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INTERPOLATION PROBLEM FOR ℓ^1 AND A UNIFORM ALGEBRA

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Abstract

Let A be a uniform algebra and $M(A)$ the maximal ideal space of A. A sequence $\{a_n\}$ in $M(A)$ is called ℓ^1-interpolating if for every sequence (α_n) in ℓ^1 there exists a function f in A such that $f(a_n) = \alpha_n$ for all n. In this paper, an ℓ^1-interpolating sequence is studied for an arbitrary uniform algebra. For some special uniform algebras, an ℓ^1-interpolating sequence is equivalent to a familiar ℓ^∞-interpolating sequence. However, in general these two interpolating sequences may be different from each other.

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1. Introduction

Let A be a uniform algebra on a compact Hausdorff space X and $M(A)$ the maximal ideal space of A. Throughout this paper we assume that $\{a_n\}$ is an infinite sequence of distinct points in $M(A)$. For $1 \le p \le \infty$, a sequence $\{a_n\}$ is called ℓ^p*-interpolating* if for every sequence (α_n) in ℓ^p there exists a function f in A such that $f(a_n) = \alpha_n$ for all n.

For $A = H^\infty(D)$, the set of all bounded analytic functions on the unit disc D in $\mathbb{C}$, an ℓ^∞-interpolating sequence was studied by Carleson [2] and Izuchi [4]. Carleson [2] determined an ℓ^∞-interpolating sequence when $\{a_n\}$ is in D, Izuchi [4] studied the general situation. Recently, Hatori [3] showed that an ℓ^1-interpolating sequence is equivalent to an ℓ^∞-interpolating sequence when $\{a_n\}$ is in D. In this paper we study

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an ℓ^1-interpolating sequence for an arbitrary uniform algebra A when $\{a_n\}$ is in $M(A)$. For $\{a_n\}$ in $M(A)$ put

$$J = \{f \in A;\ f = 0 \text{ on } \{a_n\}\}, \qquad J_k = \{f \in A;\ f = 0 \text{ on } \{a_n\}_{n\neq k}\}$$

and

$$\rho_k = \sup\{|f(a_k)|;\ f \in J_k,\ \|f\| \leq 1\}.$$

For a, b in $M(A)$

$$\sigma(a, b) = \sup\{|f(a)|;\ f(b) = 0,\ \|f\| \leq 1\}.$$

When $A = H^\infty(D)$ and $\{a_n\}$ is in D,

$$\sigma(a_k, a_n) = \left|\frac{a_k - a_n}{1 - \bar{a}_k a_n}\right| \quad \text{and} \quad \rho_k = \prod_{n\neq k}\left|\frac{a_k - a_n}{1 - \bar{a}_k a_n}\right|.$$

In general, we do not know whether

$$\rho_k = \prod_{n\neq k} \sigma(a_k, a_n).$$

However, under some mild condition (Hypothesis I in Section 4), we can show that

$$\rho_k \geq \prod_{n\neq k} \sigma(a_k, a_n).$$

In general, $\rho_k > 0$ if and only if $J_k \supsetneq J$. Hence $\rho_k > 0$ if and only if there exists a function f_k in A such that $f_k(a_n) = \delta_{nk}$. In this paper, for $\{a_n\}$ in $M(A)$ we assume that $\rho_k > 0$ for all k.

In Section 2, for an arbitrary uniform algebra we show that $\{a_n\}$ is an ℓ^1-interpolating sequence if and only if $\inf_k \rho_k > 0$. In Section 3, we define a finite ℓ^1-interpolating sequence and give a necessary and sufficient condition to characterize it. In Section 4, we show that if $\inf_k \prod_{n\neq k} \sigma(a_k, a_n) > 0$, then $\{a_n\}$ is always a finite ℓ^1-interpolating sequence and under some mild condition it is an ℓ^1-interpolating sequence. In some sense, this type of theorem for an ℓ^∞-interpolating sequence was conjectured in [1]. In Section 5, we apply the results from the previous sections to concrete uniform algebras. In Section 6, we comment on an ℓ^∞-interpolating sequence.

2. ℓ^1-interpolating sequence

In this section we show that $\{a_n\}$ is an ℓ^1-interpolating sequence if and only if $\inf_k \rho_k > 0$. The argument in the 'only if' part of Lemma 1 is similar to the one which was used by Hatori [3] when $A = H^\infty(D)$.

LEMMA 1. *$\{a_n\}$ is an ℓ^1-interpolating sequence if and only if there exists a sequence $\{f_n\}$ in A such that $f_n(a_k) = \delta_{nk}$ $(n \geq 1,\ k \geq 1)$ and* $\sup_n \|f_n + J\| < \infty$.

PROOF. Suppose $M = \sup_n \|f_n + J\| < \infty$ and $f_n(a_k) = \delta_{nk}$. Let ε be an arbitrary positive constant. For each n there exists g_n in J such that $\|f_n + g_n\| \leq M + \varepsilon$. If $(\alpha_n) \in \ell^1$, put

$$f = \sum_{n=1}^{\infty} \alpha_n (f_n + g_n).$$

Then f belongs to A and $f(a_n) = \alpha_n$ for $n = 1, 2, \ldots$. Suppose $S = \{a_n\}$ is an ℓ^1-interpolating sequence. Then there exists a sequence $\{f_n\}$ in A such that $f_n(a_k) = \delta_{nk}$. For $(\alpha_n) \in \ell^1$, put

$$T(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n f_n | S,$$

then by hypothesis there exists a function f such that $T(\alpha_n) = f|S$. Since $A|S$ is algebraically isomorphic to the quotient algebra A/J, we use the quotient norm of A/J in $A|S$. By the closed graph theorem, T is bounded from ℓ^1 to $A|S$ and so

$$\|f_k + J\| = \|f_k|S\| \leq \|T\|$$

because $T(\{\delta_{nk}\}) = f_k|S$. □

LEMMA 2. *Suppose $\{f_n\}$ is a sequence in A such that $f_n(a_k) = \delta_{nk}$. Then*

$$\|f_n + J\| = 1/\rho_n \quad \textit{for } n = 1, 2, \ldots.$$

PROOF. Since $(\rho_n f_n)(a_k) = \rho_n \delta_{nk}$, $\|\rho_n f_n + J\| \geq 1$. By definition of ρ_n, for each $l \geq 1$ there exists $g_l \in A$ such that $\|g_l\| = 1$, $g_l(a_n) = 0$ for $n \neq k$ and

$$\rho_k - 1/l \leq g_l(a_k) \leq \rho_k.$$

Put $G_l = g_l / g_l(a_k)$, then $G_l \in A$ and

$$\frac{1}{\rho_k} \leq \|G_l\| = \frac{1}{|g_l(a_k)|} \leq \frac{1}{\rho_k - 1/l}.$$

Moreover, $G_l(a_k) = 1$, $G_l(a_n) = 0$ for $n \neq k$ and so $G_l \in f_k + J$. Since $\|f_k + J\| \leq (\rho_k - 1/l)^{-1}$ for any $l \geq 1$, $\|\rho_k f_k + J\| \leq 1$. □

THEOREM 1. *Let A be an arbitrary uniform algebra and let $\{a_n\}$ be in $M(A)$. Then $\{a_n\}$ is a ℓ^1-interpolating sequence if and only if* $\inf_k \rho_k > 0$.

PROOF. The proof follows from Lemma 1 and Lemma 2. □

3. Finite ℓ^1-interpolating sequence

We say that $\{a_n\}$ is a *finite ℓ^1-interpolating sequence* if there exists a finite positive constant γ which satisfies the following: For any finite $l \geq 1$ and for any (α_n) in the unit ball of ℓ^1, there exists a function F_l in A such that

$$F_l(a_n) = \alpha_n \quad \text{for } 1 \leq n \leq l$$

and $\|F_l\| \leq \gamma$.

For $\{a_n\}$ in $M(A)$ and $1 \leq k \leq l < \infty$, put

$$J^l = \{f \in A;\ f(a_n) = 0 \text{ if } 1 \leq n \leq l\},$$
$$J^l_k = \{f \in A;\ f(a_n) = 0 \text{ if } 1 \leq n \leq l,\ n \neq k\}$$

and

$$\rho_{k,l} = \sup\{|f(a_k)|;\ f \in J^l_k,\ \|f\| \leq 1\}.$$

Then $\rho_{k,l} \geq \rho_{k,l+1}$ and $\lim_{l\to\infty} \rho_{k,l} \geq \rho_k$.

LEMMA 3. *$\{a_n\}$ is a finite ℓ^1-interpolating sequence if and only if for each $l \geq 1$ there exists a sequence $\{f_{l,n}\}_{n=1}^l$ in A such that $f_{l,n}(a_k) = \delta_{nk}$ for $1 \leq k \leq l$ and* $\sup_l \sup_{1\leq n\leq l} \|f_{l,n} + J^l\| < \infty$.

PROOF. (α_n) denotes an element in the unit ball of ℓ^1. Suppose

$$M = \sup_l \sup_{1\leq n\leq l} \left\| f_{l,n} + J^l \right\| < \infty$$

and $f_{l,n}(a_k) = \delta_{nk}$ for $1 \leq k \leq l$, then for any finite $l \geq 1$

$$\left\| \sum_{n=1}^{l} \alpha_n f_{l,n} + J^l \right\| \leq \left(\sum_{n=1}^{l} |\alpha_n| \right) M.$$

If $\gamma = M+1$, then for any $l \geq 1$ there exists $g_l \in J^l$ such that $\left\| \sum_{n=1}^l \alpha_n f_{l,n} + g_l \right\| \leq \gamma$. Set $F_l = \sum_{n=1}^l \alpha_n f_{l,n} + g_l$, then $F_l(a_n) = \alpha_n$ for $1 \leq n \leq l$ and $\|F_l\| \leq \gamma$. Suppose $\{a_n\}$ is a finite ℓ^1-interpolating sequence. Since $\{a_n\}$ is an infinite sequence of distinct points in $M(A)$, for each $l \geq 1$ there exists a sequence $\{f_{l,n}\}_{n=1}^l$ in A such that $f_{l,n}(a_k) = \delta_{nk}$ for $1 \leq k \leq n$. Put

$$T^l(\alpha_n) = \sum_{n=1}^{l} \alpha_n f_{l,n} + J^l;$$

then $\|T^l(\alpha_n)\| \leq \|T^l\| \big(\sum_{n=1}^l |\alpha_n| \big)$. If $\|T^l\| \to \infty$ as $l \to \infty$, then there exists (α_n) in the unit ball of ℓ^1 such that $\|T^l(\alpha_n)\| \to \infty$ as $l \to \infty$. On the other hand, by hypothesis $\|T^l(\alpha_n)\| \leq \gamma < \infty$ for all l. This contradiction implies that $M = \sup_l \|T^l\| < \infty$. This shows that for any $l \geq 1$ and any $k \geq 1$ with $k \leq l$,

$$\|f_{l,k} + J^l\| = \|T^l(\{\delta_{kn}\})\| \leq M. \qquad \square$$

LEMMA 4. *For* $l = 1, 2, \ldots$ *and* $1 \le k \le l$, $\| f_k + J^l \| = 1/\rho_{k,l}$.

Proof is almost the same as the proof of Lemma 2.

THEOREM 2. *Let* A *be an arbitrary uniform algebra and let* $\{a_n\}$ *be in* $M(A)$. *Then* $\{a_n\}$ *is a finite* ℓ^1*-interpolating sequence if and only if* $\inf_k \lim_{l\to\infty} \rho_{k,l} > 0$.

PROOF. The statement of the theorem follows from Lemma 3 and Lemma 4. □

4. Uniformly separated sequence

When $A = H^\infty(D)$ and $\{a_n\}$ is in D, for any $k \ge 1$

$$\rho_k = \prod_{n \ne k} \sigma(a_k, a_n) = \lim_{l\to\infty} \rho_{k,l}.$$

When $\{a_n\}$ is in $M(A)$, Izuchi [4] showed essentially that $\inf_k \rho_k > 0$ implies $\inf_k \prod_{n\ne k} \sigma(a_k, a_n) > 0$. However, this is not true in general. If $\sum_{n=1}^{\infty}(1 - \rho_n) < \infty$, then $\inf_k \prod_{n\ne k} \sigma(a_k, a_n) > 0$. In fact, $\rho_n \le \sigma(a_k, a_n)$ for $n \ne k$ and so $\prod_{n=1}^{\infty} \rho_n \le \prod_{n\ne k} \sigma(a_k, a_n)$ for any $k \ge 1$. When $\sum_{n=1}^{\infty}(1 - \rho_n) < \infty$, $0 < \prod_{n=1}^{\infty} \rho_n$ and so $\inf_k \prod_{n\ne k} \sigma(a_k, a_n) > 0$. In this section, we study these three quantities.

LEMMA 5. (1) *For any* $l \ge 1$, $\rho_{k,l} \ge \prod_{n\ne k}^{l} \sigma(a_k, a_n)$. *Hence for any* $k \ge 1$

$$\lim_{l\to\infty} \rho_{k,l} \ge \prod_{n\ne k} \sigma(a_k, a_n).$$

(2) *For* $1 \le n \le l$ *and* $n \ne k$, $\rho_{k,l} \le \sigma(a_k, a_n)$. *Hence for any* $k \ge 1$

$$\lim_{l\to\infty} \rho_{k,l} \le \inf_{n\ne k} \sigma(a_k, a_n).$$

PROOF. (1) Fix any positive constant $\varepsilon > 0$. For each n with $l \ge n \ge 1$ and $n \ne k$, there exists $F_n^\varepsilon \in A$ such that $\|F_n^\varepsilon\| \le 1$, $F_n^\varepsilon(a_n) = 0$ and $\sigma(a_k, a_n) \ge |F_n^\varepsilon(a_k)| \ge \sigma(a_k, a_n) - \varepsilon$. Then $F^\varepsilon = \prod_{n\ne k}^{l} F_n^\varepsilon$ belongs to $J_{l,k}$, $\|F^\varepsilon\| \le 1$ and

$$\rho_{l,k} \ge |F^\varepsilon(a_k)| \ge \prod_{n\ne k}^{l} \{\sigma(a_k, a_n) - \varepsilon\}.$$

As $\varepsilon \to 0$, $\rho_{l,k} \ge \prod_{n\ne k}^{l} \sigma(a_k, a_n)$ for any $l \ge 1$ and hence

$$\lim_{l\to\infty} \rho_{k,l} \ge \prod_{n\ne k} \sigma(a_k, a_n).$$

(2) is clear by the definitions of $\rho_{k,l}$ and $\sigma(a_k, a_n)$ for $1 \le n \le l$ and $n \ne k$. □

THEOREM 3. *Let A be an arbitrary uniform algebra and let $\{a_n\}$ be in $M(A)$.*

(1) *If* $\inf_k \prod_{n\neq k} \sigma(a_k, a_n) > 0$, *then $\{a_n\}$ is a finite ℓ^1-interpolating sequence.*
(2) *If $\{a_n\}$ is a finite ℓ^1-interpolating sequence, then* $\inf_{n\neq k} \sigma(a_k, a_n) > 0$.

PROOF. (1) By Lemma 5 (1), $\inf_k \lim_{l\to\infty} \rho_{k,l} > 0$ and so, by Theorem 2, $\{a_n\}$ is a finite ℓ^1-interpolating sequence.
(2) By Theorem 2 $\inf_k \lim_{l\to\infty} \rho_{k,l} > 0$ and so, by Lemma 5 (2), $\inf_{n\neq k} \sigma(a_k, a_n) > 0$. □

HYPOTHESIS I. *Let A be a uniform algebra and let $\{a_n\}$ be in $M(A)$. If g_l is a function in A and $\|g_l\| \leq 1$ for $l = 1, 2, \ldots$, then there exist a subsequence $\{g_{l(j)}\}_j$ of $\{g_l\}_l$ and a function g in A such that $\|g\| \leq 1$ and $\lim_{j\to\infty} g_{l(j)}(a_n) = g(a_n)$ for any $n \geq 1$.*

HYPOTHESIS II. *Let A be a uniform algebra and let $\{a_n\}$ be in $M(A)$. For any a, b in $\{a_n\}$ with $a \neq b$, if the function f in A satisfies $f(a) = f(b) = 0$ and $\|f\| \leq 1$, then for any $\varepsilon > 0$ there exist two functions g and h in A such that $\|g\| \leq 1+\varepsilon$, $\|h\| \leq 1+\varepsilon$, $g(a) = 0$, $h(b) = 0$ and $f = gh$.*

LEMMA 6. *Let A be an arbitrary uniform algebra and let $\{a_n\}$ be in $M(A)$. If $\{a_n\}$ satisfies Hypothesis I, then $\lim_{l\to\infty} \rho_{k,l} = \rho_k$ for any $k \geq 1$, and hence a finite ℓ^1-interpolating sequence is an ℓ^1-interpolating sequence.*

PROOF. $\lim_{l\to\infty} \rho_{k,l} \geq \rho_k$ for any $k \geq 1$. If $\lim_{l\to\infty} \rho_{k,l} > \varepsilon > 0$, then for each l there exists $g_l \in J_k^l$ such that $\|g_l\| \leq 1$ and $|g_l(a_k)| \geq \varepsilon > 0$. By hypothesis, there exists $g \in J_k$ such that $\|g\| \leq 1$ and $|g(a_k)| \geq \varepsilon > 0$. Thus $\lim_{l\to\infty} \rho_{k,l} \leq \rho_k$ and so $\lim_{l\to\infty} \rho_{k,l} = \rho_k$. This together with Theorem 1 and Theorem 2 also imply that a finite ℓ^1-interpolating sequence is an ℓ^1-interpolating sequence. □

LEMMA 7. *Assume Hypothesis II. If f is a function in $J_{k,l}$ with $\|f\| \leq 1$, then for any $\varepsilon > 0$, $f = \prod_{n\neq k}^{l} f_n$, $f_n(a_n) = 0$ $(n \neq k)$ and $\|f_n\| \leq (1+\varepsilon)^{l-1}$.*

PROOF. We may assume $k = 1$. Fix any $\varepsilon > 0$. By Hypothesis II, $f = g_2 g_3$, $\|g_j\| \leq 1+\varepsilon$ $(j = 2, 3)$ and $g_2(a_2) = g_3(a_3) = 0$. Since $f(a_4) = 0$, $g_2(a_4) = 0$ or $g_3(a_4) = 0$. We may assume $g_2(a_4) = 0$. By Hypothesis II, $g_2 = g_{22} g_{24}$, $\|g_{2j}\| \leq (1+\varepsilon)^2$ $(j = 2, 4)$, and $g_{22}(a_2) = g_{24}(a_4) = 0$. Hence there exist h_2, h_3, h_4 such that $f = h_2 h_3 h_4$, $\|h_j\| \leq (1+\varepsilon)^2$ $(j = 2, 3, 4)$ $h_2(a_2) = h_3(a_3) = h_4(a_4) = 0$. This argument implies the proof. □

LEMMA 8. *Let A be an arbitrary uniform algebra and let $\{a_n\}$ be in $M(A)$. If $\{a_n\}$ satisfies Hypothesis II, then for $1 \leq k \leq l$, $\rho_{k,l} = \prod_{k\neq n}^{l} \sigma(a_k, a_n)$. Moreover, if $\{a_n\}$ satisfies Hypothesis I, then $\rho_k = \prod_{k\neq n} \sigma(a_k, a_n)$.*

PROOF. By (1) of Lemma 5 it is sufficient to show that $\rho_{k,l} \le \prod_{k\ne n}^{l} \sigma(a_k, a_n)$. If $0 < \delta < \rho_{k,l}$, then there exists $f \in J_{k,l}$ with $\| f\| \le 1$ such that

$$\rho_{k,l} - \delta \le | f(a_k)| \le \rho_{k,l}.$$

For any $\varepsilon > 0$, by Lemma 7, f can be factorized as $f = \prod_{n\ne k}^{l} f_n$, $\| f_n\| \le (1+\varepsilon)^{l-1}$ and $f_n(a_n) = 0$ for $n \ne k$. Hence

$$\prod_{n\ne k}^{l} | f_n(a_k)| \le (1+\varepsilon)^{(l-1)(l-1)} \prod_{n\ne k}^{l} \sigma(a_k, a_n).$$

As $\varepsilon \to 0$, $\rho_{k,l} - \delta \le \prod_{n\ne k}^{l} \sigma(a_k, a_n)$. Since δ is arbitrary, $\rho_{k,l} \le \prod_{n\ne k}^{l} \sigma(a_k, a_n)$. □

THEOREM 4. *Let A be an arbitrary uniform algebra and let $\{a_n\}$ be in $M(A)$.*
(1) *Under Hypothesis II, $\{a_n\}$ is a finite ℓ^1-interpolating sequence if and only if $\inf_k \prod_{n\ne k} \sigma(a_k, a_n) > 0$.*
(2) *Under Hypotheses I and II, $\{a_n\}$ is an ℓ^1-interpolating sequence if and only if $\inf_k \prod_{n\ne k} \sigma(a_k, a_n) > 0$.*

PROOF. Theorem 1, Theorem 2 and Lemma 8 imply the theorem. □

When $A = H^\infty(D)$ and $\{a_n\}$ is in D, $\{a_n\}$ satisfies Hypotheses I and II. Let A be a disc algebra. Then if $\{a_n\}$ is in D, then $\{a_n\}$ satisfies Hypothesis II (see Section 5). On the other hand, it is easy to see that there exists a sequence $\{a_n\}$ in D which does not satisfy Hypothesis I.

5. Special uniform algebras

When $A = H^\infty(D)$ and $\{a_n\}$ in D, Hatori [3] showed that $\{a_n\}$ is an ℓ^1-interpolating sequence if and only if $\inf_k \prod_{n\ne k} \sigma(a_k, a_n) > 0$. Since it is clear that $\{a_n\}$ in D satisfies Hypotheses I and II, this is a corollary of (2) of Theorem 4. Corollary 3 is also a result of Hatori [3]. We give another proof of it. Hatori [3] also shows this type of theorem for a Hardy space H^p $(1 \le p < \infty)$ on a finite open Riemann surface and generalizes a theorem of Shapiro and Shields [7].

COROLLARY 1. *Let A be a uniform closed algebra between the disc algebra $\mathscr{A}$ and $H^\infty(D)$, and let $\{a_n\}$ be in D. Suppose that f/z belongs to A for f in A with $f(0) = 0$. Then $\{a_n\}$ is a finite ℓ^1-interpolating sequence if and only if $\inf_k \prod_{n\ne k} \sigma(a_k, a_n) > 0$.*

PROOF. If $f \in A$ and $f(a) = 0$ for some $a \in D$, then $f(z)/(z-a)$ belongs to A (see [5]). Hence

$$\frac{1-\bar{a}z}{z-a} f(z) \quad \text{belongs to} \quad A$$

and $(z-a)/(1-\bar{a}z)$ is a unimodular function in $\mathscr{A}$. Therefore, $\{a_n\}$ satisfies Hypothesis II and so (1) of Theorem 4 implies the corollary. □

COROLLARY 2. *Let $A = H^\infty(D^m)$ and let $\{a_n\}$ be in D^m. Suppose $a_n = (a_n^1, a_n^2, \dots, a_n^m)$ and $\sum_{n=1}^\infty (1-|a_n^l|) < \infty$ for $1 \le l \le m$. Then $\{a_n\}$ is an ℓ^1-interpolating sequence if and only if $\inf_k \prod_{n\neq k} \sigma(a_k, a_n) > 0$.*

PROOF. By Theorem 2 and Lemma 6, the 'if' part is proved. We will prove the 'only if' part. Put

$$B_k = B_k(z_1, \dots, z_m) = \prod_{l=1}^m \prod_{n\neq k}^\infty \frac{-a_n^l}{|a_n^l|} \frac{z_l - a_n^l}{1-\bar{a}_n^l z_l},$$

then B_k belongs to $H^\infty(D^m)$ because $\sum_{n=1}^\infty (1-|a_n^l|) < \infty$ for $1 \le l \le m$. If $F_k = B_k/B_k(a_k)$, then $F_k(a_n) = \delta_{nk}$ and

$$\|F_k + J\| = |B_k(a_k)|^{-1} \|B_k + J\| = |B_k(a_k)|^{-1};$$

thus $\rho_k = |B_k(a_k)|$. Theorem 1 implies that $\inf_k |B_k(a_k)| = \inf_k \rho_k > 0$. Since

$$\sigma(a_k, a_n) = \max\left(\left| \frac{a_k^1 - a_n^1}{1-\bar{a}_n^1 a_k^1} \right|, \dots, \left| \frac{a_k^m - a_n^m}{1-\bar{a}_n^m a_k^m} \right| \right)$$

(see [1, page 162]),

$$|B_k(a_k)| \le \prod_{k\neq n} \sigma(a_k, a_n).$$

This proves the corollary. □

COROLLARY 3. *Let R be a finite open Riemann surface and $A = H^\infty(R)$ the set of all bounded analytic functions on R. Then $\{a_n\}$ in R is an ℓ^1-interpolating sequence if and only if $\inf_k \prod_{n\neq k} \sigma(a_k, a_n) > 0$.*

PROOF. It is known [8] that $\{a_n\}$ is an ℓ^∞-interpolating sequence if and only if $\inf_k \prod_{n\neq k} \sigma(a_k, a_n) > 0$. If $\{a_n\}$ is an ℓ^1-interpolating sequence, then $\inf_k \rho_k > 0$ by Theorem 1 and so by [8, Theorem 5.9] $\{a_n\}$ is a ℓ^∞-interpolating sequence. □

Let $D_n = \{z \in \mathbb{C}; |z - c_n| < r_n\}$, $c_n > 0$ as $\overline{D_n \cap D_m} = \emptyset$ $(n \neq m)$, $D_n \subset D\backslash\{0\}$ $(n = 1, 2, 3, \dots)$ and $\sum_{n=1}^\infty r_n/c_n < \infty$. $U = D\backslash \bigcup_n D_n$ is called a *Zalcman domain* [9]. When $A = H^\infty(U)$ and $\{a_n\}$ is in U, if $\inf_k \prod_{n\neq k} \sigma(a_k, a_n) > 0$, then $\{a_n\}$ is an ℓ^1-interpolating sequence by (1) of Theorem 3 and Lemma 6 because $\{a_n\}$ satisfies Hypothesis I but $\{a_n\}$ is not necessarily an ℓ^∞-interpolating sequence by [6].

6. ℓ^∞-interpolating sequence

When $\{a_n\}$ in $M(A)$ satisfies Hypothesis I, it is interesting to give a sufficient condition or a necessary condition for an ℓ^∞-interpolating sequence. Berndtsson, Chang and Lin [1] give the following problem: Let $A = H^\infty(Y)$ and let $\{a_n\} \subset Y$ be a bounded domain $Y \subset \mathbb{C}^n$. Suppose $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$. Is $\{a_n\}$ an ℓ^∞-interpolating sequence? In Proposition 1, $\sum_{n=1}^\infty (1 - \rho_n) < \infty$ and so by the remark above Lemma 5, $\inf_k \prod_{n \neq k} \sigma(a_k, a_n) > 0$.

PROPOSITION 1. *Let A be an arbitrary uniform algebra and let $\{a_n\}$ be in $M(A)$. Suppose $\{a_n\}$ satisfies Hypothesis I. If $\rho_n \geq 2(n^t+1)(n^t+2)/\{(n^t+1)^2 + (n^t+2)^2\}$ for $n = 1, 2, 3, \ldots$ and some $t > 1$, then $\{a_n\}$ is an ℓ^∞-interpolating sequence.*

PROOF. By Hypothesis I there exists a sequence $\{F_n\}$ in A such that $\|F_n\| \leq 1$, $F_n(a_k) = 0$ if $k \neq n$ and $|F_n(a_n)| = \rho_n$ for $n = 1, 2, \ldots$. Izuchi [4, Theorem 1] has essentially proved the theorem. We use the notation from [4, Theorem 1]. Set $\rho_n = 2(1-\delta_n)/\{1 + (1-\delta_n)^2\}$ with $0 < \delta_n \leq 1/(n^t + 2)$; this is possible by the hypothesis on ρ_n. If $\varepsilon_n = 1/n^\rho$, then $\sum_{n=1}^\infty \varepsilon_n < \infty$ and so $\prod_{n=1}^\infty (1 + \varepsilon_n) < \infty$. Then

$$\delta_n < 1 - \frac{1}{\sqrt{1 + 2\varepsilon_n}}.$$

By the proof of [4, Theorem 1], there exists a sequence $G_n \in A$ such that

$$\sum_{n=1}^\infty |G_n| \leq \sum_{n=1}^\infty (1 + \varepsilon_n) < \infty \text{ on } X.$$

Hypothesis I implies that $\{a_n\}$ is an ℓ^∞-interpolating sequence. □

PROPOSITION 2. *Let A be an arbitrary uniform algebra and let $\{a_n\}$ be in $M(A)$. Suppose $\{f_k\}_k$ is a sequence in A such that $f_k(a_n) = \delta_{nk}$. Then $\{a_n\}$ is an ℓ^p-interpolating sequence if and only if*

$$\sup_{\phi \in A^* \cap J^\perp, \|\phi\| \leq 1} \left(\sum_{n=1}^\infty |\phi(f_n)|^q \right)^{1/q} < \infty,$$

where $1/p + 1/q = 1$ and $A^* \cap J^\perp = \{\phi \in A^*; \phi = 0 \text{ on } J\}$. For $p = 1$ and $q = \infty$ we assume that

$$\sup_\phi \left(\sum_{n=1}^\infty |\phi(f_n)|^q \right)^{1/q} = \sup_\phi \sup_n |\phi(f_n)| = \sup_n \|f_n + J\|.$$

PROOF. Suppose that

$$\sup_{\phi\in A^*\cap J^\perp,\|\phi\|\le 1}\left(\sum_{n=1}^{\infty}|\phi(f_n)|^q\right)^{1/q}=\gamma_q<\infty.$$

For any $\phi \in A^* \cap J^\perp$ with $\|\phi\| \le 1$ and any $l < \infty$,

$$\left|\phi\left(\sum_{n=1}^{l}\alpha_n f_n\right)\right| \le \left(\sum_{n=1}^{l}|\alpha_n|^p\right)^{1/p}\left(\sum_{n=1}^{l}|\phi(f_n)|^q\right)^{1/q}$$

and so

$$\left\|\sum_{n=1}^{\infty}\alpha_n \tilde{f}_n\right\| \le \gamma_q\left(\sum_{n=1}^{\infty}|\alpha_n|^p\right)^{1/p},$$

where $\tilde{f}_n = f_n + J$. Thus if $(\alpha_n) \in \ell^p$ then $\tilde{f} = \sum_{n=1}^{\infty}\alpha_n \tilde{f}_n$ belongs to A/J. Then $f(a_n) = \alpha_n$ for $n = 1, 2, \ldots$ and so $\{a_n\}$ is an ℓ^p-interpolating sequence. Conversely, suppose $S = \{a_n\}$ is an ℓ^p-interpolating sequence. For $(\alpha_n) \in \ell^p$, set

$$T(\alpha_n) = \sum_{n=1}^{\infty}\alpha_n f_n|S;$$

then there exists a function f such that $T(\alpha_n) = f|S$. Since T turns out to be bounded from ℓ^p to A/J (see Lemma 1), for $\phi \in A^*/J^\perp$ with $\|\phi\| \le 1$ we have

$$|\phi(f)| = \left|\sum_{n=1}^{\infty}\alpha_n\phi(f_n)\right| \le \|T\|\left(\sum_{n=1}^{\infty}|\alpha_n|^p\right)^{1/p}.$$

Hence $\sup_{\phi\in A^*\cap J^\perp,\|\phi\|\le 1}(\sum_{n=1}^{\infty}|\phi(f_n)|^q)^{1/q} < \infty$. □

Hatori [3] is interested in when an ℓ^1-interpolating sequence is an ℓ^∞-interpolating sequence. He showed that if $A = H^\infty(R)$ and $\{a_n\}$ in R, then $\{a_n\}$ is such a sequence (see Corollary 3). In general, Proposition 2 gives a necessary and sufficient condition for this to happen.

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