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# CALDERÓN-TYPE REPRODUCING FORMULAE ON LIPSCHITZ CURVES AND SURFACES

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**Abstract**

Calderón type reproducing formulae with applications have been studied on one- and higher-dimensional Lipschitz graphs. In this note we study higher order Calderón reproducing formulae on star-shaped and non-star-shaped closed Lipschitz curves and surfaces.

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## 0. Introduction

Function spaces and singular integrals on curves and surfaces (see [5, 6, 3, 2, 8, 7]) are closely related to boundary value problems on the same type of curves and surfaces. There have been growing interests in non-smooth, viz. Lipschitz-types, curves and surfaces (see [14] and [9]). Of technical importance in dealing with the above mentioned problems is Littlewood-Paley decomposition of functions. In our notation it is continuous (integral) and discrete types of Calderón reproducing formulae. Besides the direct use of the integral type Calderón's reproducing formulae, one can construct discrete type wavelet frames on curves and surfaces using the integral formulae. Examples of this approach can be found in [15] and [13] concerning function spaces and operator theory on Lipschitz graphs. The latter solves a long standing open problem on giving a constructive proof of the result that any BMO function on a Lipschitz graph can be decomposed into a sum of two functions of which one is a bounded function and the other is the Cauchy singular integral of a bounded function.

This generalizes Uchiyama's constructive proof of the famous Fefferman-Stein result. The study in [13] deals with one dimensional graphs only. In this paper we generalize the methods of [13] to closed, star- and non-start-shaped, Lipschitz curves, and further to higher dimensional surfaces.

## 1. On star-shaped Lipschitz curves

Throughout this paper $l$ is arbitrary, but fixed positive integer representing the order of the Calderón reproducing formula under study. Calderon's reproducing formula on Lipschitz graphs in relation to Cauchy's formula is studied in our context in [3] and [6]. The formula on one-dimensional Lipschitz graphs reads as follows. Let $\tilde{G}$ be the graph of a Lipschitz function defined on the whole real line. For $f \in L^p(\tilde{G})$, $1 < p < \infty$, there holds

$$\text{p.v.} \int_{-\infty}^{\infty} \tilde{J}_t^2 f(z) \frac{dt}{t} = (-1)^l C_l f(z), \quad \text{a.e. } z \in \tilde{G},$$

where $C_l = 2^{-2l}(2l-1)!$, $\tilde{J}_t f(z) = t^l \tilde{F}^{(l)}(z+it)$, $\tilde{F}^{(j)} = d^l \tilde{F}/dz^l$, and

$$\tilde{F}(z) = \frac{1}{2\pi i} \int_{\tilde{G}} \frac{1}{\eta - z} f(\eta)\, d\eta, \quad z \in \mathbb{C} \setminus \tilde{G},$$

and $\tilde{J}_t$ is the convolution integral operator on the graph with the kernel

$$\tilde{J}_t(z, w) = \frac{l!}{2\pi i} \frac{t^l}{(w - z - it)^{l+1}}.$$

We have the alternative form for the above formula:

$$\text{p.v.} \int_{-\infty}^{\infty} t^{2l-1} \tilde{F}^{(2l)}(z + 2it)\, dt = (-1)^l C_l f(z), \quad \text{a.e. } z \in \tilde{G}.$$

Let $G$ be the graph of a continuous function $G(x) = x + iA(x)$, $-\pi \le x \le \pi$, where $A(-\pi) = A(\pi)$ and $A$ is a Lipschitz function, that is, $A'(x) \in L^\infty([-\pi, \pi])$. Denote by $\gamma$ the star-shaped Lipschitz curve given by the parametric equation $\gamma(x) = e^{iG(x)}$, $-\pi \le x \le \pi$. It is easy to see that $\gamma$ is star-shaped Lipschitz with pole $z = 0$, if and only if it is of this form. Fourier analysis on star-shaped Lipschitz curves is studied in [10].

Denote the Cauchy integral

$$F(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\eta)}{\eta - z}\, d\eta, \quad z \notin \gamma.$$

To stress that $F$ is being induced by the boundary data $f$, we write $F = F(f)$. It has the alternative form as a convolution operator on $\gamma$ using the natural multiplicative structure in $\mathbb{C}$,

$$F(z) = \int_\gamma \phi(z\eta^{-1}) f(\eta)\frac{d\eta}{\eta}, \quad z \notin \gamma,$$

where

$$\phi(z) = \frac{1}{2\pi i}\frac{1}{1-\eta},$$

and the measure $1/2\pi i\ d\eta/\eta$ is the normalized complex measure on $\gamma$

$$\frac{1}{2\pi i}\int_\gamma \frac{d\eta}{\eta} = 1$$

as a consequence of Cauchy's Theorem. For $f \in L^1(\gamma)$, denote

$$I(f) = F(0) = \frac{1}{2\pi i}\int_\gamma f(\eta)\frac{d\eta}{\eta}.$$

Define the circular Dirac operator by

$$\Gamma_\theta f(z) = z\frac{d}{dz} f(z),$$

which is the differential operator along the circle. Indeed, on the circle using $z = e^{i\theta}$, we have

$$\Gamma_\theta F(z) = z\frac{d}{dz}F(z) = i\frac{d}{d\theta}F(e^{i\theta}).$$

The following decomposition is consistent with the Dirac operator decomposition in Section 2,

$$dz = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{1}{r}\Gamma_\theta\right).$$

Introduce the operator

$$J_r f(\xi) = (\ln r)^l (\Gamma_\theta^l F)(\xi r),$$

which is a convolution operator using the natural multiplicative structure of the complex number field. The kernel $J_r(\xi,\eta) = (\ln r)^l(\Gamma_\theta^l \phi)(\eta^{-1}\xi r)$ can be explicitly computed using the expression of $\phi$. It is holomorphic in both $\eta$ and $\xi$, and its mixed $k$th-derivatives in $\eta$ and $\xi$ are dominated by $C_k(\ln r)^l|w - r\xi|^{-l-k-1}$ on $\gamma$. We have

$$(J_r^2 f)(\xi) = (\ln r)^{2l}(\Gamma_\theta^{2l} f)(\xi r^2).$$

Below we will use the notation 'p. v. $\int$'. It does not denote the conventional principal value of the integral, but has the meaning

$$\lim_{\epsilon\to 1+}\int_{\epsilon}^{\infty}+\lim_{\epsilon'\to 1-}\int_{0}^{\epsilon'}.$$

Throughout this note we will adopt this less strict meaning of 'p. v.' appropriate to the context.

The continuous type Calderón reproducing formula for star-shaped Lipschitz curves is given by

THEOREM 1. *Let* $f\in L^p(\gamma)$, $1<p<\infty$. *Then*

$$\text{p. v.}\int_0^{\infty}(J_r^2 f)(\xi)\frac{dr}{r\ln r}=C_l(f(\xi)-I(f)),\quad \text{a. e. } \xi\in\gamma, \tag{1}$$

*where the* 'p. v.' *integral is with respect to* $r=1$.

PROOF. Changing variable $e^{-t}=r$, it suffices to prove

$$\text{p. v.}\int_{-\infty}^{\infty}t^{2l-1}(\Gamma_\theta^{2l}F)(\xi e^{-2t})\,dt=C_l(f(\xi)-I(f)),\quad \text{a. e. } \xi\in\gamma. \tag{2}$$

A direct calculation gives

$$(\Gamma_\theta F)(\xi e^{-2t})=\left(z\frac{d}{dz}F\right)(\xi e^{-2t})=-2^{-1}\left(\frac{d}{dt}\right)(F(\xi e^{-2t})),$$

and therefore,

$$(\Gamma_\theta^k F)(\xi e^{-2t})=(-2)^{-k}\left(\frac{d}{dt}\right)^{2k}(F(\xi e^{-2t})),\quad k\in\mathbb{Z}^+. \tag{3}$$

Using integration by parts, the left-hand side of (2) is equal to

$$\begin{aligned}
&2^{-2l}\,\text{p. v.}\int_{-\infty}^{\infty}t^{2l-1}d\left(\frac{d}{dt}\right)^{2l-1}(F(\xi e^{-2t}))\\
&\quad=2^{-2l}\Bigg[\lim_{t\to+\infty}t^{2l-1}\left(\frac{d}{dt}\right)^{2l-1}(F(\xi e^{-2t}))-\lim_{t\to-\infty}t^{2l-1}\left(\frac{d}{dt}\right)^{2l-1}(F(\xi e^{-2t}))+\\
&\qquad+\lim_{t\to 0-}t^{2l-1}\left(\frac{d}{dt}\right)^{2l-1}(F(\xi e^{-2t}))-\lim_{t\to 0+}t^{2l-1}\left(\frac{d}{dt}\right)^{2l-1}(F(\xi e^{-2t}))-\\
&\qquad-(2l-1)\,\text{p. v.}\int_{-\infty}^{\infty}t^{2l-2}d\left(\frac{d}{dt}\right)^{2l-2}(F(\xi e^{-2t}))\Bigg].
\end{aligned}$$

Now we show that the four limits are all zero. Note that for some constants $c_k$, $k = 1, 2, \dots, 2l-1$,

$$\Gamma_\theta^{2l-1} = \left(z\frac{d}{dz}\right)^{2l-1} = \sum_{k=1}^{2l-1} c_k z^k \frac{d^k}{dz^k} \tag{4}$$

and

$$\left(\frac{d^k}{dz^k}\right) F(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f(\eta)}{(\eta - z)^{1+k}} d\eta.$$

The relation (3) implies

$$t^{2l-1}\left(\frac{d}{dt}\right)^{2l-1}(F(\xi e^{-2t})) = (-2)^{2l-1}\sum_{k=1}^{2l-1}\frac{c_k k! t^{2l-1}}{2\pi i}\int_\gamma \frac{(\xi e^{-2t})^k}{(\eta - \xi e^{-2t})^{1+k}} f(\eta)\, d\eta. \tag{5}$$

Using the Lebesgue Convergence Theorem to each of the entries in the summation, we conclude that the first limit is zero as $t \to +\infty$.

Now we study the second limit corresponding to $t \to -\infty$. Using Hölder's inequality, we have, for $q \in (1, \infty)$, $1/q + 1/p = 1$ and $t$ being sufficiently close to $-\infty$,

$$\begin{aligned}|t^{2l-1}\int_\gamma \frac{(\xi e^{-2t})^k}{(\eta - \xi e^{-2t})^{1+k}} f(\eta)\, d\eta| &\le t^{2l-1}\left(\int_\gamma \frac{|\xi e^{-2t}|^{kq}}{|\eta - \xi e^{-2t}|^{(1+k)q}}|\, d\eta|\right)^{1/q} \|f\|_{L^p(\gamma)} \\ &\le c_{k,q,\gamma} t^{2l-1} e^{2t}\|f\|_{L^p(\gamma)}, \quad 1 \le k \le 2l-1.\end{aligned}$$

This shows that the second limit is zero.

Next we prove that the third and fourth limits are zero. Let $\mathscr{A}$ denote the class of the functions holomorphic in some annulus containing $\gamma$. It can be shown that $\mathscr{A}$ is dense in $L^p(\gamma)$, $1 < p < \infty$ [10]. We show that $\Gamma_\theta$ commutes with $F$, that is, $\Gamma_\theta F(f) = F(\Gamma_\theta f)$. In fact, using Laurent series expansions of functions in $\mathscr{A}$ and that of the function $\phi$, and invoking Cauchy's theorem for $z \notin \gamma$, we have

$$\begin{aligned}(\Gamma_\theta F)(z) &= \frac{1}{2\pi i}\int_\gamma (\Gamma_\theta)_z \phi(z\eta^{-1}) f(\eta)\frac{d\eta}{\eta} \\ &= \frac{1}{2\pi i}\int_\gamma \phi(z\eta^{-1})(\Gamma_\theta f)(\eta)\frac{d\eta}{\eta} = \frac{1}{2\pi i}\int_\gamma \frac{(\Gamma_\theta f)(\eta)}{\eta - z}\, d\eta.\end{aligned}$$

Denote by $f^+$ and $f^-$ the two parts, corresponding to the positive and negative powers, in the Laurent expansion of $f$. Hence, $f^+$ is holomorphic inside a disc containing $\gamma$ and $f^-$ is holomorphic outside a disc containing $\gamma$. For $t > 0$,

$$(\Gamma_\theta^{2l-1} F)(\xi e^{-2t}) = \frac{1}{2\pi i}\int_\gamma \frac{(\Gamma_\theta^{2l-1} f^+)(\eta)}{\eta - z}\, d\eta$$

and, for $t < 0$, we have

$$(\Gamma_\theta^{2l-1}F)(\xi e^{-2t}) = \frac{1}{2\pi i}\int_\gamma \frac{(\Gamma_\theta^{2l-1}f^-)(\eta)}{\eta - z}\,d\eta.$$

Letting $t \to 0\pm$, respectively, by virtue of the Plemelj theorem in the context, we have

$$\lim_{t\to 0\pm}(\Gamma_\theta^{2l-1}F)(\xi e^{-2t}) = \lim_{t\to 0\pm}\frac{1}{2\pi i}\int_\gamma \frac{(\Gamma_\theta^{2l-1}f^\pm)(\eta)}{\eta - z}\,d\eta = \Gamma_\theta^{2l-1}f^\pm(\xi), \quad \xi \in \gamma.$$

This implies that, for $f \in \mathscr{A}$,

$$\lim_{t\to 0\pm} t^{2l-1}\left(\frac{d}{dt}\right)^{2l-1}(F(\xi e^{-2t})) = 0.$$

Now we consider $f \in L^p(\gamma)$. We prove that for $t \in (0, \delta)$, where $\delta$ is a fixed positive number less than 1, the operator

$$T^+_{2l-1}f(\xi) = \sup_{0<t<\delta} t^{2l-1}\left(\frac{d}{dt}\right)^{2l-1}(F(\xi e^{-2t}))$$

is dominated by the maximal function $Mf^+$, where $f = f^+ + f^-$, $f^\pm$ are the Hardy space components of $f$ inside or outside $\gamma$, respectively. To prove this, we use the relation (5), and show that each entry of the summation is dominated by the maximum function. Notice that every entry in (5)

$$\left|t^{2l-1}\int_\gamma \frac{(\xi e^{-2t})^k}{(\eta - \xi e^{-2t})^{1+k}}f(\eta)d\eta\right| \le c_{k,\delta}\int_\gamma \left|\frac{t^k}{(1-\eta^{-1}\xi e^{-2t})^{1+k}}\right| |f^+(\eta)|\,d\eta,$$

it is sufficient to show that if $\eta$ and $\xi$ are close enough, then the kernel of the above integral is dominated by a Poisson type kernel. In fact, using the parametric expression of $\gamma$, we have $\eta = e^{i(x+iA(x))}$, $\xi = e^{i(y+iA(y))}$. If $x$ and $y$ are close enough modulo $2\pi$, then

$$\left|1 - e^{i(x-y)-(A(x)-A(y)+2t)}\right| \ge c\left((x-y)^2 + (A(x)-A(y)+2t)^2\right)^{1/2}.$$

If the Lipschitz constant $N$ of $\gamma$ is less than 1, then it is easy to show, using the elementary inequality $a^2 \ge (a+b)^2/2 - b^2$, that

$$(x-y)^2 + (A(x)-A(y)+2t)^2 \ge c_N[(x-y)^2 + t^2].$$

If the Lipschitz constant $N$ of $\gamma$ is greater than or equal to 1, then using the same inequality we have

$$\begin{aligned}(x-y)^2 + (A(x)-A(y)+2t)^2 &\ge (x-y)^2 + (A(x)-A(y)+2t)^2/(2N)\\ &\ge c_N[(x-y)^2 + t^2].\end{aligned}$$

And so

$$\left|\frac{t^k}{(1-\eta^{-1}\xi e^{-2t})^{1+k}}\right| \leq c\,\frac{t^k}{((x-y)^2+t^2)^{(1+k)/2}}.$$

The right-hand side is a Poisson type kernel. An analogous argument gives that for some $\delta' \in (-1, 0)$, the operator

$$T^{-}_{2l-1} f(\xi) = \sup_{0>t>\delta'} t^{2l-1}\left(\frac{d}{dt}\right)^{2l-1}(F(\xi e^{-2t}))$$

is dominated by $Mf^{-}$.

Denote by $m_\gamma$ the arc-length measure on $\gamma$. Let $\lambda > 0$, $f \in L^p(\gamma)$, $g \in \mathscr{A}$ and $F(g)$ be the Cauchy integral of $g$. We have

$$\begin{aligned}
m_\gamma&\left(\left\{\overline{\lim_{\delta>t\to 0+}}\left|t^{2l-1}\left(\frac{d}{dt}\right)^{2l-1}(F(\xi e^{-2t}))\right| > \lambda\right\}\right)\\
&\leq m_\gamma(\{T^{+}_{2l-1}(f-g)(\xi) > \lambda/2\})\\
&\quad + m_\gamma\left(\left\{\overline{\lim_{\delta>t\to 0+}}\left|t^{2l-1}\left(\frac{d}{dt}\right)^{2l-1}(F(g)(\xi e^{-2t}))\right| > \lambda/2\right\}\right)\\
&\leq m_\gamma(\{cM(f-g)^{+}(\xi) > \lambda/2\}) \leq c_p\left(\|f-g\|_{L^p(\gamma)}/\lambda\right)^p.
\end{aligned}$$

Since $\mathscr{A}$ is dense in $L^p(\gamma)$, the last entry may be made as small as we want. This concludes that the fourth limit is zero. Similarly, we can show that the third limit is zero.

Repeating this argument, we have

$$\begin{aligned}
2^{-2l}\,\mathrm{p.v.}&\int_{-\infty}^{\infty} t^{2l-1} d\left(\frac{d}{dt}\right)^{2l-1}(F(\xi e^{-2t}))\\
&= 2^{-2l}[-(2l-1)]\,\mathrm{p.v.}\int_{-\infty}^{\infty} t^{2l-2} d\left(\frac{d}{dt}\right)^{2l-2}(F(\xi e^{-2t})) = \cdots\\
&= 2^{-2l}[-(2l-1)!]\,\mathrm{p.v.}\int_{-\infty}^{\infty} d(F(\xi e^{-2t})) = 2^{-2l}(2l-1)!(f(\xi)-I(f)),
\end{aligned}$$

by involving the Plemelj theorem in the context. The proof is complete. □

## 2. Star-shaped Lipschitz surfaces

Denote by $\mathbb{R}^n = \{x_1e_1 + \cdots + x_ne_n : x_i \in \mathbb{R},\ i = 1, \ldots, n\}$, where $e_i^2 = -1$, $e_ie_j = -e_je_i$, $i \neq j$, $i, j = 1, \ldots, n$. Let $\Sigma$ be a star-shaped Lipschitz surface in

$\mathbb{R}^n$ whose pole is at the origin. We recall that both the real $2^n$-dimensional Clifford algebra $\mathbb{R}^{(n)}$ and the complex $2^n$-dimensional Clifford algebra $\mathbb{C}^{(n)}$ have basis vectors $e_S$, where $S = \langle j_1, \ldots, j_s \rangle$, $1 \le s \le n$, $1 \le j_1 < \cdots < j_s$, $j_k \in \mathbb{Z}$ and $e_S = e_{j_1} \cdots e_{j_s}$. For $S = \emptyset$, we identify $e_\emptyset$ with $e_0 = 1$.

Let $F$ be a real or complex Clifford-valued function defined in an open set $\Omega \in \mathbb{R}^n$ and $F(x) = \sum_S e_S F_S$, where $S$ runs over the above described ordered subsets of $\{1, \ldots, n\}$. $F$ is said to be *left-monogenic* in $\Omega$, if

$$DF(x) = 0, \quad x \in \Omega,$$

where $D = e_1\partial_1 + \cdots + e_n\partial_n$, and $DF(x) = \sum_i \sum_S e_i e_S (\partial F_S/\partial x_i)$ (see, for example, [1] or [4]). Similarly we can define *right-monogenic* functions.

A surface is called a *star-shaped Lipschitz surface*, if it is star-shaped and locally Lipschitz (see [9]). Fourier analysis on star-shaped Lipschitz surfaces is studied in [11, 12].

For $f \in L^1(\Sigma)$, the function

$$F(x) = F(f)(x) = \frac{1}{\omega_{n-1}} \int_\Sigma E(y-x)n(y)f(y)\,d\sigma(y), \quad x \in \mathbb{R}^n \setminus \Sigma, \tag{6}$$

is left-monogenic, where $n(y)$ is the outward normal of $\Sigma$ at $y \in \Sigma$, $d\sigma(y)$ is the surface area on $\Sigma$, $E$ is the Cauchy kernel $E(y) = -y/|y|^n$, and $\omega_{n-1}$ the surface area of the $(n-1)$-dimensional unit sphere.

The function $F(x)$ may be re-written as

$$F(x) = \frac{1}{\omega_{n-1}} \int_\Sigma H(y^{-1}x)E(y)n(y)f(y)\,d\sigma(y), \tag{7}$$

where $H(x) = E(x-1)$. Formally the above is a convolution integral with kernel $H$ using the multiplication of Clifford numbers.

The spherical Dirac operator $\Gamma_\eta$ is defined through the decomposition

$$D = \eta\left(\partial_r + \frac{1}{r}\Gamma_\eta\right)$$

for the polar coordinate $x = r\eta$, $|\eta| = 1$. Note that the decomposition is obtained by expressing the Dirac operator $D$ in the spherical coordinates of $\mathbb{R}^n$. For any left-monogenic function $f$ the above decomposition gives $r\partial_r f = \Gamma_\eta f$.

The Calderón reproducing formula for star-shaped Lipschitz surfaces has the same form.

THEOREM 2. *For $f \in L^p(\Sigma)$, $1 < p < \infty$, we have*

$$\text{p. v.} \int_0^\infty J_r^2 f(x) \frac{dr}{r \ln r} = C_l(f(x) - F(0)), \quad \text{a. e. } x \in \Sigma, \tag{8}$$

*where $J_r f(x) = (\ln r)^l (\Gamma_\eta^l F)(xr)$, and the* ‘p. v.’ *integral is with respect to $r = 1$.*

Note that $J_r$ is a convolution integral operator on the surface. The mixed $k$th derivatives of its kernel $J_r(x, y) = (\ln r)^l(\Gamma_\eta^l H)(y^{-1}xr)$ is left-monogenic in the $x$-variable and dominated by $C_k(\ln r)^l|y - rx|^{-l-k-n+1}$ on the surface. The explicit expression of the kernel may be computed using the relation $\Gamma_\eta = r\partial_r$ on left-monogenic functions.

PROOF. By changing variable $e^{-t} = r$, we note that it is equivalent to show

$$\text{p.v.} \int_{-\infty}^{\infty} t^{2l-1}(\Gamma_\eta^{(2l)} F)(xe^{-2t})\, dt = C_l(f(x) - F(0)), \quad \text{a.e. } x \in \Sigma.$$

Since $F$ is left-monogenic, we have $(\Gamma_\eta F)(xe^{-2t}) = (r\partial_r F)(xe^{-2t})$. For a fixed $t$, it is easy to verify that, for $x = r\eta$, $|\eta| = 1$,

$$(r\partial_r F)(xe^{-2t}) = (r\partial_r)(F(xe^{-2t}))$$

and

$$(r\partial_r)(F(\eta re^{-2t})) = -2^{-1}\frac{d}{dt}(F(\eta re^{-2t})).$$

Therefore, we have

$$(\Gamma_\eta^{2l} F)(xe^{-2t}) = 2^{-2l}\left(\frac{d}{dt}\right)^{2l}(F(xe^{-2t})).$$

So it suffices to prove

$$2^{-2l}\, \text{p.v.} \int_{-\infty}^{\infty} t^{2l-1}\left(\frac{d}{dt}\right)^{(2l)}(F(xe^{-2t}))\, dt = C_l(f(x) - F(0)), \quad \text{a.e. } x \in \Sigma. \tag{9}$$

The proof of Theorem 1 can be adapted to show (9). We only note that the formula (4) should be replaced by

$$\Gamma_\eta^{2l-1} = \left(r\frac{d}{dr}\right)^{2l-1} = \sum_{k=1}^{2l-1} c_k r^k \frac{d^k}{dr^k} \tag{10}$$

on monogenic functions; the proof of the commutativity between $F$ and $\Gamma_\eta$ follows the same line ([11, page 624] also [12]); and the local Poisson kernel property is proved using local coordinates. □

## 3. Non-star-shaped closed Lipschitz surfaces

Let $\Omega$ be a simply-connected bounded Lipschitz domain in $\mathbb{R}^n$ and $\Sigma$ its boundary. In this section we assume $\Sigma$ is not star-shaped. Let $f \in L^1(\Sigma)$, then, as before,

$$F(x) = \frac{1}{\omega_{n-1}} \int_\Sigma E(y - x)n(y)f(y)\, d\sigma(y) \tag{11}$$

is well defined and left-monogenic for $x \in \mathbb{R}^n \setminus \Sigma$.

It is easy to show (see also [7]) that there exists a constant $h > 0$, depending on the Lipschitz constant of $\Sigma$, such that for every $x \in \Sigma$ at which there exists a tangent plane to $\Sigma$, the interval segment $(x, x - 4hn(x))$ is entirely contained in $\Omega$ and the interval segment $(x + 4hn(x), x)$ is entirely contained in $\mathbb{R}^n \setminus (\Sigma \cup \Omega)$.

Denote by $\partial_{n(x)}$ the directional derivative in the direction $n(x)$. Introducing the pseudo-differential operator

$$J_{(x,t)} f(x) = t^l (\partial^l_{n(x)} F)(x - tn(x)),$$

whose kernel is dominated by $Ct^l |y - x - tn(x)|^{-l-n+1}$, we have

THEOREM 3. *For $f \in L^p(\Sigma)$, $1 < p < \infty$, we have*

$$(-1)^l C_l f(x) = \text{p. v.} \int_{-h}^{h} \left(J^2_{(x,t)} f\right)(x - tn(x)) \frac{dt}{t} + \frac{1}{\omega_{n-1}} \int_\Sigma G(x, y) n(y) f(y)\, d\sigma(y), \quad \text{a. e. } x \in \Sigma,$$

*where*

$$G(x, y) = (-1)^{l+1} 2^{-2l} \sum_{k=1}^{2l} (-1)^{k-1} h^{2l-k} \left(\frac{d}{dt}\right)^{2l-k} (E(y - x + 2tn(x)))|_{t=h} + h^{2l-k} \left(\frac{d}{dt}\right)^{2l-k} (E(y - x + 2tn(x)))|_{t=-h},$$

*is an integrable kernel in $y$ for* a. e. $x \in \Sigma$ *at which there exists a tangent plane to* $\Sigma$.

PROOF. Taking into account that $n(x)^2 = -1$, a similar argument as in Section 2 gives

$$\begin{aligned}
&\text{p. v.} \int_{-h}^{h} t^{2l-1} (\partial^{2l}_{n(x)} F)(x - 2tn(x))\, dt \\
&= (-1)^l 2^{-2l}\, \text{p. v.} \int_{-h}^{h} t^{2l-1} \left(\frac{d}{dt}\right)^{2l} (F(x - 2tn(x))\, dt \\
&= (-1)^l 2^{-2l} \Bigg[ \sum_{k=1}^{2l} (-1)^{k-1} h^{2l-k} \left(\frac{d}{dt}\right)^{2l-k} (F(x - 2tn(x)))\, \Big|_{t=h} \\
&\quad + h^{2l-k} \left(\frac{d}{dt}\right)^{2l-k} (F(x - 2tn(x)))\, \Big|_{t=-h} + (2l - 1)!\, f(x) \Bigg]
\end{aligned}$$

owing to the Plemelj theorem in the context. Thus we have

$$(-1)^l C_l f(x) = \text{p. v.} \int_{-h}^{h} t^{2l-1} (\partial^{2l}_{n(x)} F)(x - tn(x))\, dt$$

$$+(-1)^{l+1}2^{-2l}\sum_{k=1}^{2l}(-1)^{k-1}h^{2l-k}\left(\frac{d}{dt}\right)^{2l-k}(F(x-2tn(x)))\,\Big|_{t=h}$$
$$+h^{2l-k}\left(\frac{d}{dt}\right)^{2l-k}(F(x-2tn(x)))\,\Big|_{t=-h}.$$

Using the definition of $J_{(x,t)}$ we obtain the desired formula. □

The reproducing formula obtained in Theorem 3 is not of the same type as that in Theorem 2 but involves a remainder as a well-behaved convolution integral operator. It may be a shortcome that the differential directions, $n(x)$, in the definition of $J_{(x,t)}$ is not a smooth vector field on $\Sigma$. For non-star-shaped closed surfaces there does not exist a uniform spherical coordinate system so that the radial direction is non-tangential to almost all points on the surface. Using a *spherical covering* of $\Sigma$, however, we can deduce a similar result to Theorem 2. We proceed as follows.

Denote by $\{S_\alpha\} = \{(p_\alpha, r_\alpha, \eta_\alpha)\}$ a set of spherical coordinate system, where for a fixed $\alpha$, $p_\alpha$ is the pole and $(r_\alpha, \eta_\alpha)$ is the spherical coordinate system with respect to $p_\alpha$. We say that $\{S_\alpha\}$ is a *spherical covering* of $\Sigma$, if the following conditions hold:

(i) for every $\alpha$, $p_\alpha \in \Omega$;

(ii) for every $\alpha$ there exists a simply-connected open set $U_\alpha \subset \mathbb{R}^{n-1}$ such that $x_\alpha(r_\alpha, \eta_\alpha)$, $0 < r_\alpha < \infty$, $\eta_\alpha \in U_\alpha$ is a local parameterization of a part of $\mathbb{R}^n$ in which $\Sigma$ is star-shaped for $\eta_\alpha \in U_\alpha$, and the radial direction is uniformly non-tangential for almost all $\eta_\alpha \in U_\alpha$ (that is, the angle between the radial and the normal directions is dominated by a constant less than $\pi/2$ for almost all points in $x_\alpha(\mathbb{R}^+, U_\alpha) \cap \Sigma$); and

(iii) for the $U_\alpha$'s specified in (ii), $\bigcup_\alpha x_\alpha(\mathbb{R}^+, U_\alpha) \cap \Sigma = \Sigma$.

Owing to (ii) we also have $S_\alpha = (p_\alpha, r_\alpha, \eta_\alpha, U_\alpha)$.

The existence of spherical coverings for non-star-shaped closed Lipschitz surfaces can be easily justified from the definition of Lipschitz domains in terms of local coordinate systems (see [9]).

For any point $p \in \mathbb{R}^n$, denote by $f_p$ the function $f(\cdot + p)$ and by $\Sigma_p$ the surface $\{y - p \mid y \in \Sigma\}$. We have the following theorem.

THEOREM 4. *Let $\Sigma$ be a non-star-shaped closed Lipschitz surface, $f \in L^p(\Sigma)$, $1 < p < \infty$ and let $\{S_\alpha\}$ be a spherical covering of $\Sigma$. Then*

$$\text{p. v.} \int_0^\infty J_r^2 f(x) \frac{dr}{r \ln r} = C_l(f(x) - F(p_\alpha)), \tag{12}$$

*where $r = r_\alpha$ is the radial parameter of $S_\alpha = (p_\alpha, r_\alpha, \eta_\alpha, U_\alpha)$, $x \in x_\alpha(\mathbb{R}^+, U_\alpha) \cap \Sigma$,*

$$J_r f(x) = (\ln r)^l (\Gamma_\eta^l F_{p_\alpha})((x - p_\alpha)r),$$

*where* $\Gamma_\eta = \Gamma_{\eta_\alpha}$, $F$ *is defined by* (11), *the* 'p. v.' *integral is with respect to* $r = 1$.

PROOF. For $x \in x_\alpha(\mathbb{R}^+, U_\alpha) \cap \Sigma$, the formula (10) can be rewritten as

$$F_{p_\alpha}(x - p_\alpha) = \frac{1}{\omega_{n-1}} \int_{\Sigma_{p_\alpha}} E(y - (x - p_\alpha))n(y) f_{p_\alpha}(y)\, d\sigma(y).$$

In the proof of Theorem 2 only local Lipschitz property of $\Sigma$ is used. So we can use the same proof to show

$$\text{p. v.} \int_0^\infty J_r^2 f(x) \frac{dr}{r \ln r} = C_l(f(x) - F_{p_\alpha}(0)).$$

Since $F_{p_\alpha}(0) = F(p_\alpha)$, we obtain the desired formula. $\square$

Note that for $n = 2$, Theorem 3 and Theorem 4 provide the formulae for the boundaries of simply-connected bounded Lipschitz domains in the complex plane.

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