Mathellaneous

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More Unsolved Problems for Young and Old

Unsolved problems are the lifeblood of mathematics. Unfortunately, as the frontiers of our field are being pushed ever further, it is becoming more and more difficult for mathematicians from even mildly disparate research areas to find common ground. However, there is a wealth of elementary unsolved problems which any person with a working knowledge of mathematics should be able to understand and even begin to play around with. I will give concise expositions on four such problems with the hope that one of the readers of this article will perhaps lay claim to a solution further down the track. Remember that all maths problems were, at some stage, unsolved![1]

1 Unfolding a Polyhedron

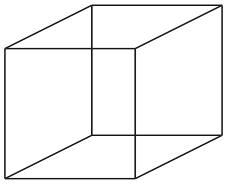
Throughout history, one of the major conundrums faced by artists is the problem of how to render a three-dimensional object in a two-dimensional medium. Whereas some budding painters presumably turned their attention toward sculpture, those artists who persisted developed various techniques, such as the method of perspective. Now suppose that I wanted to accurately convey a simple three-dimensional shape, such as a polyhedron, to the readers of this article. One technique, well-known to cereal box manufacturers and even to primary school children, is to use what is called a net. To form a net, I would take a paper model of my polyhedron and cut along some of the edges so that the faces can be unfolded into a single flat piece of paper. It would then be simple matter to transmit the picture of the net, accompanied by gluing instructions[2]. The reader could then cut out the net and reconstruct the original polyhedron. For example, the following diagram shows the net for a cube.

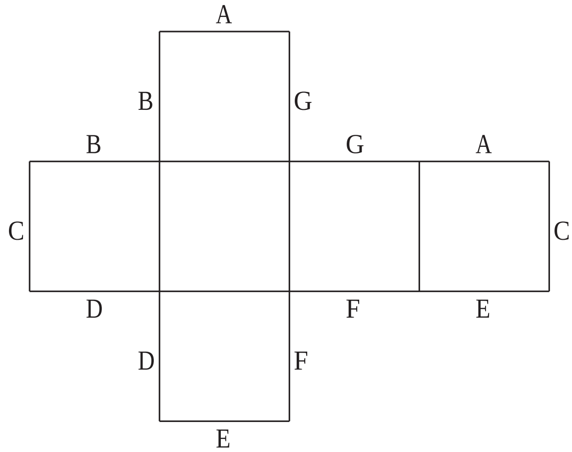
Now it turns out that we have been somewhat presumptuous in our description of a net — could it be the case that not every polyhedron has a net? It turns out that things can go

[1] This is, in fact, a sequel to the Mathellaneous article titled "Unsolved Problems for Young and Old" published in the May 2005 edition of the *Gazette*. Readers may wish to note that John Conway's angel problem, one of the unsolved problems discussed in that article, now appears to be solved. Very recently, Brian Bowditch and Oddvar Kloster have both claimed to have found solutions entirely independently. Bowditch has proven that an angel of power greater than or equal to four can always escape the devil while Kloster asserts the stronger result that an angel of power greater than or equal to two can do the same. For more details, their respective proofs can be downloaded from

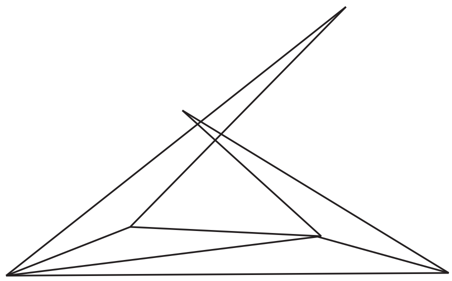
`http://www.maths.soton.ac.uk/staff/Bowditch/papers/bhb-angel.pdf`
and
`http://home.broadpark.no/~oddvark/angel/Angel.pdf`

[2] It is a non-trivial exercise to show that gluing instructions are indeed necessary. To accomplish this, it suffices to find a net which can be folded in two distinct ways to create two distinct polyhedra.





awry in this seemingly simple process. For example, the following diagram represents a tetrahedron which can be unfolded to give a polygon which overlaps with itself! It would certainly be difficult for someone to reconstruct the original polyhedron out of paper from the given net. However, not all is lost in this example — cutting along a different set of edges allows the original tetrahedron to be unfolded flat and without overlap.



Our exploration now begs the following question: does there exist a polyhedron which cannot be cut along some of its edges and unfolded flat to produce a polygon without overlap? The answer, surprisingly enough, is in the affirmative. However, the only counterexamples known are non-convex polyhedra, somewhat exotic beasts in the world of polyhedra. Restricting our attention to convex shapes, we arrive at the following problem, beautiful in its simplicity, though unsolved more than thirty years after being posed by Geoffrey C. Shephard in 1975.

Does every convex polyhedron have a net? In other words, can every convex polyhedron be cut along some of its edges and unfolded flat into a single polygon without overlap?

The experts in the field seem to think that the answer is in the affirmative; even despite the overwhelming evidence gathered by Catherine Schevon to suggest the following conjecture.

Conjecture: The probability that a random unfolding of a random convex polyhedron contains an overlap approaches 1 as the number of vertices of the polyhedron approaches infinity[3].

[3] A proof of this conjecture would, of course, require further clarification on the expressions “random unfolding” as well as “random convex polyhedron”.

If we are to believe the experts, it seems that we are in the curious position of believing that almost every unfolding of a polyhedron will produce an overlap, but that every single polyhedron has an unfolding without one!

Problem: Find a net which, with two different sets of gluing instructions, yields two distinct not necessarily convex) polyhedra.

2 The Union-Closed Sets Conjecture

Perhaps one of the simplest objects in the field of mathematics is the humble finite set. Nevertheless, the study of finite sets manages to produce fiendishly difficult problems. One particular example is the following conjecture which was posed by Peter Frankl in 1979 and has eluded combinatorialists ever since.

> A family of finite sets is said to be *union-closed* if the union of any two members of the family is also a member. In a finite union-closed family, must some element appear in at least half of the sets?

For example, consider the union-closed family which consists of all of the subsets of $\{1, 2, \ldots, n\}$. A moment's thought reveals that each of the elements appears in exactly half of the sets in the family, since for each set in which the element k appears, there is the complement in which k does not appear. Or for a more sporadic example, consider the family which consists of the sets

$$\emptyset, \{1\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}, \{2,3\}, \{3,4\}, \{4,5\}, \{1,2,3\}, \{1,3,4\}, \{1,4,5\},$$
$$\{2,3,4\}, \{3,4,5\}, \{1,2,3,4\}, \{1,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}.$$

A routine check can be used to verify that these 18 sets are indeed union-closed and that the element 3 appears in 12 of the sets, which is certainly more than half.

Despite the simplicity of the conjecture and the overwhelming evidence to support it, there has been little progress made. Amazingly enough, for a finite union-closed family of N sets, it is not even known whether there must be some element which appears in some proportion c of the sets for any constant $c > 0$. In fact, the best results give a proportion of approximately $\frac{1}{\log_2 N}$ of the sets, as shown in the following theorem.

Theorem 1 *In a union-closed family of N sets, some element must appear in at least $\frac{N-1}{\log_2 N}$ sets.*

Proof. For ease of argument, we may assume without loss of generality that the family $\mathcal{F}$ includes the empty set. Now let X be the smallest possible set which has at least one element in common with each non-empty set from $\mathcal{F}$. Construct another family of sets $\mathcal{G}$ consisting of the intersections of X with the sets in $\mathcal{F}$. We now make the following three observations.

- The empty set is in the family $\mathcal{G}$.
- The minimality of X guarantees that every element that appears in a set of $\mathcal{G}$ must also appear in a set of $\mathcal{G}$ with one element.
- The family $\mathcal{G}$ is union-closed.

In particular, it follows that $\mathcal{G}$ consists of all of the subsets of X. This, in turn, implies that

$$2^{|X|} \leq |\mathcal{G}| \leq N \Rightarrow |X| \leq \log_2 N.$$

Since each of the $N-1$ non-empty subsets of $\mathcal{F}$ must include at least one of the elements of X, a simple application of the pigeonhole principle yields the fact that some element must appear in at least

$$\frac{N-1}{|X|} \geq \frac{N-1}{\log_2 N}$$

of the sets, as required. □

This argument by Emanuel Knill appears in a paper of Piotr Wojcik [4] who shows that with quite a bit more work, the result can be strengthened by a factor of $\frac{\log 2}{\log 4 - \log 3} \approx 2.409$ for large n. The majority of progress on this problem has occurred in the past fifteen or so years and has involved determining properties that a counterexample to the conjecture might have. For example, we now know that the conjecture holds for all union-closed families with up to 40 sets.

Problem: Prove that the union-closed sets conjecture holds for families which include a set with only one or two elements.

3 Hadamard Matrices

Consider colouring the squares of a $k \times k$ grid either black or white. Suppose that the colouring is such that any two rows agree in half of their cells and disagree in the other half. Then the colouring is referred to as a Hadamard matrix of order k. For what values of k is there a Hadamard matrix of order k?

The name Hadamard matrix comes from their relation with the Hadamard determinant bound, which states that if M is an $n \times n$ matrix with complex entries of magnitude at most 1, then

$$|\det(M)| \leq n^{n/2}.$$

Note that if we take a Hadamard matrix and interpret the black cells as $+1$ and the white cells as -1, then we have a matrix which satisfies

$$MM^t = nI_n$$

where I_n denotes the $n \times n$ identity matrix. It now follows that

$$M^tM = nI_n,$$

so that if the Hadamard property holds for the rows of M, then it also holds for the columns of M. Furthermore, we can deduce from the equation $MM^t = nI_n$ that $|\det(M)| = n^{n/2}$ so that Hadamard matrices are real matrices which yield inequality in the determinant bound. So rather than being a frivolous exercise in colouring, the study of Hadamard matrices stems from analysis and the conjecture above is, in fact, regarded as important by coding theorists.

It is a simple matter to exhibit Hadamard matrices for $n = 1, 2, 4$.

$$H_1 = \begin{bmatrix} 1 \end{bmatrix} \quad H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

It turns out that there is no Hadamard matrix of order 3 and, in fact, we have the following theorem.

Theorem 2 *The order of a Hadamard matrix must be* 1, 2*, or a multiple of* 4.

Proof. Observe that the following two operations preserve Hadamard matrices.

- Swap all of the colours in a row or column.
- Permute the rows or columns.

Therefore, if we have a Hadamard matrix of order $n \geq 3$, then it is possible to swap the colours in some of the columns and permute the columns so that the first row is all black, the left half of the second row is coloured black, and the right half of the second row is coloured white. Certainly, this requires n to be even if $n \geq 3$.

Now suppose that the third row has k black squares in the left half. It follows that it must have $n-k$ black squares in the right half and hence, k white squares in the right half. Therefore, the number of squares in which the second and third rows coincide is simply $2k$. Since we require $2k = \frac{n}{2}$, it follows that $n = 4k$, as desired. □

The Hadamard matrix conjecture asserts that a Hadamard matrix exists of order $4k$ for all positive integers k. Constructions for the known Hadamard matrices use a whole variety of techniques. For example, Hadamard matrices of order $2n$ can be built from Hadamard matrices of order n using the following construction.

$$H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$$

It is also known that Hadamard matrices of orders m and n can be used to construct a Hadamard matrix of order mn. An interesting construction by Raymond Paley which uses the properties of finite fields yields the following theorem.

Theorem 3 *A Hadamard matrix of order $4k$ exists when*

- *$4k$ is of the form $q+1$ where q is a perfect power of a prime; and*
- *$4k$ is of the form $2q+2$ where q is a perfect power of a prime.*

Due to the sporadic nature of the various constructions of Hadamard matrices, it seems that a solution to the Hadamard matrix conjecture may require ideas from various different areas of mathematics. Most of the recent work in the area has been in using the computer to search for Hadamard matrices of various orders. Presently, the orders less than 2000 for which there is no known construction of a Hadamard matrix are 668, 716, 764, 892, 956, 1004, 1132, 1244, 1388, 1436, 1676, 1772, 1852, 1912, 1916, 1948 and 1964.

Problem: Prove that if there exist Hadamard matrices of orders m and n, then there exists a Hadamard matrix of order mn.

4 Thrackles

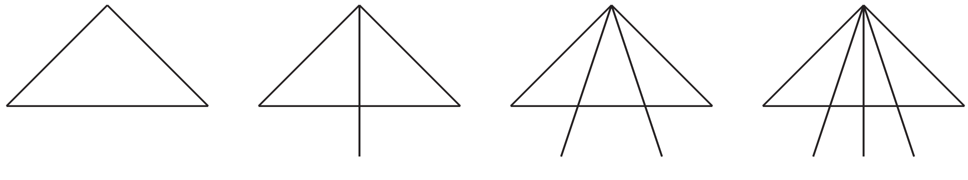
The following problem was posed in the 1960's by John Conway, who offers $1000 for its solution.

A *thrackle* is a drawing in the plane consisting of vertices, represented by points, and edges, represented by curves, such that

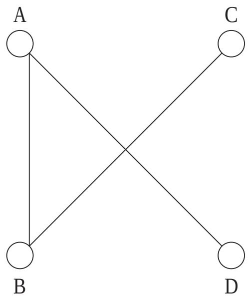
- every edge has two ends which coincide with two distinct vertices;
- each edge does not intersect itself and does not pass through a vertex; and
- each pair of distinct edges intersects precisely once, either at a vertex or at a crossing.

Does there exist a thrackle with more edges than vertices?

The following examples of thrackles shows that the bound is tight — in other words, for every $N \geq 3$ there exists a thrackle with N vertices and N edges.



Let us say that a graph can be thrackled if it has a representation in the plane as a thrackle. One observation is that the converse of the thrackle conjecture is far from true — for example, the 4-cycle is a graph with the same number of edges and vertices but cannot be thrackled. To see this, observe that removing edges from a thrackle leaves a thrackle. If we denote the cycle by the vertices $ABCD$ and remove the edge CD, it is not too difficult to see that the remaining thrackle must look something like the following. The edge CD must be drawn in to cross AB precisely once without touching the interior of the edges AD and BC. Now even the least discerning reader should be able to convince themselves that this cannot be the case[4].



Despite the childish simplicity of thrackles, no one has managed to disprove the conjecture that thrackles must always have at least as many vertices as edges. However, we do know that if the thrackle conjecture is false, then there must exist a counterexample with a very simple structure.

Theorem 4 *If there exists a counterexample to Conway's thrackle conjecture, then there exists a counterexample consisting of two even cycles which intersect at exactly one vertex.*

Perhaps the strongest result towards the thrackle conjecture is the following theorem by Grant Cairns and Yuri Nikolayevsky.

Theorem 5 *If a graph with E edges and V vertices can be thrackled, then $E \leq \frac{3}{2}(V-1)$.*

Problem: Prove that every cycle of length greater than or equal to 5 can be thrackled.

References

[1] G. Cairns and Y. Nikolayevsky, *Bounds for Generalized Thrackles*, Discrete and Computational Geometry **23** (2000), 191–206.

[2] J. Malkevitch, *Nets: A tool for representing polyhedra in two dimensions*, American Mathematical Society Feature Column, (February 2002). http://www.ams.org/featurecolumn/archive/nets.html

[3] M. Miyamoto, *A construction of Hadamard matrices*, Journal of Combinatorial Theory A **57** (1991), 86–108.

[4] P. Wojcik, *Union-Closed Families of Sets*, Discrete Mathematics **199** (1999), 173–182.

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[4]Those more dubious may wish to invoke the Jordan Curve Theorem.