian Institute of Physics and the SIGSAM group of the ACM. I could not justify the cost of so many separate memberships, let alone have time to actively contribute to these associations.
(23) J. Rigden and J. Stith, *The business of academic physics*, Physics Today, November 2003. Their 'business model' has the two outputs, "new knowledge" and "new physicists", produced from respective inputs of "existing knowledge" and "trainee physicists". They explore reasons, and possible remedies, for shortfall in the second input. As an AGU member, I get online (and hard copy) access to Physics Today. Legally, or otherwise, it seems that the article is available at other web sites. I commend it to you.
(24) Douglas Adams, *Life, the universe and everything* (Pan Books London and Sydney 1982).
(25) Note that Hiawatha majored in 'applied statistics', not 'Banach spaces'.
(26) Treating statistics as a particular application-oriented sub-branch of mathematics in the same way as meteorology or theoretical physics exacerbates the disjunction.
(27) I. Enting, *Inverse Problems in Atmospheric Constituent Transport* (CUP 2002).

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Mathematics and bodysurfing

Neville de Mestre

Abstract

Bodysurfing is an art that many people can enjoy, particularly in Australia where the ocean is relatively warm and waves break regularly near a sandy shoreline. In its purest form a bodysurfer catches a wave some distance from the water's edge by swimming onto it, just as it is about to break. The bodysurfer is then propelled by the broken surf front (breaker) towards the shoreline. Rides of 50 to 100 metres are normal for experienced bodysurfers.

This paper will discuss bodysurfing in general and consider simple mathematical models for catching a wave, riding a wave and falling off the wave at the end of the ride.

1 Introduction

No one knows who the first bodysurfers were! Captain Cook noticed South Sea Islanders frolicking in the surf on one of his trips to the Sandwich Islands, now the Hawaiian Islands [3]. Australian aborigines were observed to enjoy the surf near Caves Beach, Newcastle in the 19th century. Tommy Tanna from the Marshall Islands was working in Manly, Sydney in 1889 and began introducing white Australians to the skills of bodysurfing. But such activity was against the law, which at that time said that it was illegal to bathe in waters exposed to views from any wharf, street, public place or dwelling house between the hours of 6a.m. and 7p.m.

Neville de Mestre

Enter William Gocher, a Manly newspaper editor, in 1902. He announced that he would swim in the ocean at noon on various Sundays. Although arrested, he was told that no charges would be laid as long as he wore neck-to-knee bathers. The law was rescinded in November 1903, and surfing became a popular pastime [4].

Undoubtedly the best early bodysurfers were the Hawaiians in the 18th and 19th centuries. The Australians began at the start of the 20th century followed by the Californians, New Zealanders and South Africans around 1920. My interest in the subject arose because I was taught to bodysurf by my father in 1946, and then became a research mathematician specialising in fluid dynamics. As many readers know, I still compete at the Masters' level in the Australian Surf Life Saving Championships each year, and hence I find a scientific understanding of the skills involved in bodysurfing still helpful and intriguing.

2 Waves on the ocean

Waves breaking near the shore are generated by storms and wind at sea. The waves thus generated eventually move out from the storm area and travel as "ground swell" towards a distant shoreline. The average characteristics of a wave within this ground swell depend on the action time of the storm, the fetch (or distance from the storm centre) and the intensity of the storm or wind. In deep water the water motion set up by these waves as they pass can be modelled by the equations of inviscid, irrotational fluid dynamics.

The governing equation is Laplace's equation

$$\nabla^2\phi = 0$$

which for a 2-dimensional wave travelling at speed U on the surface can be solved by separation of the variables to yield

$$\phi = (U/k)e^{ky}\sin\{k(Ut - x)\}.$$

The $X(= x - Ut)$ axis and the y axis (vertically upwards) travel with the wave, while k is the wave number. Therefore, the water particle velocity components at any point on the surface or beneath the wave are

$$\dot{X} = Ue^{ky}\cos\{k(Ut - x)\}$$

$$\dot{y} = Ue^{ky}\sin\{k(Ut - x)\}.$$

Clearly $\dot{X}$ and $\dot{y}$ decrease rapidly as points deeper and deeper are considered. Thus submarines can dive beneath a disturbed ocean surface to avoid the pitching, rolling and yawing motions caused by surface waves.

When the wind blows on the surface of the ocean at more than 20 knots (approximately 17 ms^{-1}), "white caps" appear. These are generated by the wind altering the symmetrical ground swell into an asymmetrical wave with a steepened slope on the front section of the wave form. Eventually the slope reaches a critical value and the wave breaks. But these surf fronts soon diminish, and the wave reforms as a swell and travels on with reduced energy. It is difficult for a bodysurfer to ride these "white caps", because he or she would have to position themselves way out in the ocean exactly where the "white caps" break and they would also have to accelerate to wave speed. On the other hand, ships or craft have been known to surf the bigger white caps in deep ocean.

3 Waves near the shore

Besides the wind, there is another effect which causes waves to steepen on the front side and eventually break. This is the diminishing depth beneath the wave as the shoreline is approached. The equations governing the motion are still the Navier-Stokes equations for an incompressible, inviscid and irrotational fluid but the boundary condition on the bottom fluid/solid interface must now be included. Essentially, $\partial\phi/\partial n = 0$ on the bottom and the rate at which the bottom profile changes determines the type of breaking wave that occurs.

The analysis of a breaking wave has been carried out by many researchers. An excellent summary is given by Peregrine [8]. When the depth of the water beneath the wave reaches less than half the wavelength, the wave starts to change with its height increasing, its wave speed and wavelength decreasing, and its period remaining the same. Also, the wave crest at the surface gradually assumes a higher speed than the wave trough in front of it. The front or forward slope between this crest and trough becomes increasingly steeper, the crest eventually becomes unstable, and it spills over forming a breaker. Because this occurs near the shoreline, and within a regularly defined region, humans can take advantage of this physical phenomenon and ride the broken surf front (the breaker) to the shore.

The mathematical analysis for the formation of the breaker has only been obtained through an approximation to the Navier-Stokes equations, known as the Boussinesq shallow-water approximation [10, Section 13.11]. The relevant equations are

$$u_t + uu_x + g\eta_x = 0$$

within the wave, while the free-surface (water/air) boundary condition is

$$\eta_t + u\eta_x + \eta u_x = 0.$$

Here u denotes the horizontal velocity component in the x direction, t denotes the time, and $y = \eta(x,t)$ is the unknown wave profile. It was noticed by Stoker [9] that these are the same equations that determine the behaviour of a one-dimensional compressible gas flow with η as the variable density. The breaking of the wave is equivalent to the unstable behaviour of a compressible gas forming shock waves. These shallow-water equations are hyperbolic and solvable by the method of characteristics. One solution is the solitary wave [1]. Although the Boussinesq approximation breaks down as soon as the wave is about to break, this shallow-water theory has nevertheless proved useful in describing the behaviour of breaking ocean waves.

4 Catching a wave

Galvin (1972) [5] describes four types of breakers: spilling, plunging, collapsing and surging. Only two of these are useful for bodysurfing, namely spilling and plunging. For spilling waves, the bottom profile changes gradually and only the neighbourhood of the crest becomes unstable initially. The surf front of foam, bubbles and water starts to tumble down the front face of the sloping wave front. These rollers or gently breaking waves are ideal for bodysurfing, and they frequently occur near the high point of each tide cycle.

Plunging waves (or dumpers) overturn along the whole front of the wave with a jet of water plunging to the toe of the wave and trapping air inside the overturning tube. They occur if the depth changes quickly, particularly near the time for low tide. If refraction of the wave front occurs at the same time (for example, around headlands or along the edge of sandbars), the wave can be ridden obliquely. For surfboard riders, these are the “barrels” or “tubes” that they love to ride, but for bodysurfers, the sideways breaking of the wave along the wave front is usually too fast to enable the bodysurfer to stay level with the break, although swim fins (or flippers) can assist sometimes.

In order to ride the surf front, a bodysurfer has to accelerate up to wave speed, and float within the travelling turbulent surf front of air and water. The surfer’s motion is clearly governed by Newton’s laws. The vertical components have buoyancy balancing the surfer’s weight, while the horizontal motion shorewards has the force generated within the turbulent front and accompanying wave balanced almost by the drag on the surfer. There is a slight decrease in the surf front speed as the depth becomes shallower within the surf zone, but essentially, the surfer is carried along within the surf front at almost constant wave speed.

Beginning bodysurfers find that riding the surf front is not one of the main difficulties in bodysurfing, but catching a wave at the start or staying on the wave near its finish are the more difficult skills to acquire.

The problem in catching a wave is that most rideable waves travel at speeds faster than a person can swim, even faster than the best Olympic swimmer. When the wave has already broken and the surfer can stand in leg-deep water ahead of the wave, he or she can launch themselves up to wave speed by a huge thrust of their legs against the sandy bottom, at the same time propelling themselves into a horizontal position. Timing is important as the window of opportunity for catching the front of the wave is small. Too early, and the speed developed by the leg thrust is lost before the surf front reaches the surfer; too late, and the surfer falls off the back of the wave as it passes.

When the water depth is such that the surfer cannot stand on the bottom, it is almost impossible to catch a broken wave, although a few experienced and strong swimmers have sometimes been able to do this.

More frequently experienced bodysurfers swim out to where the waves are consistently breaking, and attempt to swim onto an unbroken wave just as it is about to break. Mathematically, this position can be determined by stipulating a limiting value on the steepness of the wave front for a train of periodic waves. However, studies have now indicated that it is better to consider each wave crest as an independent entity like a solitary or cnoidal wave rather than as part of a periodic wave train. Grimshaw [6] has provided a theoretical framework for a more general study of these waves.

For a bodysurfer to catch a wave in deep water, it is clear that he or she must be competent enough to accelerate quickly to wave speed. Timing is important once again, so that this must be accomplished just as the wave is passing. Some surfers can do this with one or two strokes only, if they position themselves correctly in the wave-breaking area.

A mathematical quantitative examination of this skill is of interest. For a person swimming in a pool or lake, the forces acting in the vertical direction are just gravity and buoyancy which balance each other, and hence physically explain the swimmer's ability to remain at the surface interface. In the horizontal direction the forces are the propulsive forward force P due to the swimmer's technique and a quadratic drag force.

The governing equation is therefore

$$m\ddot{x} = P - k\dot{x}^2$$

where m is the swimmer's mass, and the resistance coefficient is

$$k = \frac{1}{2}\rho A C_{\mathrm{D}}$$

with ρ as the water density, A the swimmer's area of cross-section normal to the direction of motion, and C_{D} as the drag coefficient.

The maximum speed that can be attained by the swimmer is therefore

$$\dot{x} = \sqrt{P/k}$$

occurring when $\ddot{x} = 0$. This value can only be improved by increasing P (technique training), decreasing A (weight loss) or decreasing C_{D} (special swimming suits).

Consider now a similar one-dimensional model of a wave travelling with speed $U(> \sqrt{P/k})$ towards a surfer. As the wave approaches its breaking point there is an increase in the water particle speeds near the crest and a strong shoreward surging force is experienced within the wave as it passes.

This force rises sharply from zero as the wave approaches, reaches a maximum at the crest, and dies quickly as the back of the wave passes. A simple model for it could be $F\cos[W(x - Ut)]$ where F is the maximum force at the crest on breaking and W is a wave property related to the field of influence $-\frac{\pi}{2} < W(x - Ut) < \frac{\pi}{2}$ of this force about the crest. Outside this region the generated force is zero until the next wave arrives. It is this force which enables the surfer to accelerate up to wave speed.

The dynamics of catching a wave within this limited window of opportunity are based on the forces shown in Figure 1.

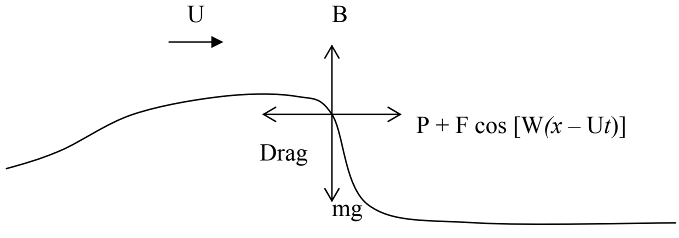

Figure 1

The horizontal forces for this simple model yield

$$m\ddot{x} = F\cos[W(x - Ut)] + P - k(\dot{x} - U)^2 \tag{1}$$

where $(\dot{x} - U)$ is the speed of the swimmer relative to the passing travelling wave. Using the co-ordinate transformation $X = x - Ut$, as before, equation (1) becomes

$$m\ddot{X} = F\cos(WX) + P - k\dot{X}^2.$$

Since $\ddot{X} = \frac{\mathrm{d}}{\mathrm{d}X}(\frac{1}{2}\dot{X}^2)$, this differential equation can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}X}(\dot{X}^2) + \frac{2k}{m}\dot{X}^2 = \frac{2F}{m}\cos(WX) + \frac{2P}{m}. \tag{2}$$

If the surfer starts to swim at the beginning of the field of influence of the wave force then

$$\dot{x} = 0, \quad \dot{X} = -U, \quad \text{when } t = 0, \quad WX = \frac{\pi}{2}. \tag{3}$$

Using the integrating factor $\exp(2kX/m)$, the solution of equation (2) with conditions (3) is

$$\dot{X}^2 = \frac{P}{k} + \frac{2F}{4k^2 + m^2W^2}\, 2k\cos(WX) + mW\sin(WX) \\ + U^2 - \frac{P}{k} - \frac{2FWm}{4k^2 + m^2W^2}\exp\left(\frac{k}{m}\left(\frac{\pi}{W} - 2X\right)\right). \tag{4}$$

Now the surfer will catch the wave when $\dot{X} = 0$ (i.e., $\dot{x} = U$), and this is only possible if there is a solution of the right hand side of equation (4) equal to zero in the range $0 < X < \pi/(2W)$, the front face of the wave.

Typical values for a surfer are $m = 75$ (kg), $k = 15$ (kg m^{-1}), $P = 15$ (N), while, for a wave about to break, typical values are $U = 3$ (ms^{-1}), $W = 2$ (m^{-1}), $F = 1000$ (N).

Equation (4) then particularises to

$$0 = 1 + [2.56\cos 2X + 12.8\sin 2X - 4.82]\exp(0.31 - 0.4X)$$

which has a solution 0.11 lying in the range $0 < X < 0.79$. In this case, the surfer will catch the wave.

Stronger swimmers have a higher value for P than weaker swimmers enabling them to catch some waves that others can't. Swim fins enable all swimmers to raise their P value.

5 Riding in the surf zone

Once a wave has been caught, the surfer is transported through the surf zone. The dynamics of this region have been surveyed by Battjes [2]. Basically, the surf front appears to be quasi-steady shortly after the initial breaking of the wave, with the turbulent surf front dissipating slowly as the wave travels towards the shoreline. The detail of the turbulent front is not well known yet, nor the dissipating mechanisms, but the turbulence seems independent of whether the wave is of the spilling or plunging type.

Of interest to the bodysurfer is how he or she falls off the wave as the surf front diminishes. In particular surfers can stay longer on the wave by kicking their legs, reducing their drag by streamlining the body with arms in a diving position, and by stroking with one arm. As the horizontal depth of the turbulent front decreases, there comes a point where the surfer's legs are no longer being carried along within the wave front even though the upper torso still is. The drag starts to increase dramatically and the wave force $F \cos[W(x - Ut)]$ is diminishing through a reduction in its maximum value F. Hence, the surfer decelerates and falls off the back of the wave.

6 Bodysurfing skills

There are many aspects of bodysurfing that have not yet been researched scientifically.

If an inert floating object, such as a log or floating surf craft, enters the surf zone, it moves with the surf front in a transverse orientation. Humans cannot do this, and they ride waves in the longitudinal orientation. A new toy has been developed which is a small-scale plastic model of a human on a surf mat with a keel at the back. This also rides waves in the longitudinal orientation. But humans do not have a built-in keel, so it is intriguing why they can ride waves in the longitudinal orientation and not turn sideways as all inert floating objects do.

Riding a wave is generally accomplished by bodysurfers with either their arms by their sides and their heads up so that they can see where they are going, or with their arms held in the diving position and their heads down. Variations include hydroplaning, where the arms are straight ahead and the palms of the hand form hydrofoils on the front slope of the wave. Hand boards have been developed to assist bodysurfers to do this.

The legs are usually held in the stiff prone position to enhance the streamline shape of the surfer, but some surfers in earlier days bent one leg at the knee, so that the foot was vertically above the knee. It is thought that this may have assisted the surfer on large turbulent surf fronts, but no evidence is available yet concerning this, and the practice seems to have died out. Additionally, no one yet seems to have investigated bodysurfing with both legs bent at the knees.

Another fascinating aspect of bodysurfing is that waves can be ridden by humans in the longitudinal orientation with their hands at the front (either face down or face up), but not with their feet first. Practical experiments conducted so far indicate that it is difficult to get up to wave speed in this orientation and, of course, the surfer finds it difficult to breathe because his or her face is deep within the turbulent surf front.

A useful skill is to practise riding obliquely across the face of a wave about to break. The lateral speed of the surfer can be increased by using swim fins, and these are used in bodysurfing events in Hawaii and California. The advantage of learning how to slide across the face of a wave is put to practical use when inadvertently catching a dumper (plunging wave). Surfers who try to catch these waves normal to the wave front may suffer severe back or neck injuries when they hit the shallow water in front of the dumper. Sliding across the face of these dumpers shoots the surfer out the back of the wave as it dumps its crest onto the shallow water below. The surfer is trapped inside the tube and suffers no injuries at all.

More advanced bodysurfing skills include “cork-screwing” down the face of a wave, “porpoising” through the face of a wave just before it breaks, and “piggy-backing” with two people on the same wave, usually with the lighter person on top.

There is much still to investigate about the scientific aspects of bodysurfing. The forces within a nearly breaking wave and within a broken surf front need further analysis. The length and depth of the turbulence within the moving surf front also needs closer scrutiny, although Longuet-Higgins and Turner [7] considered a model for this for a spilling breaker.

Of course, bodysurfers don’t need to understand the mathematics and physics of bodysurfing to enjoy the thrill of riding a wave shorewards. However, a more detailed knowledge of what is happening can enhance one’s ability by developing new skills, becoming more efficient in the various techniques, and perhaps finding new ways to enjoy the surf.

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Totally Goldbach numbers and related conjectures

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Goldbach's famous conjecture is that every even integer n greater than 2 is the sum of two primes; to date it has been verified for n up to 10^{17}; see [10, 13]. In order to establish the conjecture for a given even integer n, one optimistic approach is to simply choose a prime $p < n$, and check to see whether $n - p$ is prime. Of course, one has to make a sensible choice of p; if $n - 1$ is prime, one should not choose $p = n - 1$, and there is obviously no point choosing a prime p which is a factor of n. In this paper we examine the set of numbers n for which every "sensible choice" of p works:

Definition 1 A positive integer n is *totally Goldbach* if for all primes $p < n - 1$ with p not dividing n, we have that $n - p$ is prime. We denote by A the set of all totally Goldbach numbers.

It turns out that there are very few totally Goldbach numbers. We find:

$$A = \{1, 2, 3, 4, 5, 6, 8, 10, 12, 18, 24, 30\}.$$

At first sight, it would seem very plausible that A is a small finite set. As everyone knows, the primes tend to become rarer as one proceeds along the real line; if $\pi(n)$ denotes the number of primes no greater than n, then one expects $\pi(n) \leq 2\pi(n/2)$ for all $n \geq 6$. Indeed, this was conjectured by Landau and proved by Rosser and Schoenfeld [16]. For n to be a member of A we require as many "sensible" primes p with $p < n/2$ as there are primes p with $n/2 < p < n - 1$. So we would have $n \notin A$ if we could show that $\pi(n)$ is less than $2\pi(n/2)$ minus the number of prime divisors of n. The Prime Number Theorem tells us that the density of the primes falls off on average with $1/\log(n)$. So for big n, there will tend to be considerably more primes between 1 and $n/2$ than there are between $n/2$ and n; in fact, the difference is approximately $(2n \log 2)/(\log n)^2$. The number of prime divisors of n is more difficult to describe, but it grows much more slowly with n [14]. So we expect that large integers n will not belong to A. However, individual numbers seem to care little for expected "average" behaviour. Consistent with the falling frequency of prime numbers, Hardy and Littlewood conjectured (see for example [6]) that $\pi(x + y) \leq \pi(x) + \pi(y)$ for all sufficiently large x, y, but there are strongly held contrary views [19].

Before explaining how A can be determined, we first make some connections with three other closely related sets. Consider the set B of positive integers n such that every positive integer $r < n$ which is coprime to n is prime or 1. The members of B are called "very round numbers"; B appears as integer sequence A048597 in Sloane's Integer Sequences web site [17]. Obviously $B \subseteq A$. Knowing A, one finds easily that

$$B = A \backslash \{5, 10\}.$$

According to [12, p. 281], the composition of B was first determined by Schatunowsky (1893) and independently by Wolfskehl (1901). Apparently, it was also obtained by Bonse

This work was commenced while the first author, an undergraduate student at La Trobe University, was a vacation scholar funded by the Australian Mathematical Sciences Institute.

(1907); see [11] for an account of this proof, which is elementary, and makes use of "Bonse's inequality":

$$p_{n+1} < \sqrt{p_1 p_2 \cdots p_n},$$

where p_n denotes the n^{th} prime.

Another closely related set is $C = \{n \in \mathbb{N} : \varphi(n) \leq \tau(n)\}$, where φ is Euler's Totient function and τ is the divisor counting function. Using the simple formulae for φ and τ (see for instance [15, p. 19]), one finds that

$$C = A \backslash \{5\}.$$

This set appears as integer sequence A020490 in Sloane's Integer Sequences web site [17]. Despite the striking similarity between A and C, there is no obvious logical relation between the two sets; is their similarity merely a remarkable coincidence?

When examining Goldbach's conjecture, for a given integer n, it is common to study the number $g(n)$ of ways of representing n as the sum of two primes. Obviously $g(n)$ is less than or equal to the number of primes p with $n/2 \leq p < n-1$. Let the set D consist of those n for which $g(n)$ equals this maximum. Obviously $A \subseteq D$. In [4], Deshouillers, Granville, Narkiewicz and Pomerance showed that the maximum element of D is 210. It is easy to verify then that

$$D = A \cup \{7, 14, 16, 36, 42, 48, 60, 90, 210\}.$$

Of course, the determination of A is a simple consequence of the determination of D; one just checks the elements of D to see which are totally Goldbach.

In their 1993 paper [4], two strategies are given for finding the maximal element n_0 of D. The first strategy relies on the following simple idea: if one can find primes p, q with $n/2 \leq p < n-q$ such that $p \equiv n \pmod q$, then $n-p$ would be a multiple of q and $n-p > q$; in this case, $n-p$ would not be prime and so n could not belong to D. According to [4], using estimates for the number of primes $p \leq x$ with $p \equiv a \pmod q$, this strategy shows that D is finite and gives $n_0 \leq 10^{520}$. Unfortunately, this leaves too many cases to check, even by computer. Abandoning this approach, the authors of [4] then adopt a different strategy; using an argument involving sieve estimates, they obtain $n_0 \leq 2 \cdot 10^{24}$. Finally, using a computer to check the cases $n \leq 2 \cdot 10^{24}$, they arrive at $n_0 = 210$.

We will show that the first strategy of [4] is sufficient for the determination of our set A; i.e., A can be determined once one has the bound 10^{520}. Our motivation for doing this is two-fold. Firstly, since A is a simpler set, it is only fitting that it have a simpler determination. (This was the original motivation for this work). Secondly, and perhaps more importantly, we will see that this leads us naturally to interesting questions concerning primes in a fixed residue class.

We proceed as follows. First show that A has no element n with $30 < n \leq 2 \cdot 10^6$ by directly applying the definition; this is easily accomplished by computer. Then suppose that $n \in A$ and $n > 2 \cdot 10^6$. Obviously n must be even. Assume first that $n \equiv 1 \pmod 3$. If q is prime, $q < n-3$ and $q \equiv 1 \pmod 3$, then $n-q$ is divisible by 3 and hence not prime; as $n \in A$, we conclude that q is a factor of n. Thus

$$n \geq 2 \prod_{\substack{q \text{ prime} \\ q<n-3 \\ q\equiv 1 \bmod 3}} q \;\geq 2 \prod_{\substack{q \text{ prime} \\ q<2\cdot 10^6-3 \\ q\equiv 1 \bmod 3}} q \;\geq 10^{1000},$$

where the last calculation is performed by computer. Similarly, if $n \equiv 2 \pmod 3$, then n is at least twice the product of those primes $q < 2 \cdot 10^6 - 3$ for which $q \equiv 2 \pmod 3$. Once again, one finds that $n \geq 10^{1000}$. So it remains to consider the case where n is divisible by 3 (and hence 6). Arguing as above, for each $a \in \{1, 2, 3, 4\}$,

$$n \geq 6 \prod_{\substack{q \text{ prime} \\ q<n-5 \\ q\equiv a \bmod 5}} q \geq 6 \prod_{\substack{q \text{ prime} \\ q<2\cdot 10^6-5 \\ q\equiv a \bmod 5}} q,$$

for $n \equiv a \pmod 5$ and once again this gives $n \geq 10^{1000}$ in each case. So we may assume that n is divisible by 5 (and hence 30). Proceeding in this manner, we find that for each prime q up to the 351-st prime, 2371, and for each $a = 1, \ldots, 2370$, one has $n \geq 10^{1000}$ for $n \equiv a \pmod q$. So we may assume that n is divisible by the product of the first 351 primes; but this also gives $n > 10^{1000}$, as claimed.

In all, the various calculations took less than 24 hours running Maple 9 on a Pentium IV 2.4GHz; the calculations were verified in a little over 2 days, running Mathematica 4 on a Macintosh G3.

Notice that the above argument used the assumption that $n > 2 \cdot 10^6$ to show that $n \geq 10^{1000}$. A complete determination of A, without recourse to [4], would be obtained if the above method could be extended indefinitely and thus turned into an induction argument; that is, assuming that n is greater than some sufficiently large number K, one could try to use the above method to show that n is greater than some larger number K'. Loosely speaking, this approach would work providing the primes q, in any given residue class, are not too sparse. What this asks for is effectively a modular version of Euclid's theorem; recall that Euclid's proof of the infinitude of primes can be rephrased as follows:

$$p_{n+1} < p_1 \cdot p_2 \ldots p_n, \text{ for all } n \geq 2$$

where p_i is the i-th prime. This can be regarded as a weak version of Bonse's inequality [11], and a very weak version of Bertrand's postulate [1]. The simplest modular version of Euclid's theorem would be that for all primes q and for all $a = 1, 2, \ldots, q-1$,

$$r_{n+1} < r_1 \cdot r_2 \ldots r_n, \text{ for all } n \geq 2 \tag{1}$$

where r_i is the i-th prime that is congruent to $a \pmod q$. Unfortunately, this doesn't hold in general. For example, the primes congruent to $3 \pmod{13}$ are $3, 29, 107, \ldots,$ but $107 \not< 3 \times 29$, and the primes congruent to $5 \pmod{61}$ are $5, 127, 859, \ldots,$ but $859 \not< 5 \times 127$, etc. In fact, if the twin prime conjecture is true, there are infinitely many counterexamples to (1); indeed, if $q, q+2$ are twin primes, then the first two primes congruent to $2 \pmod q$ are $r_1 = 2, r_2 = q+2$, and since $2q+2$ is not prime, we must have $r_3 \geq 3q+2$. Hence $r_3 \geq r_1 r_2$.

Nevertheless, computer calculations do seem to show that the following is true for small values of q and n.

Conjecture 1 *For all primes q and for all $a = 1, 2, \ldots, q-1$,*

$$r_{n+1} < r_1.r_2 \ldots r_n, \text{ for all } n \geq 3$$

where r_i is the i-th prime that is congruent to a $\pmod q$.

We have been unable to find a statement to this effect in the literature. There are known modular versions of Bertrand's postulate (see [7, 18, 9]), but these results are typically of the form: "for sufficiently large n, ...", and moreover, they are usually not uniform in q and a. The above conjecture is certainly consistent with the prime number theorem modulo q, which gives the asymptotic behaviour of the number $\pi(x; q, a)$ of primes at most x which are congruent to a modulo q:

"*The expected asymptotic formula* $\pi(x; q, a) \sim x/\varphi(q)\log x$ *as* $x \to \infty$ *has long been known to hold but in all proofs given so far the dependence of the error term on the parameter* q *is rather poorly understood. For all we know it might even be the case that the asymptotic formula begins to represent the true state of affairs only after* x *is (almost) exponentially large compared to* q*.*" from John Friedlander's MathSciNet review of [2].

Notice that Conjecture 1 would follow by induction if we could prove:

Conjecture 2 *For all primes* q *and for all* $a = 1, 2, \ldots, q-1$,

(1) $r_4 < r_1 r_2 r_3$,
(2) $r_{n+1} < r_n^2$, *for all* $n \geq 4$,

where r_i *is the* i*-th prime that is congruent to* $a \pmod q$.

Computer calculations appear to support part (1) of Conjecture 2, and in fact for (2), they seem to indicate that $r_{n+1} < r_n^2$, for all $n \geq 3$. Notice that $r_n \geq (2n-3)q + a$. This is simply because the numbers congruent to $a \pmod q$ are:

$$a, q + a, 2q + a, 3q + a, \ldots$$

so if a is odd, the smallest possible r_i would be

$$a, 2q + a, 4q + a, 6q + a, \ldots$$

while if a is even, the smallest possible r_i would be

$$q + a, 3q + a, 5q + a, \ldots$$

if $a \neq 2$ and

$$a, q + a, 3q + a, \ldots$$

if $a = 2$. In each case, $r_n \geq (2n-3)q + a$. So, to establish the second part of Conjecture 2, it suffices to show that $r_{n+1} < (2n-3)^2 q^2$, or the somewhat stronger:

Conjecture 3 $r_n < 4(n-3)^2 q^2$*, for all* $n \geq 4$*, all primes* q *and all* $a = 1, \ldots, q-1$.

In fact, it is easy to see, using the same kind of elementary arguments used above, that Conjecture 3 also implies the first part of Conjecture 2. So we have

$$\text{Conjecture 3} \Rightarrow \text{Conjecture 2} \Rightarrow \text{Conjecture 1}.$$

Moreover, it is not difficult to show that by the Bombieri–Friedlander–Iwaniec Theorem [3], Conjecture 3 holds "with few exceptions". In fact, computer investigations indicate that the following may be true:

Conjecture 4 $r_n < (n + n\log n)q^2$*, for all* $n \geq 1$*, all primes* q *and all* $a = 1, \ldots, q-1$.

This conjecture is a generalization of an old conjecture of Schinzel and Sierpinski (see [12, p. 280 and p. 397]): $r_1 < q^2$ for all primes q and all $a = 1, \ldots, q-1$. At present the best result is Meng's improvement of Heath-Brown's version of Linnik's theorem [8]: $r_1 < q^{4.5}$. So Conjecture 4 may be a long way away.

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Ramanujan and Fermat's Last Theorem

Michael D. Hirschhorn

Hardy [1] relates the following anecdote. "I remember going to see him [Ramanujan] when he was lying ill at Putney. I had ridden in taxi–cab No. 1729, and remarked that the number ($7 \times 13 \times 19$) seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways.""

Indeed,

$$1729 = 9^3 + 10^3 = 12^3 + 1^3.$$

But there is another way in which this example is special. We know, since Euler, that the sum of two positive cubes is never a cube. But the above example shows that the sum of two positive cubes can do the next best thing – and that is, to miss a cube by as little as 1.

Indeed, Ramanujan left for us infinitely many examples of just that phenomenon. In his so-called "Lost Notebook" [4], he stated a result equivalent to the following.

If

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \\ 12 \end{pmatrix}, \quad \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 791 \\ 812 \\ 1010 \end{pmatrix}, \quad \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 65601 \\ 67402 \\ 83802 \end{pmatrix}$$

and

$$\begin{pmatrix} x_{n+3} \\ y_{n+3} \\ z_{n+3} \end{pmatrix} = 82 \begin{pmatrix} x_{n+2} \\ y_{n+2} \\ z_{n+2} \end{pmatrix} + 82 \begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} - \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} \tag{1}$$

then

$$x_n^3 + y_n^3 = z_n^3 + (-1)^{n+1}.$$

In two articles in the Mathematics Magazine [2], [3] I gave two proofs of this amazing statement, and gave an explanation as to how Ramanujan may have obtained this result. I will give a brief exposition below.

Recently, I was inspired to guess that the vectors $\mathbf{x}_n = (x_n, y_n, z_n)^T$ might satisfy a different type of recurrence. Let me explain. The continued fraction for $\sqrt{2}$ is

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

If we cut this off after the nth 2, we obtain a rational close to $\sqrt{2}$, which we call the nth convergent to $\sqrt{2}$, and which we denote by $\frac{p_n}{q_n}$. Thus

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

and

$$\begin{pmatrix} p_{n+2} \\ q_{n+2} \end{pmatrix} = 2 \begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} + \begin{pmatrix} p_n \\ q_n \end{pmatrix}.$$

But it is also true that

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix}$$

and so

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Inspired by this, I guessed that there may exist a 3×3 matrix M such that the vectors $\mathbf{x}_n = (x_n, y_n, z_n)^T$ given by (1) satisfy $\mathbf{x}_{n+1} = M\mathbf{x}_n$ and $\mathbf{x}_n = M^n \mathbf{x}_0$.
And indeed there is!

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \begin{pmatrix} 63 & 104 & -68 \\ 64 & 104 & -67 \\ 80 & 131 & -85 \end{pmatrix}^n \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}. \tag{2}$$

(Incidentally, we can use Ramanujan's recurrence backwards as well as forwards to obtain triples (x_n, y_n, z_n), just as we can have n negative in (2).)

This is how one might discover Ramanujan's solutions.
Suppose you notice that

$$(x^2 + 16x - 21)^3 + (2x^2 - 4x + 42)^3$$

is even.
It follows that

$$(x^2 + 16x - 21)^3 + (2x^2 - 4x + 42)^3 = (x^2 - 16x - 21)^3 + (2x^2 + 4x + 42)^3.$$

Replace x by $2x + 1$ and divide by 64 to obtain

$$(x^2 + 9x - 1)^3 + (2x^2 + 10)^3 = (x^2 - 7x - 9)^3 + (2x^2 + 4x + 12)^3.$$

Replace x by v/u, multiply through by u^6 and rearrange to obtain

$$(9u^2 + 7uv - v^2)^3 + (10u^2 + 2v^3)^3 = (12u^2 + 4uv + 2v^2)^3 + (u^2 - 9uv - v^2)^3.$$

Now comes the Ramanujan-esque touch. Set $u = h_n, \quad v = h_{n-1}$ where the sequence $\{h_n\}$ is defined by

$$h_0 = 0, \quad h_1 = 1, \quad h_{n+2} = 9h_{n+1} + h_n \quad \text{for } n \geq 0.$$

This forces

$$u^2 - 9uv - v^2 = (-1)^{n+1}$$

and if we set

$$x_n = 9u^2 + 7uv - v^2, \quad y_n = 10u^2 + 2v^3, \quad z_n = 12u^2 + 4uv + 2v^2$$

then

$$x_n^3 + y_n^3 = z_n^3 + (-1)^{n+1}.$$

These are Ramanujan's x_n, y_n and z_n.

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Visualising contingency table data

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Abstract

A geometric object, a simplex, is useful for picturing the joint, conditional and marginal distributions within a contingency table. The joint distribution is represented using weights on all vertices of the simplex, a conditional distribution by weights on vertices of a face of the simplex, and a marginal distribution by weights on the faces containing the conditional distributions. All detailed discussion is based on the simplest case, that of a two-by-two contingency table, for which all distributions are seen in a tetrahedron.

1 Introduction

A contingency table is a cross-tabulation of categorical variables. An example is given in Table 1, using data from an Australian survey of attitudes to genetic engineering of food [4]. The 894 respondents are distributed among four categories defined by income level and attitude to genetic engineering. The question of interest is whether income level and attitude to genetic engineering of food are dependent.

	Attitude	
Income	For	Against
Low	258	222
High	263	151

Table 1. A cross-tabulation of income level against acceptance of genetic engineering of food, with data drawn from a recent Australia-wide survey.

When faced with contingency table data, it is useful for the practitioner to have a quick method for visualising the associated distributions. The primary aim of this article is to bring such a method to a wider audience; the secondary aim is to provide a cameo example of the symbiosis between mathematics and statistics. The article exposits and builds on ideas first introduced by Fienberg [2] and Fienberg and Gilbert [3].

There are three distributional types associated with a contingency table: the joint distribution, conditional distributions and marginal distributions. This article pictures these three types in a simplex. For a given contingency table, the joint distribution can be represented by weights on all vertices of the simplex, a conditional distribution by weights on vertices of a face of the simplex, and a marginal distribution by weights on the faces containing the conditional distributions. All discussion is based on the contents of a two-by-two table, since such a table is complex enough to illustrate all items of interest yet simple enough to be readily pictured.

In the next section we review the three distributions, using notation of Agresti [1]. The three distributional types are described geometrically in Section 3, then the article is completed with a generalisation in Section 4 to tables of arbitrary dimension and a conclusion.

2 Distributions in a two-by-two table

We begin this section by briefly reviewing standard terminology and notation for joint, conditional and marginal distributions in a contingency table. Consider two categorical variables X_1 and X_2, each at two levels. The joint distribution of X_1 and X_2 can be represented in a 2×2 table denoted (π_{ij}), where π_{ij} is the probability of X_1 at the ith level and X_2 at the jth level, for $i = 1, 2$ and $j = 1, 2$.

The marginal distributions of X_1 and X_2 are denoted (π_{1+}, π_{2+}) and (π_{+1}, π_{+2}) respectively. Here the subscript "+" denotes summation over the associated index, so $\pi_{i+} = \sum_j \pi_{ij}$ and $\pi_{+j} = \sum_i \pi_{ij}$. Thus, the marginal distribution of X_1 (X_2) appears as the row (column) totals of the table (π_{ij}).

The distribution of X_2 conditional upon $X_1 = i$ is written as $(\pi_{1|i}, \pi_{2|i})$ so $\pi_{j|i} = \pi_{ij}/\pi_{i+}$ for all j. Symmetrically, we could define the distribution of X_1 for a given level of X_2.

These three distributions associated with a two-by-two table and a numerical example (the frequency table of the Australia survey data) are displayed in Table 2.

	X_2		
X_1	1	2	Total
1	π_{11}	π_{12}	π_{1+}
	$(\pi_{1\|1})$	$(\pi_{2\|1})$	
2	π_{21}	π_{22}	π_{2+}
	$(\pi_{1\|2})$	$(\pi_{2\|2})$	
Total	π_{+1}	π_{+2}	1.00

	Attitude		
Income	For	Against	Total
Low	0.2886	0.2483	0.5369
	(0.5375)	(0.4625)	
High	0.2942	0.1689	0.4631
	(0.6353)	(0.3647)	
Total	0.5828	0.4172	1.00

Table 2. The left panel presents the notation for joint, conditional and marginal distributions of categorical variables X_1 and X_2, each with two levels. The right panel presents the relative frequency table for the Australia survey data. Figures in brackets show the distribution of X_2 for the given level of X_1.

3 Geometry of the three distributions

The joint distribution of categorical variables X_1 and X_2 with two levels each can be represented as

$$(\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}) = \pi_{11}e_1 + \pi_{12}e_2 + \pi_{21}e_3 + \pi_{22}e_4$$

where $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$ and $e_4 = (0, 0, 0, 1)$ form the standard basis in $\mathbf{R}^4$ (points A, B, C and D respectively in Figure 1(a)). Thus the joint distribution of X_1 and X_2 can be pictured as weights π_{11}, π_{12}, π_{21} and π_{22} on A, B, C and D respectively.

Alternatively, since $\pi_{ij} \geq 0$ for all i, j and $\sum_{ij} \pi_{ij} = 1$, the joint distribution of X_1 and X_2 can be represented by the centre of mass J (more formally known as the "resultant" or "barycentre") of these weights on A, B, C and D in the three dimensional simplex given by

$$S_3 = \{(\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}) : \sum_{ij} \pi_{ij} = 1 \text{ and } \pi_{ij} \geq 0 \text{ for all } i, j\}$$

as illustrated in Figure 1(a).

The distribution of X_2 conditional on $X_1 = 1$ can be represented as $(\pi_{1|1}, \pi_{2|1}, 0, 0)$, an ordered 4-tuple in $\mathbf{R}^4$, and since we have the representation

$$C_1 = \pi_{1|1}e_1 + \pi_{2|1}e_2$$

evidently this distribution can be represented by weights $\pi_{1|1}$ and $\pi_{2|1}$ on A and B alone.

Alternatively, since $\pi_{j|1} \geq 0$ for all j with $\sum_j \pi_{j|1} = 1$, the distribution of X_2 conditional on $X_1 = 1$ is the resultant of these weights on A and B, so is a point C_1 in line segment AB. Similarly, the distribution of X_2 conditional on $X_1 = 2$ can be represented as $(0, 0, \pi_{1|2}, \pi_{2|2})$, so as a point C_2, the resultant of weights $\pi_{1|2}$ and $\pi_{2|2}$ on C and D respectively (illustrated in Figure 1(b)).

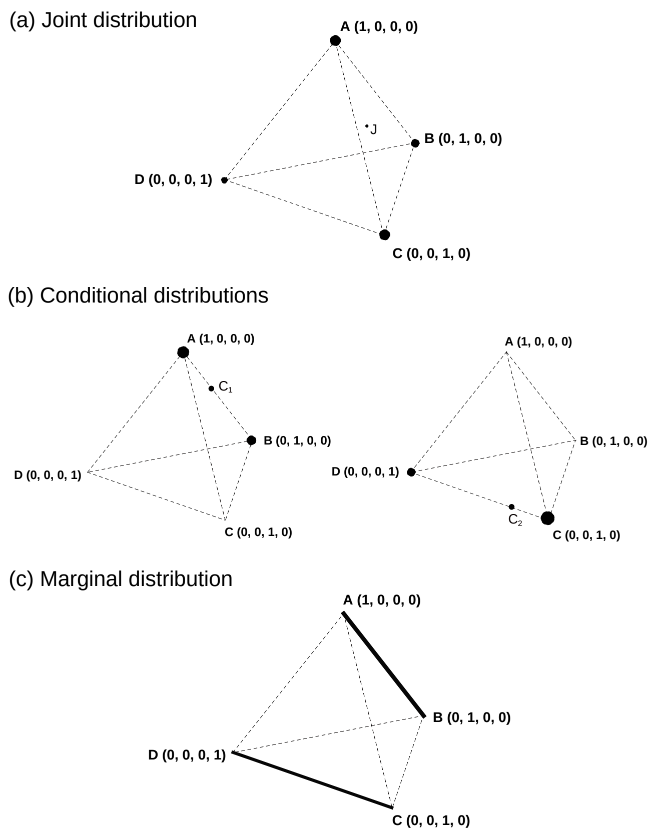

Figure 1. The three distributions of categorical variables X_1 and X_2, each with two levels. In (a) the joint distribution of X_1 and X_2 is seen as weights π_{11}, π_{12}, π_{21} and π_{22} on A, B, C and D, with resultant J. In (b) the conditional distribution of X_2 when $X_1 = 1$ is seen as weights $\pi_{1|1}$ and $\pi_{2|1}$ on A and B, having resultant C_1, while the the conditional distribution of X_2 when $X_1 = 2$ is weights $\pi_{1|2}$ and $\pi_{2|2}$ on C and D, having resultant C_2. In (c) the marginal distribution of X_1 is seen as weights π_{1+} and π_{2+} on edges AB and CD.

Joint distributions lying on AB oblige X_1 to equal one, so arguably line segment AB corresponds to $X_1 = 1$. Similarly, line segment CD corresponds to $X_1 = 2$. For this reason the marginal distribution of X_1, (π_{1+}, π_{2+}), can be represented as these weights on edges AB and CD, pictured by weighting these edges in Figure 1(c).

From the definition of conditional probability we have that

$$(\pi_{11},\ \pi_{12},\ \pi_{21},\ \pi_{22}) = \pi_{1+}(\pi_{1|1},\ \pi_{2|1},\ 0,\ 0) + \pi_{2+}(0,\ 0,\ \pi_{1|2},\ \pi_{2|2})$$

or

$$J = \pi_{1+}C_1 + \pi_{2+}C_2$$

In this special case where the joint distribution J and the conditional distributions C_1 and C_2 are known, the marginal distribution of X_1 can be represented as the weights π_{1+} and π_{2+} on C_1 and C_2 (still on AB and CD respectively) having resultant J.

Figure 1 in fact illustrates these ideas using the frequency table of the Australia survey data shown in the right panel of Table 2. Here we can represent the joint distribution of Income and Attitude as

$$(0.2886,\ 0.2483,\ 0.2942,\ 0.1689) \in \mathbf{R}^4$$

which corresponds to point J in the tetrahedron. The distributions of Attitude conditional on Income Low and Income High can be represented by $C_1 = (0.5375, 0.4625, 0, 0)$ and $C_2 = (0, 0, 0.6353, 0.3647)$ respectively. Since $J = 0.5369C_1 + 0.4631C_2$, the marginal distribution of Income, $(0.5369,\ 0.4631)$, can be specialized now as weights 0.5369 and 0.4631 on C_1 and C_2 having resultant J.

Fienberg and Gilbert [3] showed that the loci of all points corresponding to independence of rows and columns in a 2×2 table is a portion of a hyperbolic paraboloid in the tetrahedron, illustrated in Figure 2. In the figure, the point J (the joint distribution of Income and Attitude) is seen to be a small distance away from the independence surface; further analysis would confirm that, with a sample size as large as 894, this indicates dependence between Income and Attitude. Loosely speaking, for a given sample size the further J is from the independence surface, the greater the dependence between X_1 and X_2.

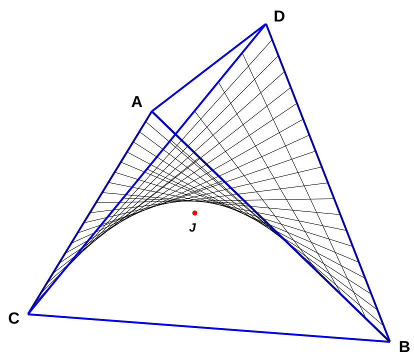

Figure 2. A graphic illustrating the locus of all points corresponding to independent 2×2 tables (a portion of a hyperbolic paraboloid) and the joint distribution J of Income and Attitude in the tetrahedron $ABCD$.

4 Tables of higher dimension

For a general contingency table, the three distributional types can be pictured in a higher dimensional simplex, having as many vertices as cells of the table. The joint distribution appears as weights on all vertices of the simplex. Conditioning on the levels of a subset of the variables partitions all vertices of the simplex; the convex hull of each partition set forms a face of the simplex. A distribution conditional on levels of the chosen variables appears as weights on vertices of the associated face. The marginal distribution of the random variables used for conditioning appears as weights on the simplicial faces determined by the partition sets. For example, for a 4×4 table with variables X_1 and X_2, the joint distribution is the weights on the sixteen vertices of the simplex S_{15}. To picture the distribution of X_2

conditional upon X_1, the vertices of S_{15} are partitioned into four sets of four using the levels of X_1. Four faces of S_{15} are then constructed as convex hulls of each set of vertices; the distribution of X_2 conditional upon a given level of X_1 is weights on the vertices of the associated face. The marginal distribution of X_1 is weights on the four faces. These ideas are illustrated in Figure 3.

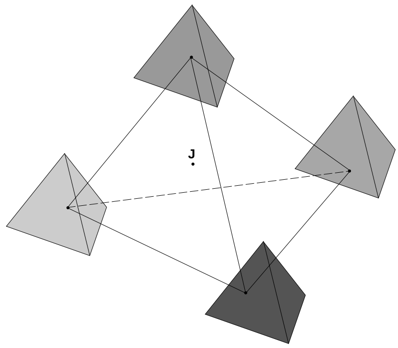

Figure 3. A schematic illustration showing that for a multi-way table the joint distribution J appears as weights on all vertices of a higher dimensional simplex; the resultant is a point in the simplex. Conditioning on values of a subset of all variables leads to a partitioning of the vertex set. Such a partition is shown as the four shaded simplexes. A conditional distribution is a weighting of the vertices of a partition set, for example, a weighting on the vertices of the upper shaded simplex. The associated marginal distribution of the subset of variables is the weighting of the facial simplexes formed by the partition, shown here using shading. The diagram presented here is strictly appropriate for a 4×4 table.

5 Conclusion

The three distributional types associated with a 2×2 table have been pictured in a tetrahedron. The joint distribution appears as weights on all vertices of the tetrahedron with resultant a point in the tetrahedron. A conditional distribution can be viewed as weights on vertices of an edge of the tetrahedron with resultant a point in the edge. A marginal distribution can be viewed as weights on the edges containing the conditional distributions. These ideas directly generalize to multi-way tables.

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Book reviews

Einstein's Heroes: Imagining the World Through the Language of Mathematics

Robyn Arianrhod
University of Queensland Press 2003
ISBN 0-7022-3408-7

Robyn Arianrhod's Einstein's Heroes is a popular-level book describing the work of some of the main intellectual influences on Einstein. The scientist that figures most prominently in the book is James Clerk Maxwell. Indeed, I think the book could also have been called something like: The Life and Ideas of Maxwell. But the book is more than just a biography: Arianrhod also has a larger aim in the book. Her aim, roughly speaking, is to explore the idea that there is "something special" about the language of mathematics that reveals hitherto inaccessible levels of reality to us. So Arianrhod is also examining broadly "philosophical" ideas in her book. Einstein's Heroes begins with a brief chapter in which the larger themes of the book are introduced. She starts with the story of a white child raised as an aboriginal, who had come to see and think about the world in a different way through learning to speak a different language. She then moves on to an account of Newton's main ideas. This takes up roughly the first quarter of the book. In the bulk of the book, Maxwell is the central figure. Arianrhod does not merely give us an exposition of his scientific ideas, but also describes his life and career, and his interactions with his contemporaries. The later chapters describe the influence that Maxwell's ideas had on Einstein, and on more recent developments. She also returns to the more general themes of language, representation and reality. Any popular book on science must avoid two potential dangers. On the one hand, it must avoid excessive dryness and technicality: that will only turn potential readers away. But on the other hand it must also avoid treating the material in such a loose, vague or merely metaphorical way that readers come away without any real understanding of the science discussed. Arianrhod has avoided both these dangers very well indeed. She explains concepts of mathematics and geometry, and many aspects of the ideas of Newton and Maxwell, with crystal clarity and a high level of rigour. But she is also careful to frequently leaven the more technical material with anecdotes about, for example, Maxwell's personal life and interesting asides in to the history of ideas. In these respects her book compares rather favourably, I think, with some other popular books about physics. This achievement is particularly impressive when one notes that the primary focus of her book – Maxwell's electromagnetism equations – do not on the face of it seem like promising material for a book of interest to the general reader. Despite all this, I do have one very minor quibble with the book. I'm afraid I found some of her claims about the nature of the relationship between language and the world, and about the special status of mathematical language, a little unconvincing. But this is a very minor blemish. If the way this book manages to combine clarity and rigour with "general interestingness" is any indication of how Arianrhod conducts her teaching, then her students at Monash are very fortunate indeed.

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Automatic Sequences: Theory, Applications, Generalizations

Jean-Paul Allouche and Jeffrey Shallit
Cambridge University Press 2003
ISBN 0-521-82332-3

Beautifully presented in a concise and scholarly manner, this book develops the fascinating theory of sequences generated by one of the most basic models of computation; namely, finite automata. Generalizations of such sequences, including Sturmian words and k-regular sequences, are also considered, and the strength of the theory is made evident through selected applications in number theory (in particular, formal power series and transcendence in positive characteristic), physics, and computer science. A topic such as this incorporates results from both mathematics and computer science, and consequently, papers on the subject are widespread in the literature, having been studied under different guises and with inconsistent notation. Allouche and Shallit, however, manage to successfully combine a myriad of concepts from a range of seemingly disparate disciplines to form a coherent and extremely informative resource for anyone from the professional researcher to the inquisitive undergraduate student.

Chapters 1 through to 5 provide us with the required background knowledge on stringology, number theory and algebra, numeration systems, finite automata, and automatic sequences. The book then delves into interesting generalizations of automatic sequences, such as the class of morphic sequences, of which automatic sequences form a sub-class. Other generalizations include characteristic words, multi-dimensional sequences, and sequences over infinite alphabets. Of particular interest to experts in this field are the relatively new results on transcendence of formal power series and automatic real numbers, given in Chapters 12 and 13. And the enthusiastic reader is sure to revel in the total of 460 exercises and 85 open problems, which, together with a very comprehensive list of references and bibliographical notes, certainly invoke the urge for further exploration.

Applicable to practically all areas of mathematics and computer science, this book is sure to become a much celebrated text on infinite sequences of symbols and their applications. A worthy addition to every mathematician's bookcase!

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Option Theory with Stochastic Analysis: An Introduction to Mathematical Finance

F.E. Benth
Springer Heidelberg 2004
ISBN 3-540-40502-X

The book under review was written as a text for an introductory level course (although we read in the Preface that it was used "in a course for students… preparing for a *master* in finance and insurance mathematics") on option theory (in continuous time and mostly within the Black–Scholes framework), with an objective to "relax the mathematical rigour to focus on ideas and techniques". There already exist several rather good books devoted to that topic and compiled with similar intentions. One could still argue, however, that financial mathematics students with various backgrounds might benefit from new approaches and simplified presentations, especially of the continuous time theory.

The main body of the text consists of five chapters: Introduction, Statistical Analysis of Data from the Stock Market, An Introduction to Stochastic Analysis, Pricing and Hedging of Contingent Claims, and Numerical Pricing and Hedging of Contingent Claims. Unfortunately, one can find rather serious deficiencies in all of them. The overall impression of the reviewer is that, despite some positive features (e.g. its small volume), the book is not well written and contains quite a few misleading and even wrong statements.

It would take too much space to give a detailed analysis of the exposition or to list all the deficiencies noticed by the reviewer in the text. The following small sample, however, could give you some flavour of what one can find there.

On p. 8, in the introduction to probability theory, we read: "We can find the expectation of a random variable X conditional on the event $A \subset \Omega$ as

$$\mathbf{E}\,[X|A] = \mathbf{E}\,[1_A X]."$$

First the reviewer decided that that must have been just a typo. However, after having seen (on p. 47) that "from Jensen's inequality (see [47, Thm. 19, p. 12]) it holds that

$$|\mathbf{E}\,[Z|\mathcal{F}_s]| \le \mathbf{E}\,[|Z|\,|\mathcal{F}_s] \le \mathbf{E}\,[|Z|],$$

which shows that the conditional expectation is finite under this moment condition", the reviewer is not so sure about that.

One could say a lot about the way the author introduces covariance/correlation and also makes claims about normally distributed random variables (*before* giving the definition of the univariate normal distribution — and that of the multivariate normal distribution he doesn't give at all). We just note that the author refers to

$$X \stackrel{d}{=} \mu + \sigma Y$$

as a "factorization (sic!) of a normal variable into a linear combination of a constant and a standard normal variable" (p. 9) and then talks about "powerful statistical distributions" (p. 11). And, referring to the sample mean and variance, the author says that "it is standard to use these two estimators for the *empirical* mean and variance of the sample".

On p. 36 we read that "the limit in (3.3) converges in variance, and thus for every $\omega \in \Omega$" (sic!). [In fact, there is a footnote commenting on the last statement, but it doesn't make things look much better.] On p. 39 we discover that, for any twice differentiable function, one can write down Taylor's expansion formula with a cubic remainder term. Furthermore, we learn on p. 41 that a semimartingale is a process that "can be decomposed into an Itô integral and a standard integral" (sic! and no further comments), and all this happens *prior* to the introduction of the notion of martingale (which is also done in quite a dangerous way).

From the author's discussion of completeness/incompleteness and arbitrage on pp. 86–91, we learn that "completeness comes from the fact that we need to be able to trade in every source of noise" and also that "the Lévy process introduces noise that cannot be traded", and also read that an $n \times m$ matrix is "non-singular exactly when $n = m$ (being quadratic)" (?!). Further, we also learn that "if we had completeness in the markets for derivatives, options and claims would not exist simply because they would be redundant. We could achieve exactly the same by entering into the claim's replicating portfolio". The last is as meaningful as the claim that, if the flour were available in all food stores, bakeries would become redundant as everybody could bake bread for themselves.

The above-mentioned examples of poor handling of even relatively basic material are typical for the book. The overall level and logic of exposition are scarcely any better.

Note in conclusion that the book appeared in the Springer *Universitext* series, that (unlike, say, the *Springer Undergraduate Mathematics Series*) apparently lacks an advisory board—at least, the reviewer failed to find anything in the book about who was responsible for selecting the text for publication.

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Mathematics for Finance: An Introduction to Financial Engineering

M. Capiński and T. Zastawniak
Springer Heidelberg 2003
ISBN 1-85233-330-8

As the authors modestly announce at the very beginning of the Preface, the book "is an excellent financial investment. For the price of one volume it teaches two Nobel Prize winning theories"—and these are the arbitrage-free pricing of derivative securities and Markowitz portfolio optimization (and the Capital Asset Pricing Model). There are eleven chapters in the book: Introduction; A Simple Market Model; Risk-Free Assets; Risky Assets; Discrete Time Market Models; Portfolio Management; Forward and Futures Contracts; Options: General Properties; Option Pricing; Financial Engineering; Variable Interest Rates; Stochastic Interest Rates. So one can see that the bulk of the text is devoted to the former theory, although, in the reviewer's opinion, the 35 pages devoted to portfolio management are quite instructive and constitute a valuable part of the book.

The level of exposition is pretty basic, with results for continuous time being mostly just outlined. That makes the book accessible to second year undergraduate students (and not only for students of mathematics, but hopefully also for students of business management, finance and economics). According to the authors, its contents could be covered in about 100 class hours. Prerequisites include elementary calculus (mostly used to find extrema of differentiable functions), some probability theory ("familiarity with the CLT would be a bonus") and elements of linear algebra (operations with matrices, solving systems of linear equations).

To keep the exposition at a low level, the authors had to state some key results (e.g. the Fundamental Theorem of Asset Pricing, even in the case of a simple discrete time market) without proving (or even properly explaining) them. This makes the task of (really) "understanding the underlying theory" pretty hard for the reader, but, on the other hand, with its rather extensive discussion of the general properties of options (Chapter 7) and a careful explanation of a large number of other important concepts, the book can still serve as a valuable introduction into the area.

There are a lot of (mostly numerical) simple examples and about 190 "doable" exercises dispersed throughout the book, solutions to all of the exercises occupying about 40 pages at the end of the text—which makes the book suitable for self-study. On the other hand, the reviewer has got an impression that the authors may have overdone it here (as the abundance of examples/exercises sometimes creates a situation where one can't see the forest for the trees).

Having said that, the overall impression of the book is quite positive. The reviewer can only congratulate the authors with successful completion of a difficult task of writing a useful textbook on a traditionally hard topic.

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Analysis on Lie Groups with Polynomial Growth

N. Dungey, A. ter Elst and D. Robinson
Progress in Mathematics Vol. 214
Springer Heidelberg 2003
ISBN 0-8176-3225-5

Consider the heat equation on $\mathbb{R}$, $\partial_t\varphi_t + H\varphi_t = 0$, where H denotes the Laplacian $\sum_{i=1}^{d}\partial_t^2$. One of the greatest achievements of Joseph Fourier was the solution of this equation to obtain $\varphi_t(x) = (G_t * \varphi_0)(x)$. Here, G is the Gaussian $(4\pi t)^{-d/2}e^{-|x|^2/(4t)}$. As is well-known, G_t can be considered as the semi-group kernel generated by the semi-group e^{tH}. Since $||G_t||_\infty = (4\pi t)^{-d/2}$, it follows that heat disperses at a rate proportional to $V(t)^{-1/2}$, where $V(t)$ is the volume of the ball of radius t.

Much effort has been expended over the past few decades to exploring these simple observations for Lie groups other than $\mathbb{R}$ and for operators other than the Laplacian. It has been understood for some time that Lie groups of polynomial growth i.e. groups where there is a G-invariant metric such that measure of the balls grows in a polynomial fashion, are the natural setting for generalisation of many of these theorems. Groups such as non-compact semisimple Lie groups, where the balls grow exponentially, require different methods.

Recent work, starting with the French school of Nicholas Varopoulos, Laurent Saloff-Coste and Thierry Coulhon, investigated the situation first for nilpotent groups, and later for more general solvable groups. Derek Robinson and his collaborators Tom ter Elst, Georgios Alexopoulos, and Nick Dungey, have made significant progress in recent years, and have given characterisations of groups where the kernels associated with general strongly elliptic second order operators satisfy Gaussian bounds, as well as giving estimates for Riesz kernels and other derivatives. This monograph is aimed at providing an up-to-date comprehensive survey of this work. It is a natural companion piece to Davies' *Heat kernels and spectral theory* (1989), to the book of Varopoulos, Saloff-Coste and Coulhon *Analysis and Geometry on Groups* (1992), or to Saloff-Coste's *Aspects of Sobolev-type inequalities* (2002).

After a brief introductory chapter, the book gives a careful outline of the theory of Lie groups, derivations, elliptic, subelliptic and strongly elliptic operators and their associated kernels. The important techniques of analysis which will be used in the text: the Carnot-Carathéodory metric, the method of transference and the de Giorgi estimates are introduced (although the proof of the latter is put off to an Appendix.) The point of the de Giorgi estimates is to use energy estimates on the stationary solutions of the heat operator associated with a subelliptic operator, in order to deduce Gaussian bounds.

Chapter III gives an analysis of the structure theory of solvable groups, introducing the nilshadow Q_N and the semi-direct shadow $G_N = Q_N \rtimes M$ associated with a solvable group G. It is proved that G can be realised as a quotient of a larger group $\tilde{G}$, whose nilshadow is a stratified group. For groups of polynomial growth, M is a compact group, and the strategy of proof is to use transference from the (stratified) nilpotent groups to get Gaussian bounds for G.

The next chapter contains, in some senses, the heart of the matter. Homogenisation theory in $\mathbb{R}^n$ is a classical method for treating subelliptic operators with oscillatory coefficients, estimating them by dilation followed by a form of averaging of the spectrum. Alexopoulos saw how to extend this to solvable groups, starting from a subelliptic operator H on G to which one associates a subelliptic operator $\hat{H}_0$ on $L^2(G_N)$: actually, this is constant in the M directions, and can be reduced to an operator $\hat{H}$ on Q_N. It turns out that $\hat{H}$ is a limit of dilates of H, and this enables one to establish Gaussian bounds, first for the nilshadow, and then for G itself. The fundamental equality is of the form

$$|K_t(g)| \leq Ct^{-D/2}e^{-b(|g|')^2/t}$$

where K_t is the kernel associated to the subelliptic operator H, c and d are constants, and $|\cdot|'$ is the Carnot-Carathéodory distance on G. This is an exact analogue of the results of Fourier quoted above! To a great extent, the methods of this chapter are based on work of Alexopoulos, who first obtained these results for sublaplacians on groups of polynomial growth, and then generalised them to solvable groups.

The stage is then set to extend this elegant theory. In Chapter V, the authors show how to obtain bounds like

$$|A^\alpha K_t(g)| \leq Ct^{-|\alpha|/2}V'(t)^{-1/2}e^{\omega t}e^{-b(|g|')^2/t}$$

where A^α is a derivative of order $|\alpha|$ of K_t. If $|\alpha| = 1$, the bound is optimal, although it can be improved to a Gaussian bound if G is near-nilpotent, i.e., a semi-direct product of a compact group and a nilpotent group. The authors prove their main structure theorem: Gaussian bounds hold for derivatives of K_t if and only if G is near-nilpotent. There are also other equivalent conditions, which I shall not detail here. In particular, this implies that it is not possible to get Gaussian bounds on the derivatives of K for a general solvable group.

The last chapter extends the theory to study the asymptotics of semigroups, using homogenisation theory and Gaussian bounds. The main theorems are recent work of Robinson, Dungey, Duong and other collaborators, in the basic direction of establishing a functional calculus for K_t. Again, the central idea is to reduce the proof to certain estimates on Q_N. The main theorems give L^p-multiplier bounds, both from above and below, on fractional powers of H. Again, there is a classification theorem: L^p bounds hold for derivatives of the semi-group S_t associated to H if and only of Gaussian bounds hold.

It is a good moment for this theory to be given a decent exposition, and who better than these authors to do it? The book contains a veritable wealth of examples, and a thorough exploration of the formidable array of analysis which has been assembled to attack these problems. It will be an invaluable research tool, and a wonderful textbook for anyone wishing to get a handle on the area.

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Graphs on Surfaces and Their Applications

S. Lando and A. Zvonkin
Encycl. Math. Sci. **141**
Springer Heidelberg 2004
ISBN 3-540-00203-0

This fascinating book is concerned with a modern approach to topological graph theory, with a particular focus on the numerous unexpected applications to, and interrelationships with, other fields of mathematics and also quantum physics, guiding the reader to the cutting edge of current research.

The basic objects of study in the book are: constellations, which are finite sequences of permutations; Riemann surfaces and their representations as ramified covering spaces of the two dimensional sphere; various classes of embedded graphs such as trees, cacti, etc.

There is a nice discussion of the combinatorial and geometric consequences of the profound Belyi theorem, relating Riemann surfaces that are defined over the algebraic numbers, to meromorphic functions having three critical values.

The method of matrix integrals turns out to have an unexpected relevance to the enumeration of graphs, and is the focus of a couple of chapters in the book. The method has its origins in certain matrix models of quantum physics, where the fields are matrix valued. It turns out that in such matrix models, the partition function is the generating function for certain classes of graphs. Some hints are given on methods of calculating these matrix integrals.

The book contains an account of the work of Harer and Zagier, which used the method of matrix integrals as a tool for computing the Euler characteristic of moduli spaces of algebraic curves. There is also a useful sketch of Kontsevich's deep proof of Witten's remarkable conjecture relating matrix models, the KdV hierarchy and the intersection theory of moduli spaces of algebraic curves.

The book also relates the enumeration of graphs to algebraic geometry and singularity theory via the Lyashko-Looijenga mapping. A beautiful extension of the Belyi theorem is discussed, involving an action of the braid group on constellations and the topological classification of meromorphic functions having four critical values.

The final chapter deals with the relationship of graphs to Vassiliev knot invariants and link invariants, via the structure of Hopf algebras on chord diagrams.

The book ends with a useful crash course on the representation theory of finite groups and their relevance to the enumeration of constellations, in the form of an appendix written by Don Zagier.

The book contains numerous diagrams, examples and exercises, making it appealing to both students and researchers.

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Chaos, A Mathematical Introduction

J. Banks, V. Dragan and A. Jones
Cambridge University Press 2003
ISBN 0-521-531047

The study of one-dimensional discrete dynamics, or first order difference equations, has shown that profound complexity can be derived from simple models. The field has matured considerably since pioneering work by Metropolis, Stein and Stein, May, Feigenbaum and others in the 1970s. One-dimensional dynamics is treated in most of the many books on dynamical systems, a classic example being [1]. Several books totally devoted to one-dimensional discrete dynamics also exist, including [2] and [3]. These two research monographs give an expert coverage of the extremely deep results in the field which have attracted the interest of many prominent mathematicians.

On the other hand, the beauty of one-dimensional dynamics is that some of its concepts and results are accessible at an elementary level. This explains why some coverage of the area is now standard in the undergraduate curriculum. Like many topics in dynamical systems, the excitement and significance of cutting-edge research results can be conveyed without an enormous amount of preparatory material. The present book fits into this category. It is pitched at the undergraduate level, for second and third year students, and is based on a course given at La Trobe University. As the authors state in the preface, chaos theory "uses many of the mathematical concepts and techniques from other parts of undergraduate mathematics". In my opinion, this is one of the great strengths of this book: it is a wonderful example of how first year calculus results can be employed to obtain nontrivial results in one-dimensional dynamics. It simultaneously reinforces an appreciation and understanding of the results themselves whilst teaching the student about the dynamics.

The first six chapters deal with standard material, introducing (periodic) orbits and their stability, cobweb diagrams and iteration. In the process, the ideas of limits, convergence and differentiability are reinforced. In particular, the treatment in Chapter 5 of stability of periodic orbits emphasizes how, around a periodic point, the map f^n is locally dominated by an affine map, the stability of which is treated separately in Chapter 4.

The unique strength of the book is revealed in its second half. In Chapter 7, the important concept of *wiggly iterates* for a one-dimensional mapping f is introduced. It means that f^n, $n \geq 1$, has 2^{n-1} humps with the base of each hump decreasing to 0 as $n \to \infty$. The case of the logistic map $x \mapsto 4x(1-x)$ is an example. The idea of wiggly iterates forms the basis of a significant discussion of the ingredients of chaos in Chapters 8 and 9. The three-faceted definition of chaos used here follows that of Devaney in [1] (although it should be noted that La Trobe mathematicians showed over a decade ago that one of these facets is redundant and implied by the other two). It is shown in Chapter 9 that wiggly iterates for a map implies sensitive dependence on initial conditions everywhere, transitivity and a dense set of periodic points. Conversely, the presence of any one of these properties implies a (symmetric) one-hump map has wiggly iterates.

Chapter 10 shows that the most significant way that f fails to have wiggly iterates is because it, or some power of it, has a *woggle.* A woggle is a fat wiggle (and so, in time, one may wonder if this term will also be appropriated to describe ageing children's entertainers). A sufficient condition to not have a woggle is for the so-called Schwarzian derivative $S(f)$ to be negative. This is a lovely Chapter which motivates the usually mysterious concept of $S(f)$ and also shows how it is preserved under composition. The mean value theorem is used extensively in the proofs.

Chapters 11 and 12 give a nice coverage of (topological) conjugacy, i.e. the idea of relating one map to another in terms of an invertible continuous coordinate transformation. This idea is very important in dynamics (and, of course, in mathematics more generally). It allows a choice of a canonical representative for the conjugacy class, or a normal form. To illustrate this, the authors prove that the tent map is representative of all one hump maps with wiggly iterates. Along the way in Chapter 12, the concepts of Cauchy sequences, completeness and uniform convergence are introduced and used.

Finally, in Chapter 13, the notion of an invariant set for a one-hump map is introduced. This allows extensions of the results of previous chapters to one-hump maps that expansively map the unit interval into a range bigger than the domain. Again, having wiggly iterates turns out to be a key concept as it implies that the largest invariant set for the one-hump map is a Cantor set and that the dynamics on this Cantor set is chaotic.

The authors should be commended for managing to steer a course through simple mathematics so as to recover some of the important results in one-dimensional dynamics. The book is well-written with a great set of illuminating and searching problems. Plus there are many great illustrations. The book is consistent with the pedagogical approach of learning mathematics through extensive problem-solving that is a hallmark of the La Trobe teaching method. I can see various uses for this book. The obvious one, consistent with its origins, is to form the basis of a stand-alone second or third year one-semester course in one-dimensional chaos. Parts of it could also be used to make nice learning modules that could be inserted into the advanced stream of a calculus course, nicely reinforcing those results with dynamical applications. Finally, for bright undergraduates who need to be extended, parts of it could be given as a reading course for independent study.

References

[1] R. Devaney, *An introduction to chaotic dynamical systems, 2nd edition*, (Addison-Wesley Redwood City 1989).

[2] L.S. Block and W.A. Coppel, *Dynamics in one dimension*, Lecture Notes in Mathematics 1513, (Springer Berlin 1992).

[3] W. de Melo and S. van Strien, *One-dimensional dynamics*, (Springer Berlin 1993).

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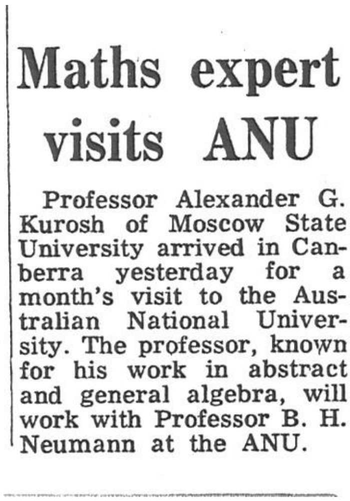
News

MathMedia

"Visiting scientist"

Back in 1964 it was big news when international scientists were visiting Australia. The following news item appeared in the very first edition of The Australian, which celebrated its 40th anniversary on the 1st of July this year. Imagine today to have your guests announced in a similar fashion.



Maths expert visits ANU

Professor Alexander G. Kurosh of Moscow State University arrived in Canberra yesterday for a month's visit to the Australian National University. The professor, known for his work in abstract and general algebra, will work with Professor B. H. Neumann at the ANU.

The Australian, 15 Jul 1964 (*Contributed by Kim Burgess*)

Completed PhDs

Deakin University:

- Dr Azmeri Khan, *Many-sample location and scale tests with quantile-function error distributions*, supervisor: Prof. Lynn Batten.

Macquarie University:

- Dr William B. Hart, *Evaluation of the Dedekind Eta function*, supervisor: Prof. Alf van der Poorten.

Murdoch University:

- Dr Alexandra Bremner, *Localised splitting criteria for classification and regression trees*, supervisor: Dr Ross Taplin.

University of Queensland:

- Dr Birgit Loch, *Surface fitting for the modelling of plant leaves*, supervisors: John Belward and Jim Hanan.

University of Western Australia:

- Dr John Bamberg, *Innately transitive groups*, supervisor: Prof. C. E. Praeger.
- Dr Devin John Kilminster, *Modelling dynamical systems via behaviour criteria*, supervisor: Dr Kevin Judd.
- Dr Maska Law, *Flocks, generalised quadrangles and translation planes from BLT-sets*, supervisor: Dr T. J. Penttila.
- Dr Sacha Daniel Roscoe, *Algorithms for detection of geometrical features*, supervisor: Prof. J. L. Noakes.
- Dr Tian Khoon Lim, *Edge-transitive homogeneous factorisations of complete graphs*, supervisors: Prof. C. E. Praeger, Dr C. H. Li.
- Dr Ricky O'Brien, *Modelling the transport and reaction of enzymes in germinating barley*, supervisors: Dr N. Fowkes, Dr S. Wang.
- Dr Tomomichi Nakamura, *Modelling nonlinear time series using selection methods and information criteria*, supervisors: Prof. A. Mees, Dr Kevin Judd.
- Dr Mahmoud El-Hirbawy, *Calculation of electromagnetic fields of power transmission lines using finite difference techniques*, supervisor: Dr Les Jennings.

New Books

R.Y. Rubinstein (Technion) and D.P. Kroese (University of Queensland), *The Cross-Entropy Method: A Unified Approach to Combinatorial Optimization, Monte Carlo Simulation, and Machine Learning*, (Springer-Verlag 2004), 320 pages, ISBN 0-387-21240-X.

A. Baddeley and E.B. Vedel Jensen, *Stereology for statisticians*, (Chapman & Hall/CRC, In press, late 2004).

Awards and other achievements

- Dr Hans Gottlieb of Griffith University has been awarded a DSc by the University of Melbourne for a thesis entitled "Studies in Vibrations and Related Phenomena".
- The book 'Einstein's Heroes' by Dr Robyn Arianrhod of Monash University has been shortlisted for the Age Book of the Year Prize in the Non-fiction section.
- Professor Adrian Baddeley of the University of Western Australia was presented the Pitman Medal for 2004 by the Statistical Society of Australia (SSA). The Pitman Medal is the highest honour that can be bestowed by the SSA, and it is awarded for achieving high distinction in Statistics which enhances the international standing of Australia in this discipline.
- Dr Akshay Venkatesh, who completed his honours degree at the University of Western Australia, has been awarded a Clay Research Fellowship.

Appointments

Curtin University:

- Professor Lou Caccetta's term as Head of the Department of Mathematics and Statistics has expired. The new head will be Professor K.L. Teo who will join the department in January 2005. During the interregnum Dr Peg-Foo Siew is acting Head.
- Mrs. Pam Hollis has resigned.

Macquarie University:

- Dr Frank Valckenborgh commenced on 21 June 2004 till the end of 2006 as MU Research Fellow, under supervision of Associate Professor John Corbett.
- Dr Sergey Panin will commence on 1 October 2004 as MU Research Fellow under supervision of Professor Paul Smith.
- Dr Thorsten Palm will commence in August 2004 as a Scott Russell Johnson Memorial Fellow with the Centre of Australian Category Theory.

Melbourne University:

- At the Department of Mathematics and Statistics, Dr Heng-Soon Gan has been appointed as Operations Research Consultant and Dr Meei Ng has been promoted to Senior Lecturer.
- At the Australian Mathematical Sciences Institute, Nancy Lane has been appointed as ICE-EM Manager, Thomas Montague has been appointed as Industry/Marketing Director, and Raoul Callaghan has been appointed as IT Manager.

University of Queensland:

- Drs Joseph Grotowski and Tony Roberts have accepted Lecturer C positions and commence in February 2005. Dr Grotowski is currently an Associate Professor at City University in New York while Dr Roberts is a Lecturer B at QUT in Brisbane.
- Dr Andrew Blinco has resigned his Lecturer A position to work as trainee actuary at SUNCORP.
- Dr Phil Isaac has been appointed as a Lecturer A until December 31, 2004.
- Dr Abdollah Khodkar has resigned his position at the University of Queensland to take up an Assistant Professorship in the Department of Mathematics State University of West Georgia, USA.

University of Western Australia:

- Dr Alice Niemeyer was promoted to Senior Lecturer in February.
- Dr Nazim Khan has been appointed as Lecturer/Consultant.
- Dr Maska Law has been appointed as a Research Associate.
- Dr John Bamberg has been appointed as a Research Fellow.

Introducing Mathematics-in-Industry Information Site

Cambridge University Press has announced the launch of a new website, Mathematics in Industry Information Site. MIIS is a joint venture of the Oxford Centre for Industrial and Applied Mathematics, The Smith Institute for Industrial Mathematics and System Engineering, and Cambridge University Press/European Journal of Applied Mathematics. It contains records of Study Groups, Workshop reports, interactive elements, a combination of preprint server, notice board, and help facility that will help mathematicians and scientists/engineers in Universities and those in industry. All this is free.

MIIS aims to be a window on what mathematics can do for industry and how industry can be a source of new ideas for mathematics. It is an on-line resource of choice for Industrial Mathematics. The ANZIAM MISGs have been invited to participate. Visit http://misg2005.massey.ac.nz.

Graeme Wake
Centre for Mathematics in Industry, Massey University
Director, MISG2005

Visiting mathematicians

Visitors are listed in the order of the last date of their visit and details of each visitor are presented in the following format: name of visitor; home institution; dates of visit; principal field of interest; principal host institution; contact for enquiries.

Etienne Ghys; ENS-Lyon; 18 September to 3 October; Geometry and Dynamical Systems; LTU; Grant Cairns

Marcel Nicolau; Universitat Autonoma de Barcelona; 18 September to 3 October; Differential Geometry; LTU; Grant Cairns

Prof. Tomaso Poggio; Massachussets Institute of Technology; 1 to 10 October; –; UWA; Prof. Lyle Noakes

Dr Soenke Blunck; University de Cergy-Pontoise; 1 September to 15 October; Analysis and Geometry; ANU; Prof. Alan McIntosh

Dr Max Neunhoeffer; University of Aachen; 4 to 30 October; –; UWA; Prof. Cheryl Praeger and Dr Alice Niemeyer

Dr Alan Camina; University of East Anglia; 4 to 31 October; –; UWA; Prof Cheryl Praeger

Prof. Liu; Oufu Normal University (China); July to October 2004; Functional Analysis and Partial Differential Equations; CUT; Dr Peg-Foo Siew and Dr Yong Hong Wu.

Prof. Ury Passy; Technion-Israel Institute of Technology; 1 August to 1 November; – ; UMB; Moshe Sniedovich

Prof. Samuel Muller; U Bern; 27 October to 1 November; Statistical science; ANU; Prof. Alan Welsh

Prof. Kirk Lancaster; Wichita State University; 18 October to 6 November 2004; Nonlinear and Applied Analysis Program; ANU; Dr Ben Andrews

Prof. H.W. Capel; University of Amsterdam; 1 October to 12 November; ARC CoE for Mathematics and Statistics of Complex Systems; LTU (Bundoora); Prof. R. Quispel

Prof. Denis White; University of Toledo; 3 to 24 November; Analysis and Geometry; ANU; Prof. Alan Carey

Mike Thorne; British Columbia Cancer Research Centre; July to November 2004; Combinatorics of Finite Sets, Bioinformatics; CDU; Dr Ian Roberts

Prof. Lincheng Zhao; University of Science & Technology of China; October to November; –; UWA, Dr Jiti Gao

Prof. Z.M. Guo; Donghua U; September to November 2004; –; UNE; –

Prof. Bill Lampe; University of Hawaii; 7 November to 11 December; Universal algebra, lattice theory; LTU; Dr Brian Davey

Prof. David Pike; Memorial University of Newfoundland; 15 October to 11 December; combinatorial designs and graph theory; UQL; Elizabeth Billington

Prof. Chengxiu Gao; Wuhan University; 15 September to 15 December; –; UMB; Dr Sanming Zhou

Dr Miroslav Haviar; M. Bel University, Slovakia; 18 October to 18 December; Universal algebra, duality theory; LTU; Dr Brian Davey

Prof. Chaiho Rim; Chonbuk National University; 1 December to 22 December; –; UMB; Paul Pearce

Dr Rachel Camina; DPMMS Centre for Mathematical Sciences, Cambridge; 17 August to 27 December 2004; –; UWA; Prof. Cheryl Praeger and Dr Alice Niemeyer

Dr Michael Levitan; USA; January to December 2004; – ; UWA

Prof. Dongsheng Tu; Queen's U; 1 February to 31 December; Statistical science; ANU; Profs. Peter Hall and Sue Wilson

Mr Bard Stove; University of Bergen; 4 October to 31 December; –; UWA; Dr Jiti Gao

Prof. Giuseppe Mussardo; SISSA, Trieste; 1 November 2004 to 11 January 2005; –; UMB; Paul Pearce

Prof. Gi-Sang Cheon; Daejin U, South Korea; 15 January 2004 to 15 January 2005; linear algebra and combinatorics; ANU; Dr Ian Wanless

Prof. Ralph Stohr; UMIST; 27 August 2004 to 27 January 2005; Algebra and Topology; ANU; Dr Laci Kovacs

Prof. Chaohua Dong; China; 1 February 2004 to 31 January 2005; – ; UWA

Prof. Karl Hofmann; Technische Universitat Darmstadt; 1 October 2004 to 31 January 2005; topological groups and semigroups; UB; Sidney A. Morris

Prof. Dale Rolfsen; University of British Columbia; 1 November 2004 to 31 January 2005; –; UMB; Hyam Rubinstein

Prof. Jean Bertoin; University Pierre et Marie Curie; 14 January 2005 to 14 February 2005; Stochastic Analysis; ANU; Prof. Ross Maller

Wahib Arroum; University of Southampton; 1 October 2004 to 31 March 2005; –; UMB; Owen Jones

Dr Alex Lindner; Technischen University; 16 January to 12 April 2005; Stochastic Analysis; ANU; Prof. Ross Maller

Prof. Akos Seress; Ohio State University; July 2004 to July 2005; –; UWA; Prof. Cheryl Praeger

Prof. Wen-Han Hwang; Feng Chia University; February to August 2005; Statistical Science; ANU; Prof. Richard Huggins

Kim Levy; Universite de Montreal; 11 August 2004 to 1 August 2005; –; UMB; Felisa Vazquez-Abad

Prof. Robert Lipster; Tel Aviv University; 1 October 2004 to 30 September 2005; Stochastic Processes; MNU; Prof. Fima Klebanar

Eloim Gutierrez; Universite de Montreal; 10 September 2004 to 1 December 2005; –; UMB; Felisa Vazquez-Abad

Dr Shenglin Zhou; Shantou University; October 2004 to October 2006; –; UWA; Prof. Cheryl Praeger

Conferences

The Centre of Excellence for Mathematics and Statistics of Complex Systems Workshop on Metapopulations

2 September, University of Queensland, Brisbane
Web: http://www.maths.uq.edu.au/~pkp/MetaPop04.html

Computational Techniques And Applications Conference: CTAC 2004

27 September – 1 October 2004, The University of Melbourne
Web: http://www.conferences.unimelb.edu.au/CTAC2004/

48th Australian Mathematical Society Annual Conference (2004)

28 September – 1 October 2004, RMIT, Melbourne
Web: http://www.ma.rmit.edu.au/austms04/

International Conference on Mathematical Inequalities and their Applications

6 – 8 December 2004, Victoria University, Melbourne
Web: http://rgmia.vu.edu.au/conference

6th International Conference on Optimization: Techniques and Applications

9 – 11 December 2004, University of Ballarat
Web: http://www.ballarat.edu.au/icota

2004 World Conference in Natural Resource Modelling

12 – 15 December 2004, RMIT, Melbourne
Web: http://www.ma.rmit.edu.au/2004RMAconference

The First International Workshop on Intelligent Finance (IWIF 1)

13 – 14 December 2004, Crown Promenade Hotel, Melbourne
Web: http://www.iwif.org

The 2004 NZIMA Conference in Combinatorics and its Applications and The 29th Australasian Conference in Combinatorial Mathematics and Combinatorial Computing (29ACCMCC) (Joint Conference)

13 – 18 December 2004, Lake Taupo, New Zealand
Web: http://www.nzima.auckland.ac.nz/combinatorics/conference.html

Geometry: Interactions with Algebra and Analysis

January – June 2005, Auckland
Web: http://www.math.auckland.ac.nz/Conferences/2005/geometry-program

Mathematics-in-Industry Study Group 2005

24 – 28 January 2005, Massey University at Albany, Auckland, New Zealand
Web: http://misg2005.massey.ac.nz

ANZIAM 2005: The annual ANZIAM Applied Mathematics Conference

30 January – 3 February 2005, Napier, New Zealand
Web: http://www.math.waikato.ac.nz/anziam05

The Annual ANZIAM Applied Mathematics Conference and Annual Meeting of ANZIAM for 2005 is sponsored by the Royal Society of New Zealand. The annual conference of ANZIAM is an established annual gathering of applied mathematicians, scientists and engineers with wide-ranging interests. It provides an interactive forum for presentation of results and discussions by students, academics and other researchers on applied and industrial problems derived in many scientific fields and amenable to quantitative description and solution.
The deadline for registration is December 1.

49th Annual Meeting of the Australian Mathematical Society

26 – 30 September 2005, The University of Western Australia
Director: Lyle Noakes
E-mail: lyle@maths.uwa.edu.au
Web: http://www.maths.uwa.edu.au/~austms05/index.html

AustMS bulletin board

AustMS Accreditation

The secretary has announced the accreditation of:

Dr Dirk Kroese, University of Queensland, as an Accredited Member (MAustMS).

Rules governing AustMS Grants for Special Interest Meetings

The Australian Mathematical Society sponsors Special Interest Meetings on specialist topics at diverse geographical locations around Australia. This activity is seen as a means of generating a stronger professional profile for the Society within the Australian mathematical community, and of stimulating better communication between mathematicians with similar interests who are scattered throughout the country.

These grants are intended for once-off meetings and not for regular meetings. Such meetings with a large student involvement are encouraged. If it is intended to hold regular meetings on a specific subject area, the organisers should consider forming a Special Interest Group of the Society. If there is widespread interest in a subject area, there is also the mechanism for forming a Division within the Society.

The rules governing the approval of grants are:

- **(a)** each Special Interest Meeting must be clearly advertised as an activity supported by the Australian Mathematical Society;
- **(b)** the organizer must be a member of the Society;
- **(c)** the meeting must be open to all members of the Society;
- **(d)** registration fees should be charged, with a substantial reduction for members of the Society. A further reduction should be made for members of the Society who pay the reduced rate subscription (i.e. research students, those not in full time employment and retired members);
- **(e)** a financial statement must be submitted on completion of the Meeting;
- **(f)** any profits up to the value of the grant are to be returned to the Australian Mathematical Society;
- **(g)** on completion, a Meeting Report should be prepared, in a form suitable for publication in the Australian Mathematical Society *Gazette*;
- **(h)** a list of those attending and a copy of the conference Proceedings (if applicable) must be submitted to the Society;

(i) only in exceptional circumstances will support be provided near the time of the Annual Conference for a Special Interest Meeting being held in another city.

In its consideration of applications, Council will take into account locations around Australia of the various mathematical meetings during the period in question. Preference will be given to Meetings of at least two days duration. The maximum allocation for any one Meeting will be $2500, with up to $12,000 being available in 2004. There will be six-monthly calls for applications for Special Interest Meeting Grants, each to cover a period of eighteen months commencing six months after consideration of applications.

Elizabeth J. Billington
AustMS Secretary

Membership and Correspondence

Applications for membership, notices of change of address or title or position, members' subscriptions, correspondence related to accounts, correspondence about the distribution of the Society's publications, and orders for back numbers, should be sent to the Treasurer. All other correspondence should be sent to the Secretary. Membership rates and other details can be found at the Society web site: http://www.austms.org.au.

THE AUSTRALIAN MATHEMATICAL SOCIETY

Publications

The Journal of the Australian Mathematical Society

Editor: Prof. C.F. Miller
Department of Mathematics and Statistics
The University of Melbourne
VIC 3010
Australia

The ANZIAM Journal

Editor: Prof. C.E.M. Pearce
Department of Applied Mathematics
The University of Adelaide
SA 5005
Australia

Bulletin of the Australian Mathematical Society

The Editorial Office
Bulletin of the Australian Mathematical Society
Department of Mathematics
The University of Queensland
QLD 4072
Australia

The *Bulletin of the Australian Mathematical Society* aims at quick publication of original research in all branches of mathematics. Two volumes of three numbers are published annually.

The Australian Mathematical Society Lecture Series

Editor-in-Chief: Prof. M. Murray
Department of Pure Mathematics
The University of Adelaide
SA 5005
Australia

The lecture series is a series of books, published by Cambridge University Press, containing both research monographs and textbooks suitable for graduate and undergraduate students.

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