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A DECOMPOSITION THEOREM FOR HOMOGENEOUS ALGEBRAS

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Dedicated to Professor D. Ž. Djoković

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Abstract

An algebra A is *homogeneous* if the automorphism group of A acts transitively on the one dimensional subspaces of A. Suppose A is a homogeneous algebra over an infinite field $\mathbf{k}$. Let L_a denote left multiplication by any nonzero element $a \in A$. Several results are proved concerning the structure of A in terms of L_a. In particular, it is shown that A decomposes as the direct sum $A = \ker L_a \oplus \operatorname{Im} L_a$. These results are then successfully applied to the problem of classifying the infinite homogeneous algebras of small dimension.

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1. Introduction

The algebras to be discussed are assumed to be finite dimensional over a field $\mathbf{k}$ and are not necessarily associative. We call an algebra A *nontrivial* if $\dim A > 1$ and $A^2 \neq 0$. Also, $\operatorname{Aut}(A)$ will denote the group of algebra automorphisms of A.

An algebra A is *homogeneous* if $\operatorname{Aut}(A)$ acts transitively on the one-dimensional subspaces of A. This is a very strong condition indeed and the known examples fall into two easily described classes. The existence of homogeneous algebras depends critically on the choice of $\mathbf{k}$, the field of scalars, and a number of results are known classifying these algebras according to the field. Kostrikin [5] showed how to construct homogeneous algebras of any dimension over the finite field GF(2). Work by

Shult [7], Gross [3] and Ivanov [4] showed that if **k** is finite, then there are no algebras other than those constructed by the method of Kostrikin. Djoković [1] completely classified homogeneous algebras over the reals and found only 3 examples, one each in dimensions 3, 6 and 7. It was shown by Sweet [10] that there are no non-trivial examples whatsoever when the scalar field is algebraically closed.

The first general study of homogeneous algebras was carried out by Sweet [9], and subsequently the authors [6, 8] have completely classified the non-trivial algebras of dimensions 2, 3 and 4 over any field. There it has been shown that no examples exist other than those found by Kostrikin and by Djoković. Recently, motivated by the examples over the reals, Djoković and Sweet [2] have shown that all non-trivial homogeneous algebras over any infinite field satisfy $x^2 = 0$ for all $x \in A$, and hence are anti-commutative.

The main purpose of this paper is to prove the following structure theorem which applies to homogeneous algebras over any infinite field. For any $a \in A$, $L_a : A \to A$ will denote left multiplication by a.

THEOREM. *Let A be a non-trivial homogeneous algebra over an infinite field. Then for any* $a \in A\backslash\{0\}$, $A = \ker L_a \oplus \operatorname{Im} L_a$.

This theorem has a number of interesting consequences regarding the possible structure of infinite homogeneous algebras which will provide important tools in our continuing program of classifying these algebras.

In all that follows, A will be a non-trivial homogeneous algebra over an arbitrary infinite field **k**.

2. Results and proofs

One of the immediate consequences of homogeneity (see [9]) is that all left multiplications are projectively similar. More precisely, for any $a, b \in A$, if $\alpha \in \operatorname{Aut}(A)$ maps a to λb, then $\alpha L_a \alpha^{-1} = \lambda L_b$. This fact has been exploited very successfully in [6] and [8] to classify the homogeneous algebras of dimensions 2, 3 and 4. In particular we use the matrix representation of L_a with respect to some suitably chosen basis. Note that if $a, b \in A\backslash\{0\}$ then $\operatorname{rank} L_a = \operatorname{rank} L_b$. Also if some coefficient of the characteristic polynomial of L_a is zero then the corresponding coefficient of the characteristic polynomial of L_b is also zero.

THEOREM 2.1. *Let A be a nontrivial homogeneous algebra over an infinite field* **k**. *If* $a \in A$, *then* L_a *has no nonzero eigenvalues in* **k**.

PROOF. In [2] it is proved that any homogeneous algebra A over an infinite field has the property that $x^2 = 0$ for all $x \in A$. The theorem then follows from Theorem 3 of [8]. □

Our main result, which is Corollary 2.3, follows from the following theorem.

THEOREM 2.2. *Let A be a nontrivial homogeneous algebra over an infinite field* $\mathbf{k}$. *If $a, b \in A \backslash \{0\}$ and $ab = 0$, then* $\operatorname{Im} L_a = \operatorname{Im} L_b$.

PROOF. Choose a basis $B = \{b_1, b_2, \ldots b_n\}$ of A such that $\{b_1, b_2, \ldots b_s\}$ is a basis of $\ker L_a$ and choose another basis $C = \{c_1, c_2, \ldots c_n\}$ such that $c_i = ab_i$ for $s < i \leq n$. Then the matrix of L_a with respect to the bases B and C is

$$\begin{bmatrix} O_s & O \\ O & I_r \end{bmatrix}$$

where $r = n - s$ is the rank of L_a. Let $x \in A$ be arbitrary and let

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$$

be the matrix of L_x with respect to B and C. Since the rank of $L_x + t L_a = L_{x+ta}$ cannot exceed r for all $t \in k$, we conclude that $X_1 = O$. Hence $x(\ker L_a) = L_x(\ker L_a) \subset \operatorname{Im} L_a$. But x is arbitrary and so $A(\ker L_a) \subset \operatorname{Im} L_a$. Since $b \in \ker L_a$, $Ab = \operatorname{Im} L_b \subset \operatorname{Im} L_a$. But $\operatorname{rank} L_a = \operatorname{rank} L_b$ and so $\operatorname{Im} L_a$ and $\operatorname{Im} L_b$ have the same dimension. It follows that $\operatorname{Im} L_a = \operatorname{Im} L_b$. □

COROLLARY 2.3. *Let A be a nontrivial homogeneous algebra over an infinite field* $\mathbf{k}$. *If $a \in A \backslash \{0\}$, then $A = \ker L_a \oplus \operatorname{Im} L_a$.*

PROOF. Let $b \in \ker L_a \cap \operatorname{Im} L_a$. If $b \neq 0$, then Theorem 2.2 implies that $\operatorname{Im} L_a = \operatorname{Im} L_b$. But then $b \in \operatorname{Im} L_b$, which contradicts Theorem 2.1. Hence $\ker L_a \cap \operatorname{Im} L_a = \{0\}$ and so $A = \ker L_a \oplus \operatorname{Im} L_a$. □

Let A be a nontrivial homogeneous algebra over a field $\mathbf{k}$ and let $a \in A$. If $\mathbf{k}$ is finite it was shown by Shult [7] that L_a is either invertible or nilpotent. If $\mathbf{k}$ is infinite the first case is impossible since $a^2 = 0$. Also, if $\mathbf{k}$ is infinite the above corollary implies that the second case is also impossible. In fact, we have a slightly stronger result.

COROLLARY 2.4. *Let A be a nontrivial homogeneous algebra over an infinite field* $\mathbf{k}$. *If $a \in A \backslash \{0\}$, then L_a cannot have a nonzero nilpotent block in its rational canonical form.*

Corollary 2.3 says that A can be written as a direct sum of the subspaces $\ker L_a$ and $\operatorname{Im} L_a$. We now show that $\ker L_a$ is actually a subalgebra.

THEOREM 2.5. *Let A be a nontrivial homogeneous algebra over an infinite field $\mathbf{k}$. If $a \in A\backslash\{0\}$, then $\ker L_a$ is a zero subalgebra.*

PROOF. Assume $a \in A\backslash\{0\}$ and let $x \in A\backslash\{0\}$ be arbitrary. Using Corollary 2.3, decompose A into $A = \ker L_a \oplus \operatorname{Im} L_a$. Then using a corresponding basis

$$L_a = \begin{bmatrix} O & O \\ O & A_1 \end{bmatrix} \quad \text{and} \quad L_x = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

Using the rank argument as in Theorem 2.2, we conclude that $X_1 = O$ for all $x \in A$. Now let $b \in \ker L_a\backslash\{0\}$. Then

$$L_b = \begin{bmatrix} O & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

But $ab = 0$ and so $\operatorname{Im} L_a = \operatorname{Im} L_b$ by Theorem 2.2. This implies that $B_2 = O$. Also Corollary 2.3 implies that A_1 is nonsingular and so B_4 is also nonsingular since L_a and L_b are projectively similar.

Assume $B_3 \neq O$. Then there exists $c \in \ker L_a$ such that $bc = d \in \operatorname{Im} L_a\backslash\{0\}$. Since B_4 is nonsingular the equation $B_4 x = d$ must have a unique solution $e \in \operatorname{Im} L_a$. But then $bc = be$ and so $b(c - e) = 0$. It follows that $\operatorname{Im} L_b = \operatorname{Im} L_{c-e}$. Since $c \in \ker L_a$, we again can assume that

$$L_c = \begin{bmatrix} O & O \\ C_1 & C_2 \end{bmatrix}.$$

On the other hand,

$$L_e = \begin{bmatrix} O & E_2 \\ E_3 & E_4 \end{bmatrix}.$$

But now $\operatorname{Im} L_{c-e} = \operatorname{Im}(L_c - L_e) = \operatorname{Im} L_b = \operatorname{Im} L_a$, and this implies that $E_2 = 0$. But $e \in \operatorname{Im} L_a$ and $e^2 = 0$ and so E_4 is singular. This is impossible since L_e is projectively similar to L_a.

Hence $B_3 = O$ and therefore $\ker L_a$ is a zero subalgebra. □

COROLLARY 2.6. *Let A be a nontrivial homogeneous algebra over an infinite field $\mathbf{k}$. If $a, b \in A\backslash\{0\}$ and $ab = 0$, then $\ker L_a = \ker L_b$. Also, denoting $\ker L_a\backslash\{0\}$ by $K_a^\star$, the sets $K_a^\star$ partition $A\backslash\{0\}$.*

PROOF. Assume $x \in \ker L_b$. Since $a \in \ker L_b$, Theorem 2.5 implies that $ax = 0$, and so $x \in \ker L_a$. Hence $\ker L_b \subset \ker L_a$ and similarly $\ker L_a \subset \ker L_b$. Hence $\ker L_a = \ker L_b$. The proof of the second conclusion is similar. □

Theorem 2.5 shows that each $\ker L_a$ is a subalgebra. We now show that each $\operatorname{Im} L_a$ is not a subalgebra.

THEOREM 2.7. *Let A be a nontrivial homogeneous algebra over an infinite field* $\mathbf{k}$. *Let $a, b \in A \backslash \{0\}$. If $\operatorname{Im} L_a = \operatorname{Im} L_b$, then $\ker L_a = \ker L_b$.*

PROOF. By using an argument similar to that found in Theorem 2.5 it can be shown that $ab = 0$ and then the result follows directly from Corollary 2.6. □

COROLLARY 2.8. *Let A be a nontrivial homogeneous algebra over an infinite field* $\mathbf{k}$. *If $a \in A \backslash \{0\}$, then $\operatorname{Im} L_a$ is not a subalgebra.*

PROOF. Assume $\operatorname{Im} L_a$ is a subalgebra. Let $A = \ker L_a \oplus \operatorname{Im} L_a$. Suppose $b \in \operatorname{Im} L_a \backslash \{0\}$. Then as before

$$L_b = \begin{bmatrix} O & O \\ B_3 & B_4 \end{bmatrix}.$$

This implies that $\operatorname{Im} L_a = \operatorname{Im} L_b$ and so $\ker L_b = \ker L_a$. Thus $b \in \ker L_a \cap \operatorname{Im} L_a$ which is impossible. □

It is natural to look at the action of an automorphism on $\ker L_a$ and $\operatorname{Im} L_a$. The next result is well known and the proof is easy.

REMARK. Let A be any algebra over a field $\mathbf{k}$. If $a \in A \backslash \{0\}$, $\alpha \in \operatorname{Aut}(A)$ and $\alpha(a) = b$, then $\alpha(\ker L_a) = \ker L_b$ and $\alpha(\operatorname{Im} L_a) = \operatorname{Im} L_b$.

COROLLARY 2.9. *Let A be a nontrivial homogeneous algebra over an infinite field* $\mathbf{k}$. *If $\alpha \in \operatorname{Aut}(A)$, $a \in A \backslash \{0\}$ and $\alpha(\ker L_a) \cap \ker L_a \neq 0$, then $\alpha(\ker L_a) = \ker L_a$ and $\alpha(\operatorname{Im} L_a) = \operatorname{Im} L_a$.*

PROOF. The proof follows easily from the above theorem using Corollary 2.6 and Theorem 2.5. □

We now show that $\operatorname{Aut}(A)$ cannot be abelian if A is a nontrivial homogeneous algebra over an infinite field $\mathbf{k}$ (the result is false when $\mathbf{k}$ is finite). Let $Z(\operatorname{Aut}(A))$ denote the center of $\operatorname{Aut}(A)$.

THEOREM 2.10. *Let A be a nontrivial homogeneous algebra over an infinite field* $\mathbf{k}$. *If $\alpha \in Z(\operatorname{Aut}(A))$ and $a \in A \backslash \{0\}$, then $\alpha(\ker L_a) = \ker L_a$ and $\alpha(\operatorname{Im} L_a) = \operatorname{Im} L_a$.*

PROOF. Let $\alpha \in Z(\mathrm{Aut}(A))$. We define a new multiplication $a \circ b$ on A to make a new algebra A^α as follows:

$$a \circ b = a\,\alpha(b).$$

Then if $\gamma \in \mathrm{Aut}(A)$

$$\gamma(a \circ b) = \gamma(a\,\alpha(b)) = \gamma(a)\,\gamma(\alpha(b)) = \gamma(a)\,\alpha(\gamma(b)) = \gamma(a) \circ \gamma(b)$$

and so $\gamma \in \mathrm{Aut}(A^\alpha)$. Thus A^α is a homogeneous algebra. Hence

$$a \circ a = a\,\alpha(a) = 0.$$

Thus $\alpha(a) \in \ker L_a$, $\forall a \in A$, and the result follows from Corollary 2.9. □

COROLLARY 2.11. *Let A be a nontrivial homogeneous algebra over an infinite field* $\mathbf{k}$. *Then* $\mathrm{Aut}(A)$ *is not abelian.*

PROOF. This follows immediately from Theorem 2.10. □

The remaining theorems use the direct sum decomposition to study the possible dimension of $\ker L_a$.

THEOREM 2.12. *Let A be a nontrivial homogeneous algebra over an infinite field* $\mathbf{k}$. *If $a \in A\backslash\{0\}$, then* $\dim(\ker L_a) < (1/2)\dim(A)$.

PROOF. Let $t = \dim(\ker L_a)$ and $n = \dim(A)$. Assume $t \geq n/2$. Let $\{a_1, a_2, \ldots, a_t\}$ be a basis of $\ker L_a$ and decompose a as $A = \ker L_a \oplus \mathrm{Im}\, L_a$. It follows from Theorem 2.5 that each L_{a_i} is of the form

$$L_{a_i} = \begin{bmatrix} O & O \\ O & A_i \end{bmatrix}.$$

where A_i is a nonsingular $(n-t) \times (n-t)$ matrix.

Since $t \geq n/2$ there exists a nontrivial $b = x_1 a_1 + x_2 a_2 + \cdots + x_t a_t$ such that

$$L_b = \left[\begin{array}{c|cccc} O & & O & & \\ \hline & \star & \cdots & & \star \\ & \star & \cdots & & \star \\ O & \vdots & & & \vdots \\ & \star & \cdots & & \star \\ & 0 & \cdots & 0 & b_{nn} \end{array}\right].$$

But then $b_{nn} = 0$ by Theorem 2.1, and hence L_b is not projectively similar to L_a. This is a contradiction and hence $t < n/2$. □

THEOREM 2.13. *Let A be a nontrivial homogeneous algebra over an infinite field* $\mathbf{k}$*. If $n = \dim A$ is odd and $n > 3$, then for $a \in A\backslash\{0\}$, $\dim(\ker L_a) < (n-1)/2$.*

PROOF. By Theorem 2.12 we know that $t = \dim(\ker L_a) < n/2$ and so it suffices to show that $t \neq (n-1)/2$. Assume otherwise. Decompose A as $A = \ker L_a \oplus \operatorname{Im} L_a$. Let $b \in \operatorname{Im} L_a$ and let $B = \{b, b_2, \ldots, b_t\}$ be a basis for $\ker L_b$. Each b_i can be written uniquely as

$$b_i = a_i + b_i',$$

where $a_i \in \ker L_a$ and $b_i' \in \operatorname{Im} L_a$. Let $B' = \{b, b_2', \ldots, b_t'\}$. We claim that B' is an independent set. For suppose $\lambda_1 b + \lambda_2 b_2' + \ldots + \lambda_t b_t' = 0$. Then

$$a(\lambda_1 b + \lambda_2 b_2' + \cdots + \lambda_t b_t') = a(\lambda_1 b + \lambda_2 b_2 + \cdots + \lambda_t b_t) = 0.$$

So $\lambda_1 b + \lambda_2 b_2 + \cdots + \lambda_t b_t \in \ker L_a \cap \ker L_b = \{0\}$. But B is an independent set, and so B' must also be an independent set.

Let c be any vector in the complement of the span of B' in $\operatorname{Im} L_a$. Then $B' \cup \{c\}$ is a basis of $\operatorname{Im} L_a$. Now $b_i b_i' = b_i a_i \in \operatorname{Im} L_a$ and so using any basis for $\ker L_a$ and $B' \cup \{c\}$ as a basis of $\operatorname{Im} L_a$, we have

$$L_{b_i} = \left[\begin{array}{c|ccccc} & 0 & 0 & \ldots & 0 & b_{i1} \\ & 0 & 0 & \ldots & 0 & b_{i2} \\ O & \vdots & \vdots & & \vdots & \vdots \\ & 0 & 0 & \ldots & 0 & b_{it} \\ \hline \star & & & \star & & \end{array}\right] \quad \text{and} \quad L_c = \left[\begin{array}{c|c} O & C_1 \\ \hline C_2 & \star \end{array}\right].$$

Since $c \notin \ker L_a$, the columns of C_2 are independent and so rank $C_2 = t$. Also $c \notin \ker L_a$ implies that $C_1 \neq 0$ and so rank $C_1 = 1$. Since $n > 3$ this implies that there exists a nonzero b' in the span of B' such that

$$L_{b'} = \left[\begin{array}{c|c} O & O \\ \hline \star & \star \end{array}\right].$$

Since $b' \notin \ker L_a$ this is impossible. $\square$

Our final result involves a lower bound for $\dim(\ker L_a)$. First we need the well-known result described in the following lemma.

LEMMA 2.14. *Let M be an $n \times n$ matrix with entries from a field* $\mathbf{k}$*. Suppose M is skew-symmetric and $m_{ii} = 0$ for $1 \leq i \leq n$. If n is odd, then M is singular.*

THEOREM 2.15. *Let A be a nontrivial homogeneous algebra over an infinite field* $\mathbf{k}$. *Let* $a \in A\backslash\{0\}$. *If* $\dim A$ *is even, then* $\dim(\ker L_a) > 1$.

PROOF. Suppose $\dim A = n$ and let $a \in A\backslash\{0\}$. Then decompose A as

$$A = \ker L_a \oplus \operatorname{Im} L_a.$$

Let $\{a, e_2, \ldots, e_n\}$ be the corresponding basis and assume $\dim(\ker L_a) = 1$. We consider the top rows of $L_{e_2}, L_{e_3}, \ldots, L_{e_n}$. Since A is anticommutative these rows are of the form

$$\begin{array}{ll|llllll} R_2: & 0 & 0 & e_{23} & e_{24} & e_{25} & \ldots & e_{2n} \\ R_3: & 0 & -e_{23} & 0 & e_{34} & e_{35} & \ldots & e_{3n} \\ R_4: & 0 & -e_{24} & e_{34} & 0 & e_{45} & \ldots & e_{4n} \\ \vdots & \vdots & & & & & & \\ R_n: & 0 & -e_{2n} & -e_{3n} & -e_{4n} & -e_{5n} & \ldots & 0\,. \end{array}$$

Consider $x_2 R_2 + x_3 R_3 + \cdots + x_n R_n = 0$. This is a homogeneous linear system of the form $Mx = 0$, where M is a $(n-1) \times (n-1)$ skew-symmetric matrix, with $m_{ii} = 0$ (we discard the first column). By Lemma 2.14, M is singular and so the system has nontrivial solutions. Thus there exists a nonzero $x \in \operatorname{Im} L_a$ such that

$$L_x = \left[\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline \star & & \star & \end{array}\right].$$

Again this is impossible since $x \notin \ker L_a$. This completes the proof. $\square$

3. Homogeneous algebras of small dimension

The general results described in the previous section are strong enough to limit the possible existence of homogeneous algebras having small dimension. Their real strength lies in the fact that they do not depend on the choice of the scalar field. These theorems allow us to dramatically shorten the work involved in classifying dimensions 2, 3 and 4, (as reported in [6] and [8]) and to make some additional useful observations.

We first briefly describe the only known examples of infinite homogeneous algebras. These exist over the real field and are described by Djoković in [1]. In that paper, he shows that there are only 3 such algebras. The first two are well-known: the 3-dimensional algebra consisting of the pure quaternions and the 7-dimensional algebra consisting of the pure octonions. In both cases the multiplication is redefined to make $x^2 = 0$. There is also a 6-dimensional algebra $T = \mathbb{C}^3$ considered as a real vector space with multiplication as follows: for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, let

$$x \cdot y = \left(\overline{x_2 y_3 - x_3 y_2},\ \overline{x_3 y_1 - x_1 y_3},\ \overline{x_1 y_2 - x_2 y_1}\right).$$

We conclude with a summary of the results of applying Theorems 2.12, 2.13 and 2.15 to algebras of dimension up to 7. It may be that any further progress on classification will depend on specifying the scalar field **k**.

- **Dimension 2** By Theorem 2.15, $\dim(\ker L_a) = 2 = \dim(A)$. Thus there are no non-trivial homogeneous algebras over any infinite field. This result was first shown in [9].
- **Dimension 3** By Theorem 2.12, we must have $\dim(\ker L_a) = 1$. Such an algebra exists as described above (also over certain other fields; see [6].
- **Dimension 4** By Theorem 2.12, $\dim(\ker L_a) < 2$, but by Theorem 2.15, $\dim(\ker L_a) > 1$. Therefore there are no non-trivial homogeneous algebras over any infinite field. This is an improvement on the authors' work in [8].
- **Dimension 5** According to Theorem 2.13, the only possibility for a homogeneous algebra is to have $\dim(\ker L_a) = 1$. This case has not yet been resolved, but we conjecture no such algebra exists over any infinite field.
- **Dimension 6** According to Theorems 2.12 and 2.15, the only possibility is for $\dim(\ker L_a) = 2$. Such an algebra does exist over the reals, as described above.
- **Dimension 7** By Theorem 2.13, there are two possibilities: $\dim(\ker L_a) = 1$ or 2. The case of $\dim(\ker L_a) = 1$ can occur: the algebra of pure octonions described above. The case of $\dim(\ker L_a) = 2$ is unresolved, but we again conjecture no such algebra exists over any infinite field.

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