

Pantographs and cyclicity

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Abstract

Every parallelogram Λ generates a doubly-infinite family of quadrilaterals called its pantograph. The types of quadrilaterals so arising can be characterised by tiling the plane. The family contains a single infinity of cyclic quadrilaterals. The locus of their circumcentres is a rectangular hyperbola or orthogonal line-pair passing through the vertices of Λ .

Any non-degenerate parallelogram $\Lambda = FGHI$ generates in its plane a doubly-infinite system of quadrilaterals called its *pantograph*, denoted by $\mathcal{Q}(\Lambda)$. Its elements are just those plane quadrilaterals having the vertices of Λ as the midpoints of their sides, taken in order. More precisely, let P be any point in the plane, and draw the straight line segment PQ having I as its midpoint, from Q draw a segment QR with F as its midpoint, draw RS with G as its midpoint, and finally join SP . It is easily verified that SP has H as its midpoint. (This is true even if P is not in the plane of Λ . Then $\Omega = PQRS$ is a quadrilateral of $\mathcal{Q}(\Lambda)$, and every element of $\mathcal{Q}(\Lambda)$ can be constructed in this way. Λ is called the *median parallelogram* of Ω . Clearly Ω is uniquely determined by specification of any one of its vertices. The diagonals of Ω are parallel to the sides of Λ respectively and twice their length, and the area of Ω is twice that of Λ . See [1], where it was shown how, in the representation of quadrilaterals explained there, the members of $\mathcal{Q}(\Lambda)$ are representable by the points of a surface Σ of degree 12 in the Euclidean space E^6 , modulo an equivalence. In what follows we suppose that the vertices of Λ and Ω are related as described above.

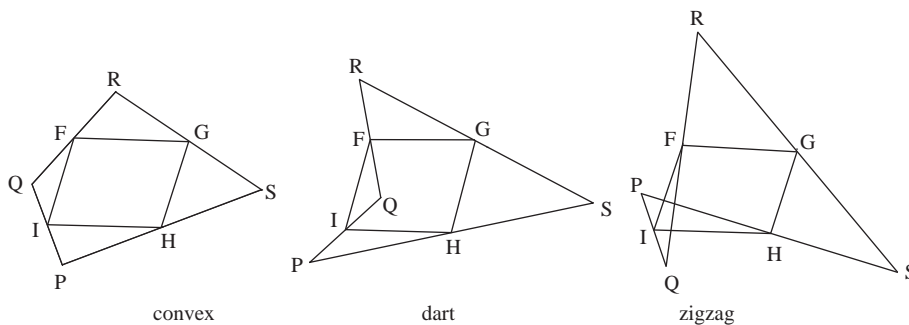


Figure 1. The three non-degenerate types of quadrilaterals and their median parallelograms.

Of interest is the location within $\mathcal{Q}(\Lambda)$ of the *cyclic* quadrilaterals; these are representable by a curve C in E^6 of degree 24 on Σ . This high degree may be a

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multiple of the effective degree, but it suggests an order of complexity which is belied by the apparent simplicity of the geometry. We show here how the cyclic quadrilaterals can be located more directly by means of a rectangular hyperbola in the plane of Λ . This does not in itself contradict the assertion of the degree of C , for the representations are different.

Tiling of the plane

In order to distinguish the different types of quadrilaterals in $\mathcal{Q}(\Lambda)$ (the nondegenerate types (convexes, darts, zigzags) and the partially degenerate types (flags, triangles, ...; see [1]), we first ask: For what locations of a specified vertex is the member of $\mathcal{Q}(\Lambda)$, having that vertex, of a particular type? It turns out that the answer can be given in terms of tiling.

We tile the plane using copies of Λ in the obvious manner. The tiles are isomorphic copies of $\Lambda \cup$ inside (Λ) , any distinct two being disjoint or having an edge or vertex as intersection. *It is sufficient, for two quadrilaterals to be of the same type, that their specified vertices are interior to one and the same tile.*

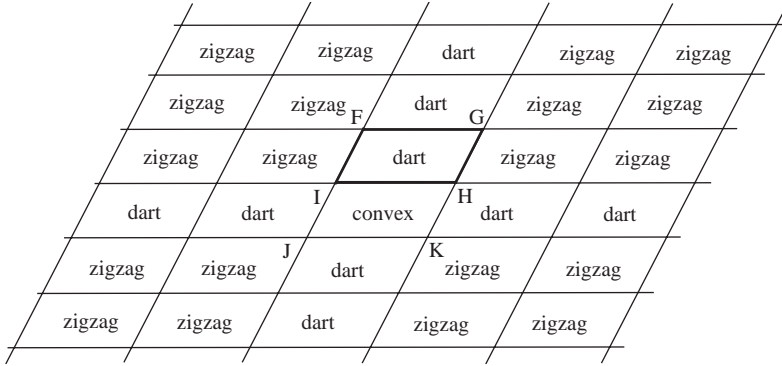


Figure 2. Tiling of the plane near Λ , when P is the specified vertex.

There is only one tile, $t(P) = HIJK$, for which Ω is convex. All other tiles on the cross through $t(P)$ produce darts, and all remaining tiles give rise to zigzags. If P belongs to the boundary of $t(P)$ then Ω is a triangular degenerate. Of course, if another vertex is specified then the labelling of tiles is different, in an obvious way.

Cyclic quadrilaterals in $\mathcal{Q}(\Lambda)$

We suppose that $\Lambda = FGHI$ is a given nondegenerate parallelogram, and seek to characterise the cyclic members of $\mathcal{Q}(\Lambda)$. They may be convex or zigzag, but cannot be a dart or degenerate.

Theorem 1. *If Ω is a cyclic quadrilateral in $\mathcal{Q}(\Lambda)$, and O is the centre of its circumcircle, then angles $\angle FOG, \angle HOI$ are*

- supplementary if O is inside Λ
- equal if O is outside Λ

and the same is true for the angles $\angle FOI, \angle GOH$.

Proof. Suppose that O is inside Λ . PQ, QR, RS and SP , being chords of the circle, have their right bisectors meet at O , creating cyclic quadrilaterals $FOGR, HOIP$, so angles $\angle FOG, \angle R$ are supplementary, and so are $\angle HOI, \angle P$. But since Ω is cyclic, angles $\angle P$ and $\angle R$ are also supplementary. The result follows. If O is outside, the proof is analogous. See Figure 3.

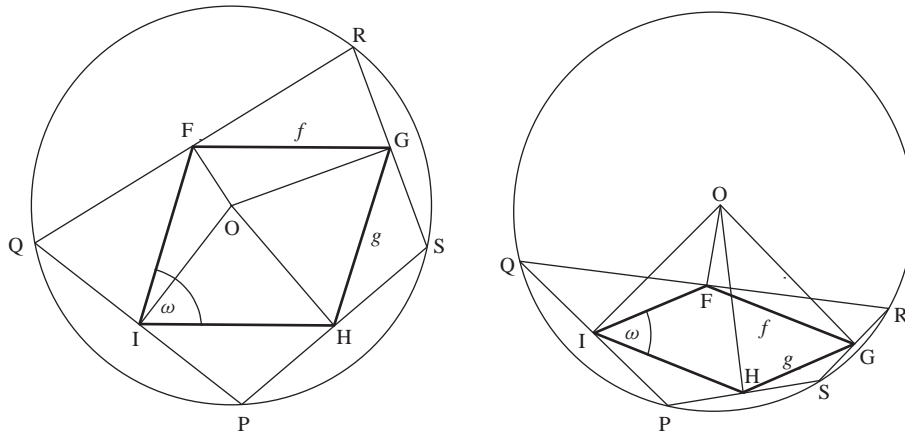


Figure 3. Illustrating Theorem 1 and Lemma 1.

Lemma 1. *If O is the centre of the circumcircle of Ω as in Theorem 1, then O is inside Ω if and only if it is inside Λ .*

Proof. When Ω is convex the parallelogram Λ is inside Ω , and the geometry is as shown in Figure 3, from which the lemma’s statement is immediate. The reader may like to draw the corresponding figures when Ω is a zigzag. If O is on a side of Ω then that side is a diameter and O coincides with a vertex of Λ . Note that O cannot be an internal point of a side of Λ .

We propose now to prove the converse of Theorem 1. The arguments here are more delicate, and involve various cases. For ease of exposition an enabling lemma is postponed until after the main result.

Theorem 2. *Let $\Lambda = FIGH$ be a parallelogram. If O is any point inside Λ and such that angles $\angle FOG, \angle HOI$ are supplementary, or outside Λ and such that these angles are equal, then O is the centre of the circumcircle of some cyclic quadrilateral Ω of $Q(\Lambda)$.*

Proof. Join O to F, G, H, I , the vertices of Λ . Through these points draw lines perpendicular to OF, OG, OH, OI respectively, and call the intersection points of adjacent lines R, S, P, Q (see Figure 4).

Case 1. The point O is inside Λ and the angles $\angle FOG, \angle HOI$ are supplementary. Then so are angles $\angle P, \angle R$, and $\Omega = PQRS$ is therefore cyclic. We shall suppose first that Ω is convex.

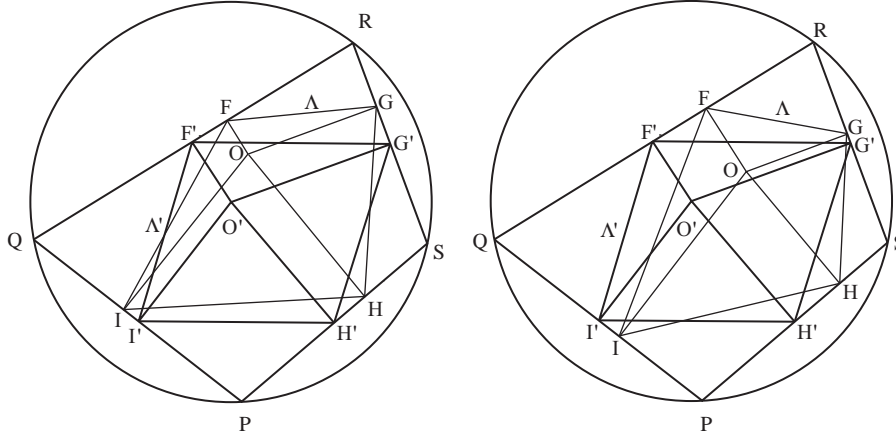


Figure 4. Cases (α) and (β) .

It remains only to prove that O is its circumcentre. This will ensure that F, G, H, I are the midpoints of the sides and that $\Omega \in \mathcal{Q}(\Lambda)$. Suppose on the contrary that the circumcentre is another point O' . Drop perpendiculars $O'F' \dots$, onto the corresponding sides with feet F', \dots , thereby making these points the midpoints of the sides of Ω and making $\Lambda' = F'G'H'I'$ the median parallelogram of Ω ; that is $\Omega \mathcal{Q}(\Lambda')$, and Λ' is inside Ω .

Suppose O' were outside Λ' , and therefore outside Ω , by Lemma 1. Then by Theorem 1 the angles $\angle F'O'I', \angle G'O'H'$ are equal. But these are equal respectively to $\angle FOI, \angle GOH$, which are supplementary by assumption. Thus all four angles $\angle P, \angle Q, \angle R, \angle S$ are rightangles, Ω is a rectangle and therefore its circumcentre O' is inside Ω and hence inside Λ' by Lemma 1 again, a contradiction. So O' is indeed inside Λ' .

The discussion now separates into several cases. Point O' determines a partition of the inside of Ω into four disjoint open regions, $\text{ins}(O'F'RG')$ etc., together with their boundaries. Setting aside for the moment the cases where O lies on a boundary, suppose without loss of generality that $O \in \text{ins}(O'F'RG')$. Then

$$FG < F'G'. \quad (1)$$

For a proof of this seemingly obvious fact see Lemma 2 below. Now, the position of O implies that O' belongs to the inside of one of $OFQI, OIPH$ or $OHSI$. We consider these cases separately.

- (α) Let $O' \in \text{ins}(OIPH)$. Then $I'H' < IH$ by Lemma 2, so $FG < F'G' = I'H' < IH$. But $FG = IH$, being opposite sides of the parallelogram Λ . Hence this case is not possible.
- (β) Let $O' \in \text{ins}(OFQI)$. This subcase is more delicate, relying on the ways in which two parallelograms can intersect. Note that FG is outside Λ' , and $FG < F'G'$. Likewise $F'I'$ does not meet Λ , and $F'I' < FI$; on the other hand $I'H$ meets $I'H'$ and GH meets $G'H'$. Moreover all vertices of Λ are outside Λ' , and vice versa: for all sides of Ω are outside both parallelograms except at their vertices. The two parallelograms cannot intersect in this way, so the case is not possible.

(γ) Let $O' \in \text{ins}(\text{OHSG})$. This case is similar to (β).

Thus in all subcases, we conclude (except where O or O' lies on the boundary) that $O \neq O'$ is impossible, and O is indeed the centre of the circumcircle. We invite the reader to verify the boundary cases (which must follow however by appeals to continuity), and then to adapt the proof above in the case where Ω is a zigzag. Then finally the reader can try his or her hand at Case 2.

Case 2. O is outside Λ and angles $\angle\text{FOG}$, $\angle\text{HOI}$ are equal. The proof in this case is in the same vein as Case 1.

Lemma 2 is the enabling lemma used in the above proof.

Lemma 2. *Let $ABCD$ be a quadrilateral in which angles $\angle A$ and $\angle C$ are right angles, and let K be any point inside $ABCD$. Let M_K and N_K be the feet of the perpendiculars from K onto CD and AD respectively. Then $M_K N_K < CA$, and the segments $M_K N_K$ and CA do not meet.*

Proof. The case where B is obtuse is illustrated in Figure 5. Let M_1 be any chosen point on CD ; the points K for which the construction gives $M_K = M_1$ all lie on the perpendicular to CD at M_1 . Suppose that this perpendicular meets DA (not DA produced) in U_1 . Then lengths $M_1 N_K$ have upper bound $M_1 U_1$, corresponding to $K = U_1$. If M_0 is the foot of the perpendicular from A to CD then $M_1 U_1 < M_0 A$ since these two lines are parallel, and

$$M_K N_K < M_1 U_1 < M_0 A < CA.$$

It is clear that $M_K N_K$ and CA cannot intersect. If K lies inside $M_0 ABC$ the proof is similar. The case where $\angle B$ is acute and $\angle D$ is obtuse is handled in a like manner.

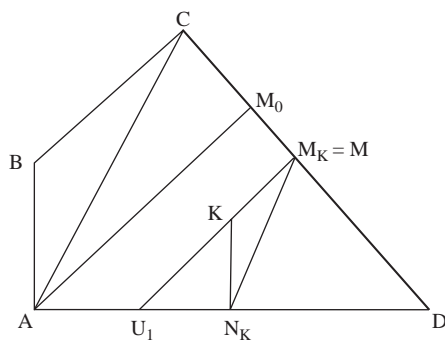


Figure 5. For Lemma 2.

Locus of circumcentres of cyclic quadrilaterals in $\mathcal{Q}(\Lambda)$

We shall call any point O which is the circumcentre of a cyclic quadrilateral in $\mathcal{Q}(\Lambda)$ a *circumcentre for Λ* , and write $\text{Cen}(\Lambda)$ for the set of all such points. The reader can easily verify the following theorem.

Theorem 3. *Each vertex of Λ is a circumcentre for Λ , and these are the only points on Λ which are circumcentres.*

For example, to find the cyclic quadrilateral with centre G , draw lines α through I perpendicular to IG , β through F perpendicular to FG , and γ through H perpendicular to HG ; then in the notation of Figure 1, $Q = \alpha \cap \beta$, $P = \alpha \cap \gamma$, and R, S are such that $RF = FQ$ and $SH = HP$.

Theorems 1, 2 and 3 characterise completely the points of $\text{Cen}(\Lambda)$. The angle properties in Theorems 1 and 2 will now allow us to calculate the locus of circumcentres. We have to find those points O inside $FGHI$ at which each pair of opposite sides subtends supplementary angles. This can be done as a somewhat irksome exercise in analytic geometry.

Let O be a circumcentre lying inside Λ . First observe that if Γ denotes the circle through O, F and G then O lies also on a circle Γ' of equal radius through I, H : this is equivalent to the property of supplementary.

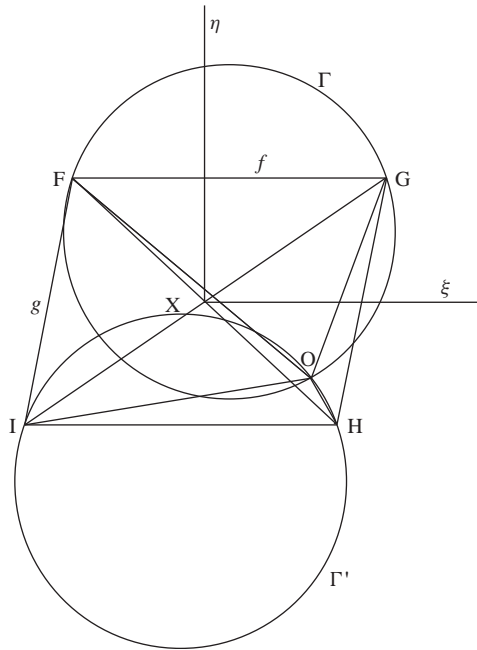


Figure 6. Circles Γ and Γ' .

Write, for the sidelengths and angle measure,

$$f = FG = HI, \quad g = GH = IF, \quad \omega = \text{angle } \angle FGH,$$

and assume without loss of generality that $g \leq f$ and $0 < \omega < \pi/2$. Let X denote the point of intersection of the diagonals of Λ , take X as the origin for coordinate axes $X\xi, X\eta$ with $X\xi$ parallel to and in the sense FG . Using the geometry of circles Γ, Γ' we find, for the coordinates (ξ, η) of O ,

$$(\xi^2 - \eta^2) \sin \omega - 2\xi\eta \cos \omega - \frac{1}{4}[(f^2 - g^2) \sin \omega] = 0. \quad (2)$$

Let the axes be rotated clockwise through an angle $(\pi - \omega)/2$ to new axes $X\lambda, X\mu$; the equation becomes

$$\lambda\mu = E^2, \quad \text{where } E := \sqrt{\frac{1}{8}(f^2 - g^2) \sin \omega}. \quad (3)$$

Λ is a rhombus if and only if $f = g$. This and a little further calculation leads to the following theorem, where we write $S = \sin(\omega/2)$, $C = \cos(\omega/2)$:

Theorem 4. *When Λ is not a rhombus, the locus $\text{Cen}(\Lambda)$ of circumcentres is a rectangular hyperbola (2) through the vertices of Λ , with centre X and asymptotes*

$$\lambda = S\xi - C\eta = 0 \quad \text{and} \quad \mu = C\xi + S\eta = 0.$$

When Λ is a rhombus, $\text{Cen}(\Lambda)$ is an orthogonal line-pair, namely the lines containing the diagonals of Λ .

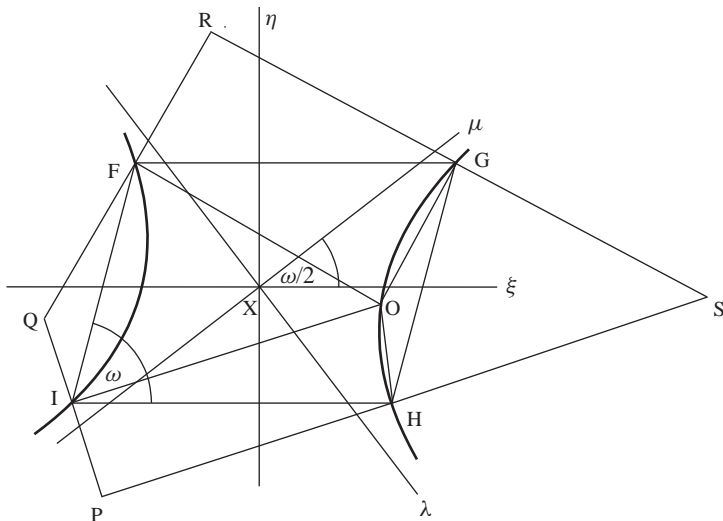


Figure 7. The rectangular hyperbola $\text{Cen}(\Lambda)$, and an arbitrarily chosen point O on it.

Now suppose that Λ is not a rhombus. A typical point on $\text{Cen}(\Lambda)$ is $O = (\xi, \eta) \equiv O(t)$ where

$$\xi = E(St + Ct^{-1}), \quad \eta = E(-Ct + St^{-1}). \quad (4)$$

As t grows from 0 to $+\infty$, O describes the right-hand arm of the hyperbola downwards; as t grows from $-\infty$ to 0 , O describes the left-hand arm downwards.

To find the radius $r(t)$ of the circumcircle (call it $\Theta(t)$), we draw the lines FR perpendicular to OF and GR perpendicular to OG : the radius we seek is OR (see Figure 3). In principle, calculating the coordinates of R is quite straightforward; but it is in fact a formidable piece of algebra, at the end requiring finding the factors of a quartic in t whose coefficients feature the five related constants E, f, g, S, C . When Λ is a rhombus the parametrisation of the line-pair is different but the subsequent calculations are simpler. The eventual result is:

Theorem 5. *When Λ is not a rhombus, the radius of the circumcircle $\Theta(t)$ having centre O with parameter t as in (4) is given by*

$$r(t) = \sqrt{E^2 C^{-2} t^2 + \frac{1}{2}(f^2 + g^2) + E^2 S^{-2} t^{-2}}.$$

When Λ is a rhombus, the circumcircle with centre $O(u)$ on diagonal line $GI(\lambda = Cu, \mu = Su)$ has radius $\sqrt{f^2 + u^2 S^{-2}}$, and that with centre $O(t)$ on diagonal line $FH(\lambda = Cu, \mu = Su)$ has radius $\sqrt{f^2 + u^2 C^{-2}}$.

We can easily prove the following converse of Theorem 4:

Theorem 6. *Let \mathcal{H} be any rectangular hyperbola in the $\xi X \eta$ plane with equation $\lambda \mu = \text{constant}$ referred to some axes $\lambda X \mu$, where $X\lambda$ is obtained by a clockwise rotation through an angle ϕ , $\pi/4 < \phi < \pi/2$. If G is any point on \mathcal{H} and in the first quadrant for $\xi X \eta$, there exists a unique parallelogram $\Lambda = FGHI$, with FG parallel to and in the direction of $X\xi$, such that $\mathcal{H} = \text{Cen}(\Lambda)$.*

Acknowledgements

I thank Emanuel Strzelecki for showing me his analytic proof of Theorem 4, Harry Calkins for advice on the use of Mathematica, and the anonymous referee for valuable comments. The figures were drawn using TurboCAD. In a subsequent paper I shall describe the curve osculated by this family of circles.

References

- [1] Miller, J.B. (2007). Plane quadrilaterals. *Gaz. Aust. Math. Soc.* **34**, 103–111.
- [2] TurboCAD Deluxe, Version 12.0, Build 39.6, IMSI.