Mathematics contest problems: Please donate generously!

Norman Do*

About once every three years, the Senior Problems Committee of the Australian Mathematical Olympiad Committee (AMOC) turns to the mathematical community to donate contest problems. Previous appeals by my predecessor Hans Lausch attracted submissions that ended up on both national and international mathematical olympiads. Often, the inspiration for composing such problems strikes while reading research papers or while carrying out research. So I encourage you to be vigilant and to submit your problems — perhaps even just kernels of ideas for problems — to me via email. As always, due credit will be given to all problem donors.

Olympiad problems rely only on pre-calculus mathematics and are often broadly classified into the following four areas: algebra, combinatorics, geometry, and number theory. The role of the AMOC Senior Problems Committee is to write the papers for two national competitions and to submit problems for consideration at two international competitions. To give some idea of what we are looking for, we briefly describe these four competitions below and present an example problem from each. These have been submitted by members of the Australian mathematical community in the past three years. The hope is that many more of you will come forward with your problem creations over the coming years.

- **AMOC Senior Contest**

  Approximately 80 Australian students up to Year 11 sit this competition in August each year. The following geometry problem was composed by Angelo Di Pasquale and appeared as Problem 1 on the 2014 AMOC Senior Contest.

  > Each point in the plane is labelled with a real number. For each cyclic quadrilateral $ABCD$ in which the line segments AC and BD intersect, the sum of the labels at A and C equals the sum of the labels at B and D.
  >
  > Prove that all points in the plane are labelled with the same number.

- **Australian Mathematical Olympiad (AMO)**

  Approximately 100 Australian students up to Year 12 sit this competition in February each year. The following combinatorics problem was composed by Andrew Elvey Price and appeared as Problem 8 on the 2015 AMO.

  > Let n be a given integer greater than or equal to 3. Maryam draws n lines in the plane such that no two are parallel. For each equilateral triangle formed by three of the lines, Maryam receives three

*Chair of the AMOC Senior Problems Committee. Email: norm.do@monash.edu

apples. For each non-equilateral isosceles triangle formed by three of the lines, she receives one apple.

What is the maximum number of apples that Maryam can obtain?

- **Asian Pacific Mathematics Olympiad (APMO)**

Approximately 30 Australians students up to Year 12 sit this competition in March each year. In 2015, the Australian contingent was placed ninth out of a total of 33 countries. The following number theory problem was composed by Angelo Di Pasquale and appeared as Problem 2 on the 2016 APMO.

A positive integer is called *fancy* if it can be expressed in the form

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}},$$

where $a_1, a_2, \ldots, a_{100}$ are non-negative integers that are not necessarily distinct.

Find the smallest positive integer n such that no multiple of n is a fancy number.

- **International Mathematical Olympiad (IMO)**

Australia sends a team of six students up to Year 12 to take part in this international competition in July each year. In 2015, the Australian team were placed sixth out of a total of 104 countries. This was Australia's best performance since first competing at the IMO in 1981. The following algebra problem was composed by Ross Atkins and Ivan Guo and appeared as Problem 6 on the 2015 IMO.

The sequence $a_1, a_2, \ldots$ of integers satisfies the following conditions:

(i) $1 \leq a_j \leq 2015$ for all $j \geq 1$;

(ii) $k + a_k \neq \ell + a_\ell$ for all $1 \leq k < \ell$.

Prove that there exist two positive integers b and N such that

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \leq 1007^2$$

for all integers m and n satisfying $n > m \geq N$.

Remarkably, this problem is inspired by juggling! The original idea for the problem came about while Ross Atkins was reading the paper 'Positroid Varieties: Juggling and Geometry' by Allen Knutson, Thomas Lam and David Speyer. (Thomas Lam was actually a member of the 1997 Australian IMO team and a recipient of an IMO gold medal.) Intuitively, each term a_i in the sequence corresponds to throwing a ball at the ith second with an air time of a_i. The inequality condition ensures that no two balls land simultaneously. The problem was considered to be very difficult, with a full solution for the problem obtained by only 11 of the 577 participants at the 2015 IMO.