Book reviews

Mathematics and Its History

John Stillwell
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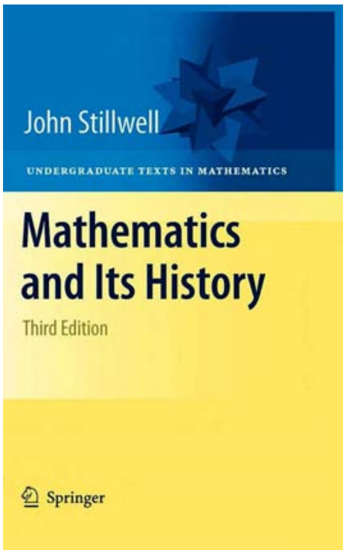
Readers of the first and second editions of this book will already know that this is *not* a book about the history of mathematics; rather, it is a book about mathematics, where history plays a guiding role. Thus the book's title is exact and apt, although the casual browser may be led to think otherwise. The notation used is modern — there is no interest here in how the Greeks used geometry to study number theory, for example, but rather in understanding that ancient number theory in modern terms. The third edition adds two new chapters, on simple groups and on combinatorics, as well as some new sections in old chapters. Overall there is an even greater cohesion between topics than in the previous editions.

The idea of presenting mathematics through its historical development is not new, and there are other books that take the same approach, such as Hairer and Wanner's *Analysis by Its History*, Bressoud's *A Radical Approach to Real Analysis*, and Weil's *Number Theory, an Approach Through History, from Hammurapi to Legendre*. In each case, the authors choose a topic (calculus, Fourier series, and number theory, respectively), and explain the mathematics through a historical perspective. What makes Stillwell's book different is the scope of his goal: 'to give a bird's eye view' of nearly *all* of the mathematics contained in the undergraduate curriculum, aiming at undergraduates. As far as I am aware, no other book in the world has comparable goals. Does he succeed? Yes. This book is brilliant. Let me illustrate how he does it.

The book contains 25 chapters, each headed by a short preview where the chapter's contents are summarised and connections are made to other relevant chapters. Each chapter focuses on one topic, for instance Greek Geometry, or Number Theory in Asia, or Infinite Series. Each chapter contains around seven or eight short sections. Each section, other than the last, makes a very specific point about a very specific topic, and comes equipped with its own exercises. The exercises usually deal with mathematical, rather than historical, questions associated with the text. The last section contains a short biographical sketch of mathematicians who contributed to the topic at hand. For example, the chapter on topology contains biographical notes on Poincaré, while the chapter on differential geometry includes notes on Harriot and Gauss. I particularly enjoyed reading about Pascal in Chapter 8.

To see how this structure translates into practice, consider Chapter 3 (Greek Number Theory), Section 3.4 (Pell's Equation). It is explained that Pell's equation is the Diophantine equation $x^2 - Ny^2 = 1$, where N is a given non-square integer, and integer solutions x, y are sought. The author places the equation in context by pointing out that if x, y are large solutions to $x^2 - 2y^2 = 1$, then x/y will be a good approximation to $\sqrt{2}$; that is why the very early Greeks were interested in it. This is followed by the presentation of an ingenious recursive method, known to the Greeks, to obtain larger solutions to Pell's equation from smaller solutions. In this way, if we start with $x = 1, y = 0$, we can successively obtain larger solutions, and therefore better approximations to $\sqrt{2}$. How did the Greeks discover this recursive method? Nobody knows, but Van der Waerden and Fowler suggest plausibly that it was through an application of the Euclidean algorithm to line segments. An explanation of how this may have been done follows. The last three paragraphs in the section chart further developments in the same vein through the ensuing centuries: how Brahmagupta obtained a recursive relation for $x^2 - Ny^2 = 1$ (see Chapter 5); how the final theoretical step in showing that Brahmagupta's formulas work was done by Lagrange in 1768; a mention is made of a relation with continued fractions (to be explored in the exercises); how there is a non-trivial relation between N, the number of steps in the Greek construction, and the size of the smallest non-trivial solution x, y; how *Archimedes' cattle problem* leads to the equation $x^2 - 4729494y^2 = 1$, the smallest non-trivial solution of which was found, in 1880, to have over 200 000 digits; and how Lenstra, in 2002, found a way to express all of the solutions to the cattle problem in a condensed form.



We see from this example how sections are structured: a topic is introduced (Pell's equation), and its relevance clearly explained. Here a neat recursive formula is presented, but the real mathematical idea is how the formula was (perhaps) derived from the Euclidean algorithm. Then connections are drawn to related topics, all the way down to current research, referenced to enable easy access. You can't get much better than that, and such examples abound in the text. To mention just one more, in Section 4.2 we are treated to Eudoxus' Theory of Proportions, where a direct link is made to Dedekind cuts. These connections help to fix in the minds of students what they already know both about Greek geometry and about the construction of the reals. The author presents mathematics as a coherent and unified endeavour, rather than as unrelated topics (in meaning and in time). Indeed, the only criticism that can be made here is that perhaps the edifice of mathematics is made to seem *too* coherent, as if there is an inherent necessity in the way mathematics is produced, driven by historical forces. (To be fair, the author never for a moment implies in the text that such is the case, but instructors should be wary.)

Of course, it is unavoidable that some topics will be missing, and that some will be better represented than others. Stillwell concentrates on the broad areas of geometry, number theory, analysis, and algebra. The first three are already present, implicitly or explicitly, in the work of the Greeks, while the latter is a more recent, but no less distinguished, arrival. In particular Stillwell weaves a very good narrative of algebra, geometry, and number theory, whenever possible relating other topics to it. For example the presentation of complex numbers and functions (on which there are three chapters) makes connections to the theory of polynomial equations (algebra), the complex projective line (algebraic geometry), and elliptic curves (number theory). A bit outside the main narrative there are chapters on mechanics (where we can find some differential equations), differential geometry, topology, sets and logic, and combinatorics. There are also some topics missing almost entirely. For example, measure theory and probability merit a passing mention in Chapter 24 (Sets, Logic, and Computation). Also there is scant reference to the work of some very influential mathematicians, including Legendre (four mentions in the index), Chebyshev (one mention), Kolmogorov (one mention), Hadamard (one mention not about his work, and one reference in the Prime Number Theorem), and Poisson (no mention). These decisions are understandable, given that the book is not driven by history, and there are all sorts of constraints in terms of choice of topics and length of the work — this is not an encyclopedia. The omissions, although noticeable, take nothing away from the book.

Who should use this book? A reviewer of the second edition (this *Gazette*, Volume 29, Number 5, December 2002) suggested two avenues for adopting it. First, a course on the *history* of mathematics should use a different primary source, but this book could be adopted as secondary reading, as it will help to illuminate the mathematics itself. Second, one could design a third-year (or honours) course around topics chosen from this book, with the aim of enriching the whole mathematical experience by drawing connections between areas. I concur with these ideas, but I have a further proposal: instructors should use this book liberally in their existing courses, as a source and as motivation. We don't have to use the whole book. If we are teaching group theory, it offers two very good chapters, on general group theory and on simple groups, which contain material sure to enhance our classes. If we are teaching number theory, then we are spoiled for choice, as there's just so much we can use. In short, we need not wait to use this book only when a 'suitable' course is offered. This book is excellent. Use it now.

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