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ABOUT A PROBLEM OF HERMITE AND BIEHLER

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Abstract

A point of departure for this paper is the famous theorem of Hermite and Biehler: If $f(z)$ is a polynomial with complex coefficients a_k and its zeros z_k satisfy $\operatorname{Im} z_k > 0$, then the polynomials with coefficients $\operatorname{Re} a_k$ and $\operatorname{Im} a_k$ have only real zeros.

We generalize this theorem for some entire functions. The entire functions in Theorem 2 and Theorem 3 are of first and second genus respectively.

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THEOREM 1. *Let* $f(z) = cz^n \prod_{k=1}^{\infty}(1 - z/z_k)$ *be an entire function, where* $c, z_k \in \mathbb{C}$, $\operatorname{Im} z_k > 0$. *Assume that* $\lim_{k\to\infty} |z_k| = \infty$ *and the Maclaurin series of* f *is* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, *where* $a_k = \alpha_k + i\beta_k$, $\alpha_k, \beta_k \in \mathbb{R}$, *and* $u(z) = \sum_{k=0}^{\infty} \alpha_k z^k$, $v(z) = \sum_{k=0}^{\infty} \beta_k z^k$. *Then all roots of* $u(z)$ *and* $v(z)$ *are real.*

PROOF. The proof is analogous to that for algebraic polynomials. □

THEOREM 2. *Let* $f(z) = cz^n \prod_{k=1}^{\infty}(1 - z/z_k)\exp(z/z_k)$ *be an entire function, where* $c, z_k \in \mathbb{C}$, $\arg(z_k) \in (0, \varphi)$, *where* $0 < \varphi < \pi/2$. *Assume that* $\lim_{k\to\infty} |z_k| = \infty$ *and the Maclaurin series of* f *is* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, *where* $a_k = \alpha_k + i\beta_k$, $\alpha_k, \beta_k \in \mathbb{R}$, *and* $u(z) = \sum_{k=0}^{\infty} \alpha_k z^k$, $v(z) = \sum_{k=0}^{\infty} \beta_k z^k$. *Then all roots of* $u(z)$ *and* $v(z)$ *satisfy* $\arg(z) \notin (\varphi + \pi/2, \pi)$.

PROOF. We have $f(z) = u(z) + iv(z) = cz^n \prod_{k=1}^{\infty}(1 - z/z_k)\exp(z/z_k)$. Let z_0 be such that $v(z_0) = 0$ or $u(z_0) = 0$. Then

$$u(z_0) + iv(z_0) = u(z_0) - iv(z_0) \quad \text{or} \quad u(z_0) + iv(z_0) = -(u(z_0) - iv(z_0)),$$

that is,

$$(1)\qquad cz_0^n\prod_{k=1}^{\infty}(1-z_0/z_k)\exp(z_0/z_k)=\pm\bar{c}z_0^n\prod_{k=1}^{\infty}(1-z_0/\bar{z}_k)\exp(z_0/\bar{z}_k).$$

Suppose that $\arg(z_0)\in(\varphi+\pi/2,\pi)$; then we prove that

$$\left|\left(1-\frac{z_0}{z_k}\right)\exp\left(\frac{z_0}{z_k}\right)\right|>\left|\left(1-\frac{z_0}{\bar{z}_k}\right)\exp\left(\frac{z_0}{\bar{z}_k}\right)\right|,$$

that is,

$$\left|\exp\left(\frac{z_0}{z_k}-\frac{z_0}{\bar{z}_k}\right)\right|>\left|\frac{z_0-\bar{z}_k}{z_0-z_k}\right|.$$

If we let $z_0=a+ib$, $z_k=x_k+iy_k$ and $\bar{z}_k=x_k-iy_k$, where $a,b,x_k,y_k\in\mathbb{R}$, $b>0$, $y_k>0$, then we have

$$\left|\exp\left(\frac{z_0}{z_k}-\frac{z_0}{\bar{z}_k}\right)\right|=\exp\left[\operatorname{Re}\left(\frac{z_0}{z_k}-\frac{z_0}{\bar{z}_k}\right)\right]=\exp\left[\frac{2by_k}{x_k^2+y_k^2}\right]\qquad\text{and}$$

$$\left|\frac{z_0-\bar{z}_k}{z_0-z_k}\right|=\sqrt{1+\frac{4by_k}{(a-x_k)^2+(b-y_k)^2}}.$$

Obviously, we have that

$$\left|\exp\left(\frac{z_0}{z_k}-\frac{z_0}{\bar{z}_k}\right)\right|^2=\exp\left[\frac{4by_k}{x_k^2+y_k^2}\right]>1+\frac{4by_k}{x_k^2+y_k^2}.$$

We fix x_k and y_k. Since $\arg(z_0)\in(\varphi+\pi/2,\pi)$ and $\arg(z_k)\in(0,\varphi)$ we have

$$(a-x_k)^2+(b-y_k)^2>x_k^2+y_k^2.$$

Thus we have

$$1+\frac{4by_k}{x_k^2+y_k^2}>1+\frac{4by_k}{(a-x_k)^2+(b-y_k)^2},$$

which means that $|\exp(z_0/z_k-z_0/\bar{z}_k)|^2>|(z_0-\bar{z}_k)/(z_0-z_k)|^2$ and the assertion is proved. Hence

$$\left|cz_0^n\prod_{k=1}^{\infty}\left(1-\frac{z_0}{z_k}\right)\exp\left(\frac{z_0}{z_k}\right)\right|>\left|\bar{c}z_0^n\prod_{k=1}^{\infty}\left(1-\frac{z_0}{\bar{z}_k}\right)\exp\left(\frac{z_0}{\bar{z}_k}\right)\right|,$$

which is impossible in view of (1). □

THEOREM 3. *Let* $f(z) = cz^n \prod_{k=1}^{\infty}(1 - z/z_k)\exp[z/z_k + (z/z_k)^2/2]$ *be an entire function, where* $c, z_k \in \mathbb{C}$, $\arg(z_k) \in (\varphi + \pi/3, \pi/2)$, $0 < \varphi < \pi/6$. *Let* $\lim_{k\to\infty}|z_k| = \infty$ *and let the Maclaurin series of* f *be* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, *where* $a_k = \alpha_k + i\beta_k$, $\alpha_k, \beta_k \in \mathbb{R}$ *and* $u(z) = \sum_{k=0}^{\infty}\alpha_k z^k$, $v(z) = \sum_{k=0}^{\infty}\beta_k z^k$. *Then all roots of* $u(z)$ *and* $v(z)$ *satisfy* $\arg(z) \notin (0, \varphi)$.

PROOF. We have

$$f(z) = u(z) + i\,v(z) = cz^n \prod_{k=1}^{\infty}\left(1 - \frac{z}{z_k}\right)\exp\left[\frac{z}{z_k} + \frac{1}{2}\left(\frac{z}{z_k}\right)^2\right].$$

Let z_0 be such that $v(z_0) = 0$ or $u(z_0) = 0$. Then $u(z_0) + i\,v(z_0) = u(z_0) - i\,v(z_0)$ or $u(z_0) + i\,v(z_0) = -(u(z_0) - i\,v(z_0))$, that is,

$$cz_0^n \prod_{k=1}^{\infty}\left(1 - \frac{z_0}{z_k}\right)\exp\left[\frac{z_0}{z_k} + \frac{1}{2}\left(\frac{z_0}{z_k}\right)^2\right] = \pm\bar{c}z_0^n \prod_{k=1}^{\infty}\left(1 - \frac{z_0}{\bar{z}_k}\right)\exp\left[\frac{z_0}{\bar{z}_k} + \frac{1}{2}\left(\frac{z_0}{\bar{z}_k}\right)^2\right]. \tag{2}$$

Arguing by contradiction, let us suppose that $\arg(z_0) \in (0, \varphi)$. Then we show that

$$\left|\left(1 - \frac{z_0}{z_k}\right)\exp\left[\frac{z_0}{z_k} + \frac{1}{2}\left(\frac{z_0}{z_k}\right)^2\right]\right| > \left|\left(1 - \frac{z_0}{\bar{z}_k}\right)\exp\left[\frac{z_0}{\bar{z}_k} + \frac{1}{2}\left(\frac{z_0}{\bar{z}_k}\right)^2\right]\right|, \tag{$*$}$$

that is,

$$\left|\exp\left[\frac{z_0}{z_k} - \frac{z_0}{\bar{z}_k} + \frac{1}{2}\left(\frac{z_0}{z_k}\right)^2 - \frac{1}{2}\left(\frac{z_0}{\bar{z}_k}\right)^2\right]\right| > \left|\frac{z_0 - \bar{z}_k}{z_0 - z_k}\right|.$$

If we put $z_0 = a + ib$, $z_k = x_k + iy_k$ and $\bar{z}_k = x_k - iy_k$, where $a, b, x_k, y_k \in \mathbb{R}$, $a > 0$, $b > 0$, $x_k > 0$, $y_k > 0$, then we obtain that

$$\begin{aligned}&\left|\exp\left[\frac{z_0}{z_k} - \frac{z_0}{\bar{z}_k} + \frac{1}{2}\left(\frac{z_0}{z_k}\right)^2 - \frac{1}{2}\left(\frac{z_0}{\bar{z}_k}\right)^2\right]\right|\\ &\quad = \exp\left\{\mathrm{Re}\left[\frac{z_0}{z_k} - \frac{z_0}{\bar{z}_k} + \frac{1}{2}\left(\frac{z_0}{z_k}\right)^2 - \frac{1}{2}\left(\frac{z_0}{\bar{z}_k}\right)^2\right]\right\} = \exp\left[\frac{2by_k}{x_k^2 + y_k^2} + \frac{4abx_ky_k}{(x_k^2 + y_k^2)^2}\right]\end{aligned}$$

and

$$\left|\frac{z_0 - \bar{z}_k}{z_0 - z_k}\right| = \sqrt{1 + \frac{4by_k}{(a - x_k)^2 + (b - y_k)^2}}.$$

Obviously,

$$\left|\exp\left[\frac{z_0}{z_k}-\frac{z_0}{\bar{z}_k}+\frac{1}{2}\left(\frac{z_0}{z_k}\right)^2-\frac{1}{2}\left(\frac{z_0}{\bar{z}_k}\right)^2\right]\right|^2$$
$$=\exp\left[\frac{4by_k}{x_k^2+y_k^2}+\frac{8abx_ky_k}{(x_k^2+y_k^2)^2}\right]>1+\frac{4by_k}{x_k^2+y_k^2}+\frac{8b^2y_k^2}{(x_k^2+y_k^2)^2}+\frac{8abx_ky_k}{(x_k^2+y_k^2)^2}.$$

We wish to prove that

$$1+\frac{4by_k}{x_k^2+y_k^2}+\frac{8b^2y_k^2}{(x_k^2+y_k^2)^2}+\frac{8abx_ky_k}{(x_k^2+y_k^2)^2}>1+\frac{4by_k}{(a-x_k)^2+(b-y_k)^2},$$

which will be true if

$$\frac{1}{x_k^2+y_k^2}+\frac{2by_k}{(x_k^2+y_k^2)^2}+\frac{2ax_k}{(x_k^2+y_k^2)^2}>\frac{1}{(a-x_k)^2+(b-y_k)^2},$$

that is, $\left[(a-x_k)^2+(b-y_k)^2\right](x_k^2+y_k^2+2by_k+2ax_k)>(x_k^2+y_k^2)^2$ or

$$(a^2+b^2)(x_k^2+y_k^2+2by_k+2ax_k)>(2by_k+2ax_k)^2.$$

Hence,

$$(3a^2-b^2)x_k^2+(3b^2-a^2)y_k^2+8abx_ky_k-(2by_k+2ax_k)(a^2+b^2)<0.$$

The equation $(3a^2-b^2)x^2+(3b^2-a^2)y^2+8abxy-(2by+2ax)(a^2+b^2)=0$ is an equation of a hyperbola. Indeed, if we make the change of the variables

$$x=\frac{ax'-by'}{\sqrt{a^2+b^2}},\quad y=\frac{bx'+ay'}{\sqrt{a^2+b^2}},$$

then we have $3x'^2-y'^2-2x'/\sqrt{a^2+b^2}=0$. If the angle of rotation is ψ, then $\cos\psi=a/\sqrt{a^2+b^2}$, that is, $\psi\in(0,\varphi)$. Hence

$$3\left(x'-\frac{1}{3\sqrt{a^2+b^2}}\right)^2-y'^2=\frac{1}{3(a^2+b^2)},$$

that is,

$$\frac{\left(x'-1/\left(3\sqrt{a^2+b^2}\right)\right)^2}{1/\left(3\sqrt{a^2+b^2}\right)^2}-\frac{y'^2}{1/\left(\sqrt{3(a^2+b^2)}\right)^2}=1.$$

After the change $X=x'-1/3\sqrt{a^2+b^2}$, $Y=y'$, we obtain

$$\frac{X^2}{p^2}-\frac{Y^2}{q^2}=1,\quad\text{where}\quad p=\frac{1}{3\sqrt{a^2+b^2}},\ q=\frac{1}{\sqrt{3(a^2+b^2)}}.$$

Thus $q/p=\sqrt{3}=\tan(\pi/3)$ and all $w=x+iy$, with $\arg w\in(\varphi+\pi/3,\pi/2)$ satisfy

$$(3a^2-b^2)x^2+(3b^2-a^2)y^2+8abxy-(2by+2ax)(a^2+b^2)<0.$$

For example, z_k satisfy this condition, which confirms the assertion (∗). Then we obtain

$$\left|cz_0^n\prod_{k=1}^{\infty}\left(1-\frac{z_0}{z_k}\right)\exp\left[\frac{z_0}{z_k}+\frac{1}{2}\left(\frac{z_0}{z_k}\right)^2\right]\right|>\left|\bar{c}z_0^n\prod_{k=1}^{\infty}\left(1-\frac{z_0}{\bar{z}_k}\right)\exp\left[\frac{z_0}{\bar{z}_k}+\frac{1}{2}\left(\frac{z_0}{\bar{z}_k}\right)^2\right]\right|.$$

which contradicts (2). The theorem is proved. □

References

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