Puzzle corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner No. 22. Each puzzle corner includes a handful of fun, yet intriguing, puzzles for adventurous readers to try. They cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites for their solution. Should you happen to be ingenious enough to solve one of them, then you should send your solution to us.

For each puzzle corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please email solutions to ivanguo1986@gmail.com or send paper entries to: Kevin White, School of Mathematics and Statistics, University of South Australia, Mawson Lakes, SA 5095.

The deadline for submission of solutions for Puzzle Corner 22 is 1 July 2011. The solutions to Puzzle Corner 22 will appear in Puzzle Corner 24 in the September 2011 issue of the *Gazette*.

Distinctive solid

Does there exist a convex polyhedron such that no two of its faces have the same number of edges?

Triple cheque

Penny cashed a cheque at the bank, but the careless teller transposed the dollar figure with the cent figure, and gave her the wrong amount of money. For example, if the cheque was for $12.34, Penny received $34.12 instead. Assume that the teller has old copper coins available, so that all positive multiples of 1 cent are possible. After buying a newspaper for 50 cents, Penny discovered the mistake, but still had three times the amount on the original cheque. What was the value of the cheque?

Last ball remaining

Mr Bored has 2011 blue balls and 2011 red balls in a gigantic bag. There is also an abundance of red and blue balls on the floor. Mr Bored randomly selects two balls from the bag and drops them onto the floor. If the two balls have the same colour, he places a blue ball into the bag, otherwise, he places a red ball into the

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bag. This is repeated until only one ball is left in the bag. What is the chance of it being red?

Midpoint madness

$ABCDE$ is a pentagon with side lengths of 2010, 2011, 2012, 2013 and 2014 in some order. Let P, Q and R be the midpoints of AC, BD and CE respectively. Then let X and Y be the midpoints of PQ and QR. If line segment XY has integer length, find this length.

Tangled tangents

Let

$$X = \sum_{n=1}^{2011} \tan^{-1}\left(\frac{1}{n^2+n+1}\right).$$

Find $\tan X$.

Radioactive rods

(1) There are eight rods, identical in appearance, but one of them is radioactive. It is possible to test for radioactivity by placing some number of rods into a super high-tech box. After the test, the box will indicate whether there was any radioactivity in its contents. Since the box is very expensive to operate, what is the minimum number of tests needed to find the radioactive rod?

(2) Now suppose that two out of the eight rods are radioactive. How many tests are needed to find them both?

Solutions to Puzzle Corner 20

The \$50 book voucher for the best submission to Puzzle Corner 20 is awarded to Mike Hirschhorn. Congratulations!

Lousy labelling

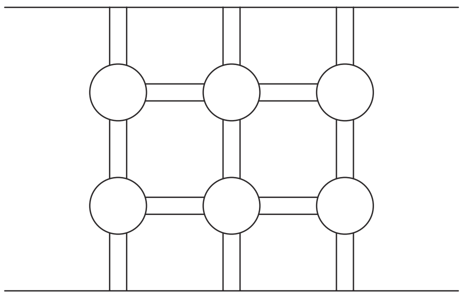
Three boxes are on the table. One has red balls, one has blue balls, and one has balls of both colours. Three labels are made for the boxes, but they are misplaced so that none of the boxes is labelled correctly. How many balls would you need to retrieve from the boxes in order to determine the correct labelling?

Solution by John Miller: Let R, B and M be the three incorrect labels, meaning red, blue and mixed. Draw one ball from the box labelled M. Without the loss of generality, let the ball be red, then the box labelled M must have red balls only since it cannot contain mixed balls. Now the box labelled B cannot have blue balls only, so it must have mixed balls. Finally the box labelled R must contain blue balls only. The argument works similarly if the initial ball was blue.

Thus only one ball is needed to identify the boxes.

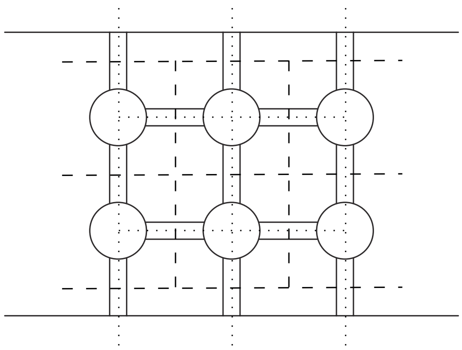
Broken bridges

There are thirteen bridges connecting the banks of River Pluvia and its six piers, as shown in the diagram below:



On an extremely stormy night, each bridge has a 50% *chance of being damaged by the rainfall. What is the probability that the locals can still cross the river using undamaged bridges the next morning?*

Solution by Robert Tang: Many boats frequent River Pluvia. Most of them can safely pass under the thirteen bridges, but have to navigate around the six piers. The following illustration shows boat paths as dashed lines and human paths as dotted lines.



Note the boat paths have the same configuration as the human paths, but rotated by 90°. Also, every boat path joining east and west cuts every human path joining north and south.

Suppose a tall sail boat can pass under a bridge if and only if the bridge is damaged. So for any bridge, the chance of the sail boat being able to pass under it would also be 50%. Since the boat paths and the human paths are symmetric, the chance of the sail boat passing these piers is equal to the chance of a person crossing the river.

However, the sail boat can pass the mess of piers and bridges if and only if people can no longer cross the river. Therefore the answer is 50%.

Trick question

Find all real solutions to the equation

$$\sqrt{x+4\sqrt{x-4}}-\sqrt{x-4\sqrt{x-4}}=4.$$

Solution by Randell Heyman: For the original equation to make sense, we require $x \geq 4$, so write $x = y^2 + 4$ for $y \geq 0$. Simplifying gives

$$\begin{aligned}\sqrt{x+4\sqrt{x-4}}-\sqrt{x-4\sqrt{x-4}} &= \sqrt{y^2+4+4\sqrt{y^2}}-\sqrt{y^2+4-4\sqrt{y^2}}\\ &= \sqrt{y^2+4+4y}-\sqrt{y^2+4-4y}\\ &= \sqrt{(y+2)^2}-\sqrt{(y-2)^2}\\ &= |y+2|-|y-2|.\end{aligned}$$

Certainly $y + 2 > 0$, so

$$y+2-|y-2|=4 \iff y-2=|y-2| \iff y \geq 2.$$

Since $x = y^2 + 4$, the required solution would be $x \geq 8$.

Comment. Squaring both sides of the original equation twice can result in $0 = 0$, which may confuse the careless, but certainly not the readers of the Puzzle Corner!

Clock shop

A clock shop has 10 accurate clocks of various sizes on display. Prove that there exists a moment in time when the sum of all pairwise distances between the tips of the minute hands is greater than the sum of all pairwise distances between the centres of the clocks.

Solution by Ross Atkins: Consider a single pair of clocks whose centres are at points C_1 and C_2. Respectively let A_1, A_2 be the tips of their minute hands half an hour ago, and let B_1, B_2 be the tips of their minute hands right now. However, since C_i is the midpoint of A_iB_i, we have the vector identity

$$\overrightarrow{A_1A_2}+\overrightarrow{B_1B_2}=2\overrightarrow{C_1C_2},$$

which by the triangle inequality yields

$$|A_1A_2|+|B_1B_2| \geq 2|C_1C_2|.$$

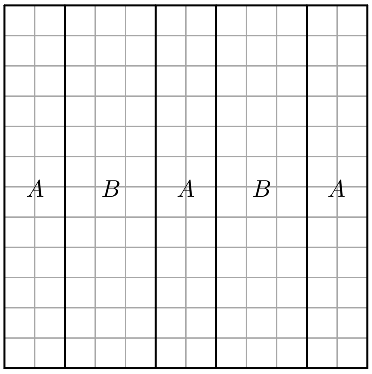
Now if we sum up this expression for all pairs of clocks, we see that the sum of all pairwise distances between the tips of the minute hands, half an hour ago, plus the sum of all pairwise distances between the tips of the minute hands now is greater than twice the sum of the distances between the centres. Therefore one of these minute sums must be larger than the centre sum.

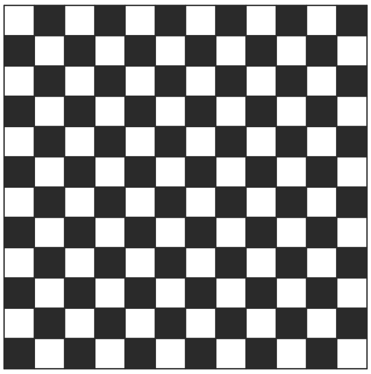
Super knight tour

In a game of super chess, a super knight can move diagonally across a 4×3 rectangle (as opposed to a standard knight which moves diagonally across a 3×2

rectangle). Can the super knight perform a knight tour on a 12×12 *super chessboard, i.e., use a sequence of moves to visit every square exactly once?*

Solution by Norman Do: Divide the super chessboard as follows.

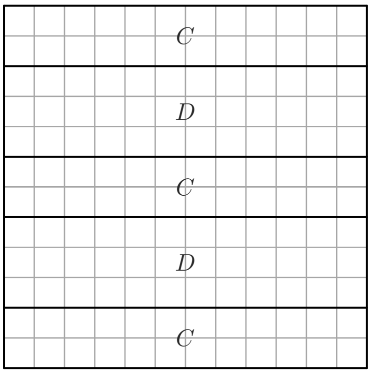




In the first diagram, there are 72 A squares and 72 B squares. A super knight cannot move between two A squares. Now under the usual black-white chessboard colouring, as shown in the second diagram, a super knight has to alternate between black and white squares. Since the A and B regions have an equal number of black squares, any super knight tour must follow the sequence

$$\underbrace{ABAB\cdots ABAB}_{72}\underbrace{BABA\cdots BABA}_{72}.$$

Now apply the same idea but in a different direction.



By similar arguments, the knight tour has to follow

$$\underbrace{CDCD\cdots CDCD}_{72}\underbrace{DCDC\cdots DCDC}_{72}.$$

But that means every C square is also an A square, which is a contradiction. Hence a super knight tour is not possible.

Consecutive sums

(1) *What is the smallest number that can be expressed as a sum of consecutive positive integers in exactly 2010 different ways? Note that a sum must contain at least two summands.*

(2) *Can you find a number which can be expressed as a sum of an even number of consecutive positive integers in exactly 2010 different ways? Can you find one that is smaller than the answer to part (1)?*

Solution by Mike Hirschhorn: We begin with the following theorem [1].

Theorem. The number of partitions of n into an odd number of consecutive positive integers is equal to the number of odd divisors of n less than $\sqrt{2n}$, while the number of partitions into an even number of consecutive positive integers is equal to the number of odd divisors greater than $\sqrt{2n}$. □

Proof. First note that, if n has an odd divisor $d = 2k+1$ and let $d' = n/d$, then

$$d^2 < 2n \iff d < 2d' \iff 2k+1 < 2d' \iff k < d'. \tag{1}$$

Suppose n is the sum of an odd number of consecutive positive integers, then it is possible to write

$$n = (a-b) + \cdots + a + \cdots + (a+b) = (2b+1)a \tag{2}$$

where $b < a$. By (1), setting $d = 2b+1$ and $d' = a$, we must have $d^2 < 2n$. Conversely, suppose $d = 2b+1$ is an odd divisor of n with $d^2 < 2n$ and let $a = n/d$. Again by (1), $b < a$, and it is possible to write n as in (2).

Next, suppose n is the sum of an even number of consecutive positive integers, then

$$n = (a+1-b) + \cdots + a + (a+1) + \cdots + (a+b) = b(2a+1) \tag{3}$$

where $a \geq b$. By the contrapositive of (1), setting $d = 2a+1$ and $d' = b$, we must have $d^2 \geq 2n$. Since d^2 is odd, equality is not possible and $d^2 > 2n$. Conversely, suppose $d = 2a+1$ is an odd divisor of n with $d^2 > 2n$ and let $b = n/d$. Again by the contrapositive of (1), $a \geq b$, and it is possible to write n as in (3). The theorem is proven. □

(1) From the theorem, the number of ways to write n as a sum of consecutive positive integers is equal the number of odd divisors of n. A sum of one term is counted in the theorem, but is not valid for the purpose of the puzzle. Hence we look for numbers with exactly 2011 odd divisors. Since 2011 is prime, the answer is in the form $2^k p^{2010}$ where p is an odd prime. The smallest possibility is $n = 3^{2010}$.

(2) Again from the theorem, we look for n with exactly 2010 odd divisors greater than $n\sqrt{2}$. One example would be $n = 3^{4020}$. It has 4021 divisors, 2010 of which are greater than $3^{2010}\sqrt{2}$.

The following solutions are less than 3^{2010} (which has 960 digits).

n	Approximate value	Number of digits
$2 \times 3^2 \times 5^{1340}$	7.50×10^{937}	938
$2 \times 3^{1340} \times 7^2$	2.16×10^{641}	642
$3^{1340} \times 7^2$	1.08×10^{641}	642
$3^{804} \times 23^4$	1.13×10^{389}	390
$3^{268} \times 79^{14}$	2.72×10^{154}	155

There are many more, but after a *tremendous* effort, the smallest solution is conjectured to be

$$3^6 \times 5^4 \times 7^4 \times 11^2 \times 13 \times 17 \times 769 = 22492916406088125.$$

References

[1] Hirschhorn, M.D. and Hirschhorn, P.M. (2005). Partitions into consecutive parts. *Math. Mag.* **78,** 396–397.

Ivan is a PhD student in the School of Mathematics and Statistics at The University of Sydney. His current research involves a mixture of multi-person game theory and option pricing. Ivan spends much of his spare time playing with puzzles of all flavours, as well as Olympiad Mathematics.