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MONO-UNARY ALGEBRAS ARE STRONGLY DUALIZABLE

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Abstract

We show that mono-unary algebras have rank at most two and are thus strongly dualizable. We provide an example of a strong duality for a mono-unary algebra using an alter ego with (partial) operations of arity at most two. This mono-unary algebra has rank two and generates the same quasivariety as an injective, hence rank one, mono-unary algebra.

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0. Introduction

Given a finite algebra $\mathbf{M}$, an *alter ego* of $\mathbf{M}$ is a topological structure $\mathbb{M}$ with the discrete topology on M, finitary operations, finitary partial operations and relations each of which is a subalgebra of an appropriate power of $\mathbf{M}$.

For each $\mathbf{A}$ in the quasivariety $\mathbf{ISP}(\mathbf{M})$, the *dual* of $\mathbf{A}$ is $\mathbb{D}(\mathbf{A}) = \mathrm{Hom}(\mathbf{A}, \mathbf{M})$, viewed as a non-empty topologically closed substructure of $\mathbb{M}^A$. Given $\mathbb{X}$ a topologically closed substructure of a power of $\mathbb{M}$, the *dual* of $\mathbb{X}$ is $\mathbb{E}(\mathbb{X}) = \mathrm{Hom}(\mathbb{X}, \mathbb{M})$, the collection of continuous operation preserving maps from $\mathbb{X}$ into $\mathbb{M}$, viewed as a subalgebra of $\mathbf{M}^X$.

For each $\mathbf{A}$ in the quasivariety $\mathbf{ISP}(\mathbf{M})$ there is a natural embedding $e_{\mathbf{A}}$ from $\mathbf{A}$ to the double dual $\mathbb{E}(\mathbb{D}(\mathbf{A}))$ that assigns to $a \in \mathbf{A}$ the evaluation map $e_{\mathbf{A}}(a)$ given by

$$e_{\mathbf{A}}(a)(\alpha) = \alpha(a).$$

An algebra $\mathbf{M}$ is *dualized by* $\mathbb{M}$ if this embedding is an isomorphism for all $\mathbf{A} \in \mathbf{ISP}(\mathbf{M})$. An algebra $\mathbf{M}$ is *dualizable* if there is an alter ego $\mathbb{M}$ that dualizes $\mathbf{M}$. A

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subset X of M^S is *term-closed* (in M^S) if for all $y \in M^S \setminus X$ there exist S-ary term functions $\sigma, \tau : M^S \to M$ on $\mathbf{M}$ that agree on X but not at y. When $\mathbb{M}$ dualizes $\mathbf{M}$ and every closed substructure of a power of $\mathbb{M}$ is term-closed, we say that $\mathbb{M}$ *strongly dualizes* $\mathbf{M}$. These concepts are elaborated on in [1] and [2].

In [8], Willard shows that a finite algebra with finite rank is strongly dualizable whenever it is dualizable. The definition of rank provided in Section 1 is equivalent to that used in [8]. By definition projections have rank 0. In Section 2 we define a type of homomorphism called a wrap that always has rank 1. Using projections and wraps we show that all relevant mono-unary homomorphisms have rank at most 2. It follows that mono-unary algebras have rank at most 2 and are thus strongly dualizable. In the last section we give an example of two mono-unary algebras, one with rank 2 and the other with rank 1, that generate the same quasivariety. We provide an alter ego that gives a strong duality for the rank 2 algebra. The construction of the alter ego would work for any rooted mono-unary algebra.

1. Definition of rank

Let $\mathbf{M}$ be a fixed finite algebra, n a positive integer, and let $\mathbf{B}$ be a subalgebra of $\mathbf{M}^n$. Let $h \in \mathrm{Hom}(\mathbf{B}, \mathbf{M})$, the homomorphisms from $\mathbf{B}$ to $\mathbf{M}$. The notation $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}'$ denotes that: $\mathbf{B}'$ is a subalgebra of $\mathbf{M}^{n+k}$ for some finite k; σ embeds $\mathbf{B}$ in $\mathbf{B}'$ by repetition of some coordinates; and $\mathbf{B} \cong \mathbf{B}'$. Let $h' = \sigma^{-1} \circ h$ be the natural extension of h to $\mathbf{B}'$. Let $\mathbf{B}' \le \mathbf{C} \le \mathbf{D} \le \mathbf{M}^{n+k}$. Moreover, assume there exists $h^+ : \mathbf{D} \to \mathbf{M}$ such that h' lifts to h^+. Throughout this paper when we refer to the commuting diagram in Figure 1 we assume the above setup holds.

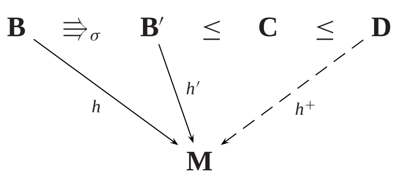
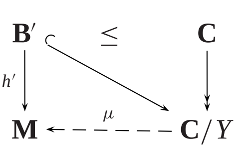


FIGURE 1. $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}'$ and $\mathbf{B}' \le \mathbf{C} \le \mathbf{D} \le \mathbf{M}^{n+k}$.

Let $Y \subseteq \mathrm{Hom}(\mathbf{D}, \mathbf{M})$, then $\mathbf{D}/Y$ is defined to be the algebra $\mathbf{D}/\bigcap\{\ker g \mid g \in Y\}$ and $\mathbf{C}/Y$ is defined to be the algebra $\mathbf{C}/\bigcap\{\ker(g|_C) \mid g \in Y\}$. The set Y *separates* $\mathbf{B}'$ if $\bigcap\{\ker(g|_{\mathbf{B}'}) \mid g \in Y\} = \mathbf{0}_{\mathbf{B}'}$. The homomorphism h' *lifts to* $\mathbf{C}/Y$ if Y separates $\mathbf{B}'$ and there exists a homomorphism μ such that the following diagram commutes.



Given a homomorphism $h : \mathbf{B} \to \mathbf{M}$, define $\operatorname{rank}(h)$ as follows: $\operatorname{rank}(h) \le 0$ if and only if h is a projection. Moreover $\operatorname{rank}(h) \le \alpha$ if and only if there exists a finite N such that for all nonnegative integers k, for all subalgebras $\mathbf{D}$ of $\mathbf{M}^{n+k}$, and for all commuting diagrams like Figure 1, where h' lifts to $\mathbf{D}$, there exists $Y \subseteq \operatorname{Hom}(\mathbf{D}, \mathbf{M})$ such that

- $|Y| \le N$;
- h' lifts to $\mathbf{C}/Y$; and
- $\operatorname{rank}(g|_{\mathbf{C}}) < \alpha$ for all $g \in Y$.

Further, $\operatorname{rank}(h) = \alpha$ if $\operatorname{rank}(h) \le \alpha$ and it is not true that $\operatorname{rank}(h) < \alpha$. Finally $\operatorname{rank}(\mathbf{M}) = \alpha$ if for all homomorphisms h from a subalgebra of a finite power of $\mathbf{M}$ into $\mathbf{M}$, $\operatorname{rank}(h) \le \alpha$ but they do not all have rank strictly less than α.

2. Ranks of finite mono-unary algebras are finite

Let $\mathbf{M} = \langle M, f \rangle$ be a finite mono-unary algebra. A *connected component* of a mono-unary algebra is a subalgebra $\mathbf{B}$ that is maximal with respect to the property that for all $a, b \in \mathbf{B}$, $f^m(a) = f^s(b)$ for some $m, s \ge 0$. For complete details on the structure of mono-unary algebras see [6, Section 3.2]. The example that we use in Section 3 is illustrated in Figure 2.

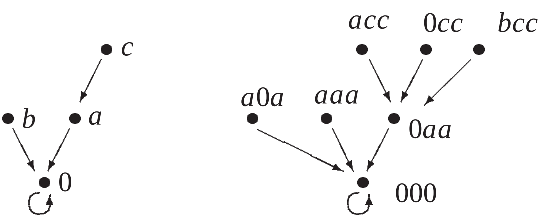


FIGURE 2. An algebra $\mathbf{M}$ and a subalgebra of $\mathbf{M}^3$.

Let $\mathbf{C}$ be a subalgebra of a finite power of $\mathbf{M}$ and let $\mathbf{A}$ be a connected component of $\mathbf{C}$. The *essential components of* $\mathbf{A}$ are the minimum set of connected components of $\mathbf{M}$ that contain $\pi_i(\mathbf{A})$ for all projections π_i. A *core* of a connected mono-unary algebra is a nonempty subalgebra on which f is a one-to-one function. In a finite, connected mono-unary algebra the unique core will be of the form $\{a, f(a), f^2(a), \dots, f^{k-1}(a)\}$ where $f^k(a) = a$. An arbitrary finite mono-unary algebra may have several cores. The *circumference* of the connected component $\mathbf{A}$ is $\operatorname{circ}(\mathbf{A})$, the least k such that for some $a \in \mathbf{A}$, $f^k(a) = a$. That is, the circumference of a finite, connected mono-unary algebra is the size of the core. Pick $x \in \mathbf{C}$ such that x is not in a core but $f(x)$ is in a core. The set $\{y \in \mathbf{C} \mid f^m(y) = x \text{ for some } m \ge 0\}$ is a *branch*. For $x \in \mathbf{C}$, x is a *branch element* if x is an element of a branch. For x a branch element, the *coheight* of x is the greatest k such that there exists a y with $f^k(y) = x$. For x a core element, coheight$(x) = \infty$.

The following lemma and the definition of wrap provide the motivation for the concept of essential component.

LEMMA 2.1. *Let* $\mathbf{A}$ *be a connected component of* $\mathbf{C}$ *and* $\mathbf{M}_\alpha$ *an essential component of* $\mathbf{A}$*. Then the circumference of* $\mathbf{A}$ *is a (positive) integral multiple of the circumference of* $\mathbf{M}_\alpha$*.*

For $\mathbf{A}$ a connected component in $\mathbf{C}$ and $\mathbf{M}_\alpha$ an essential component of $\mathbf{A}$, *to wrap* $\mathbf{A}$ around $\mathbf{M}_\alpha$ means to define an operation q from $\mathbf{A}$ to the core of $\mathbf{M}_\alpha$ recursively as follows. Pick $a \in \mathbf{A}$ and d in the core, L, of $\mathbf{M}_\alpha$. Let $q(a) = d$ and $q(f^m(a)) = f^m(d)$. If q is defined on $f(x)$ but not defined on x, then let $q(x) \in f^{-1}(q(f(x))) \cap L$. Note $f^{-1}(q(f(x))) \cap L$ has exactly one element in it. By Lemma 2.1, q is well-defined so by construction, q is a homomorphism whose image is L. The map q is called a *wrap*. An example of a wrap is any homomorphism from a core to an essential component of the core.

LEMMA 2.2. *Given* $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}' \le \mathbf{C} \le \mathbf{M}^n$*, at most* $|\mathbf{B}| - 1$ *projections from* $\mathbf{C}$ *to* $\mathbf{M}$ *are required to separate* $\mathbf{B}'$*.*

PROOF. Let $Y_0 = \emptyset$. Y_0 separates no elements so there is one equivalence class of $\mathbf{C}/Y_0$ containing elements from $\mathbf{B}'$. Given a set of projections, Y_i, and two distinct elements $x, y \in \mathbf{B}'$ that are not separated by Y_i, let π_{i+1} be a projection that separates x and y. Let $Y_{i+1} = Y_i \cup \{\pi_{i+1}\}$. The set $\{[a]_{Y_{i+1}} \in \mathbf{C}/Y_{i+1} \mid a \in \mathbf{B}'\}$ has at least one more equivalence class than $\{[a]_{Y_i} \in \mathbf{C}/Y_i \mid a \in \mathbf{B}'\}$. So to obtain $|\mathbf{B}'|$ equivalence classes for $\mathbf{B}'$ in $\mathbf{C}/Y$ we need at most $|\mathbf{B}'| - 1 = |\mathbf{B}| - 1$ projections. □

LEMMA 2.3. *Let* $\mathbf{B}$ *be a connected subalgebra of a finite power of* $\mathbf{M}$*. Let* $\mathbf{M}_\alpha$ *be an essential component of* $\mathbf{B}$*. Any wrap of* $\mathbf{B}$ *into* $\mathbf{M}_\alpha$ *is a homomorphism of rank* ≤ 1*.*

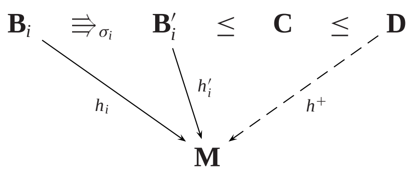
PROOF. Let $\mathbf{B} \le \mathbf{M}^n$ and $h : \mathbf{B} \to \mathbf{M}_\alpha$ be a homomorphism that wraps $\mathbf{B}$ into $\mathbf{M}_\alpha$. Let $N = |\mathbf{B}| - 1$. Assume $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}' \le \mathbf{C} \le \mathbf{D} \le \mathbf{M}^{n+k}$ and h' lifts to $\mathbf{D} \le \mathbf{M}^{n+k}$ as in the commuting diagram in Figure 1. Let Y be enough projections on $\mathbf{D}$ so that Y separates $\mathbf{B}'$. At most N are required, and they are still projections when restricted to $\mathbf{C}$. Let $\mathbf{A}_1$ be the connected component of $\mathbf{C}/Y$ containing the connected set $\{[b]_Y \mid b \in \mathbf{B}'\}$.

Let $b \in \mathbf{B}'$ and set $\mu([b]_Y) = h'(b)$. This is well-defined as Y separates $\mathbf{B}'$. Since $h'(\mathbf{B}')$ is contained in the core we may extend this to $\mu : \mathbf{A}_1 \to \mathbf{M}$ by wrapping. Fix $\pi_{i_0} \in Y$ and consider $[a]_Y$ in $\mathbf{C}/Y$ but not in $\mathbf{A}_1$. Extend μ by $\mu([a]_Y) = \pi_{i_0}(a)$. By the choice of π_{i_0} from Y this extension is well-defined. μ is a homomorphism that lifts h' to $\mathbf{C}/Y$ and Y is a set of projections of size less than or equal to N so $\operatorname{rank}(h) \leq 1$. □

LEMMA 2.4. *Let* $\mathbf{B}_1, \ldots, \mathbf{B}_t$ *be the disjoint connected components of an algebra* $\mathbf{B}$. *Let* $h_i : \mathbf{B}_i \to \mathbf{M}$ *be homomorphisms. Then* $h = \bigcup h_i$ *is a homomorphism from* $\mathbf{B}$ *to* $\mathbf{M}$.

LEMMA 2.5. *Let* $\mathbf{B}_1, \ldots, \mathbf{B}_t$ *be the disjoint connected components of an algebra* $\mathbf{B}$ *that is a subalgebra of a finite product of* $\mathbf{M}$. *Let* $h_i : \mathbf{B}_i \to \mathbf{M}$ *be homomorphisms of rank* $\leq \alpha$. *Then* $h = \bigcup h_i$ *is a homomorphism of rank* $\leq \alpha$ *from* $\mathbf{B}$ *to* $\mathbf{M}$.

PROOF. Let N_i be the maximum number of homomorphisms of rank less than α required to show $\operatorname{rank}(h_i) \leq \alpha$. Let $N = t(t-1) + |\mathbf{B}| - 1 + \sum_{i=1}^{t} N_i$. Assume $\mathbf{B} \leq \mathbf{M}^n$ and $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}' \leq \mathbf{C} \leq \mathbf{D} \leq \mathbf{M}^{n+k}$ as in the commuting diagram in Figure 1, where h' lifts by h^+ to $\mathbf{D} \leq \mathbf{M}^{n+k}$. Since $h_i = h|_{\mathbf{B}_i}$ and $h'_i = h'|_{\mathbf{B}'_i}$ we let $\sigma_i = \sigma|_{\mathbf{B}_i}$ and we have the commuting diagram



where h'_i lifts by the same h^+ to $\mathbf{D}$. As $\operatorname{rank}(h_i) \leq \alpha$, we may choose a set of homomorphisms of rank less than α, Y_i with $|Y_i| \leq N_i$ where h'_i lifts to $\mathbf{C}/Y_i$ by, say, γ_i. There is a set Y' of at most $|\mathbf{B}| - 1$ projections which separates $\mathbf{B}'$. Let $Y = Y' \cup \bigcup_{i=1}^{t} Y_i$. We construct γ a lift of h to $\mathbf{C}/Y$ using γ_i.

Let $\mathbf{C}_i$ be the connected component in $\mathbf{C}$ that contains $\mathbf{B}'_i$ and let $\mathbf{A} = \mathbf{C} \setminus \bigcup_{i=1}^{t} \mathbf{C}_i$. Let ι be the natural map from $\mathbf{B}'$ to $\mathbf{C}/Y$ where $\iota(x) = [x]_Y$. Define $\gamma : \mathbf{C}/Y \to \mathbf{M}$ by

$$\gamma(z) = \begin{cases} \gamma_i([x]_{Y_i}) & \text{if } \exists x \exists i \; z = [x]_Y \text{ and } x \in \mathbf{C}_i; \\ \gamma_1([x]_{Y_1}) & \text{otherwise.} \end{cases}$$

We need to show that γ is a well-defined homomorphism and $\gamma \circ \iota = h'$.

From the inclusion of Y' in Y it is easy to show that, for all $z \in \mathbf{C}/Y$, we have $\{x \in \mathbf{C} \mid z = [x]_Y\} \subseteq C_i \cup A$ for some $1 \leq i \leq t$. Since $[x]_Y \subseteq [x]_{Y_i}$ and $[w]_Y \subseteq [w]_{Y_i}$, if $[x]_Y = [w]_Y$ then $[x]_{Y_i} \cap [w]_{Y_i} \neq \emptyset$. This implies $[x]_{Y_i} = [w]_{Y_i}$ so γ is well defined. If $z = [x]_Y$ with $f^k(x) \in \mathbf{C}_i$ then $f^{k+1}(x) \in \mathbf{C}_i$ and

$$\begin{aligned} f(\gamma(z)) &= f(\gamma([x]_Y)) = f(\gamma_i([x]_{Y_i})) = \gamma_i(f([x]_{Y_i})) = \gamma_i([f(x)]_{Y_i}) \\ &= \gamma([f(x)]_Y) = \gamma(f([x]_Y)) = \gamma(f(z)). \end{aligned}$$

A similar argument holds in the other case. Hence γ is a homomorphism. For $x \in \mathbf{B}'$, $x \in \mathbf{B}'_i$ for some i, so

$$\gamma \circ \iota(x) = \gamma([x]_Y) = \gamma_i([x]_{Y_i}) = h'_i(x) = h'(x).$$

Thus $\operatorname{rank}(h) \leq \alpha$. □

The following lemma contains the technical details that make the major arguments in this paper work.

LEMMA 2.6. *Let* $\mathbf{B}$ *be a connected subalgebra of* $\mathbf{M}^n$ *and let* $h : \mathbf{B} \to \mathbf{M}$ *be a homomorphism. Assume* $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}' \leq \mathbf{C} \leq \mathbf{D} \leq \mathbf{M}^{n+k}$ *and* $h' = h \circ \sigma^{-1}$ *lifts to* $\mathbf{D}$*. Assume also that* $Y \subseteq \operatorname{Hom}(\mathbf{D}, \mathbf{M})$ *separates* $\mathbf{B}'$*, and, for all branch elements* $b \in \mathbf{B}'$, $\operatorname{coheight}_{\mathbf{C}/Y}([b]_Y) \leq \operatorname{coheight}_{\mathbf{M}}(h'(b))$*, then* h' *lifts to* $\mathbf{C}/Y$*.*

PROOF. We construct the homomorphism from $\mathbf{C}/Y$ to $\mathbf{M}$ that lifts h' as follows. Define $X_0 = \{[b]_Y \in \mathbf{C}/Y : b \in \mathbf{B}'\}$ and $\mu_0 : X_0 \to \mathbf{M}$ by $\mu_0([b]_Y) = h'(b)$. Note that X_0 is a connected subalgebra of $\mathbf{C}/Y$. μ_0 is well defined because Y separates $\mathbf{B}'$; and, by construction, μ_0 is a homomorphism. By hypothesis, for all $z \in X_0$, $\operatorname{coheight}_{\mathbf{C}/Y}(z) \leq \operatorname{coheight}_{\mathbf{M}}(\mu_0(z))$. We now construct a chain of subsets $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_t = \mathbf{C}/Y$ and homomorphisms $\mu_i : X_i \to \mathbf{M}$ such that $\mu_i|_{X_0} = \mu_0$ and for all $z \in X_i$, $\operatorname{coheight}_{\mathbf{C}/Y}(z) \leq \operatorname{coheight}_{\mathbf{M}}(\mu_i(z))$. Given the connected subalgebra X_i of $\mathbf{C}/Y$ containing X_0 and a homomorphism $\mu_i : X_i \to \mathbf{M}$ such that $\mu_i|_{X_0} = \mu_0$, and such that, for all $z \in X_i$, $\operatorname{coheight}_{\mathbf{C}/Y}(z) \leq \operatorname{coheight}_{\mathbf{M}}(\mu_i(z))$, construct X_{i+1} and $\mu_{i+1} : X_{i+1} \to \mathbf{M}$ as follows. Let

$$X_{i+1} = X_i \cup \{z \in \mathbf{C}/Y \mid f(z) \in X_i\}.$$

Define $\mu_{i+1}(z)$ by considering the following three mutually exclusive and exhaustive cases.

Case I. If $z \in X_i$, then $\mu_{i+1}(z) = \mu_i(z)$.

Case II. If $z \notin X_i$ but $\mu_i(f(z))$ is in a core L, then $f^{-1}(\mu_i(f(z))) \cap L = \{f^{\operatorname{circ}(L)-1}(\mu_i(f(z)))\}$. Let $\mu_{i+1}(z) = f^{\operatorname{circ}(L)-1}(\mu_i(f(z)))$. Since $f(z) \in X_i$,

$$f(\mu_{i+1}(z)) = f^{\operatorname{circ}(L)}(\mu_i(f(z))) = \mu_i(f(z)) = \mu_{i+1}(f(z)).$$

Case III. $z \notin X_i$ and $\mu_i(f(z))$ is a branch element such that

$$1 \leq \operatorname{coheight}_{\mathbf{C}/Y}(f(z)) \leq \operatorname{coheight}_{\mathbf{M}}(\mu_i(f(z))).$$

Pick $v \in \mathbf{M}$ such that $f(v) = \mu_i(f(z))$ and $\operatorname{coheight}_{\mathbf{M}}(v)$ is maximal. Let $\mu_{i+1}(z) = v$. Then

$$\begin{aligned} \operatorname{coheight}_{\mathbf{C}/Y}(z) &\leq \operatorname{coheight}_{\mathbf{C}/Y}(f(z)) - 1 \leq \operatorname{coheight}_{\mathbf{M}}(\mu_i(f(z))) - 1 \\ &= \operatorname{coheight}_{\mathbf{M}}(v) = \operatorname{coheight}_{\mathbf{M}}(\mu_{i+1}(z)). \end{aligned}$$

We have $f(\mu_{i+1}(z)) = f(v) = \mu_i(f(z)) = \mu_{i+1}(f(z))$. Thus $\mu_{i+1} : X_{i+1} \to M$ is a homomorphism satisfying the same properties as μ_i. For some s, X_s is maximal and hence a connected component of $\mathbf{C}/Y$. It is easy to check that $\mathbf{C}/Y \in \mathbf{ISP}(\mathbf{M})$. By Lemma 2.4 we may extend μ_s to a homomorphism on all of $\mathbf{C}/Y$ by wrapping each component $\mathbf{C}_\alpha$, disjoint from X_s, onto an essential component of $\mathbf{C}_\alpha$. By construction μ_s lifts h' to $\mathbf{C}/Y$. □

Figure 3 shows an example where factoring over any set Y of two or fewer projections would result in the coheight of the equivalence class of the element aaa being 1 while the coheight of aaa itself is 0, invalidating one of the hypotheses of Lemma 2.6. Adding to the set Y a homomorphism that forces the equivalence class of aaa to be a singleton would prevent this situation from occurring. We now define such homomorphisms.

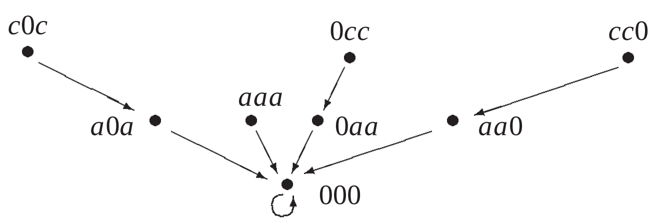


FIGURE 3. Example illustrating the need for singleton equivalence classes to maintain coheight.

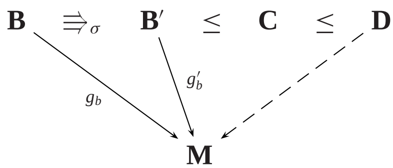
Let $\mathbf{B} \leq \mathbf{M}^n$. For a branch element $b = (b_1 \ldots, b_n) \in \mathbf{B}$, we now wish to construct a homomorphism, $g_b : \mathbf{B} \to \mathbf{M}$, which has rank less than or equal to 1. Let $\mathbf{A}_1$ be the connected component of $\mathbf{B}$ containing b. Pick u in an essential component, $\mathbf{M}_\alpha$, of $\mathbf{A}_1$ such that $\text{coheight}_{\mathbf{M}}(u)$ is finite and maximal over all elements in essential components of $\mathbf{A}_1$. Let $t = \text{coheight}_{\mathbf{B}}(b)$. Note $f(u)$ is in the core of $\mathbf{M}_\alpha$ and

$$\begin{aligned} t = \text{coheight}_{\mathbf{B}}(b) &\leq \min\{\text{coheight}_{\mathbf{M}}(b_i) \mid 1 \leq i \leq n\} \\ &\leq \max\{\text{coheight}_{\mathbf{M}}(a) \mid a \text{ is a branch element in any essential component of } \mathbf{A}_1\} \\ &= \text{coheight}_{\mathbf{M}}(u). \end{aligned}$$

The first inequality holds because any element of $\mathbf{B}$ has a pre-image only if each coordinate has a pre-image. So we may choose $v \in \mathbf{M}_\alpha$ with $f^t(v) = u$. Define $g_b(b) = u$. For any x in $\mathbf{B}$ and any s with $1 \leq s \leq t$, where $f^s(x) = b$, define $g_b(x) = f^{t-s}(v)$. Wrap the remaining elements of $\mathbf{A}_1$ around the core of $\mathbf{M}_\alpha$ specifying $g_b(f(b)) = f(u)$. By Lemma 2.4, we may extend g_b by wrapping each connected component $\mathbf{A}_2$, distinct from $\mathbf{A}_1$, around an essential component of $\mathbf{A}_2$. By construction g_b is a homomorphism whose image contains one component with a single branch and, possibly, some cores. In addition, for $b \in \mathbf{B}$, we have $g_b^{-1}(g_b(b)) = \{b\}$ so the equivalence class of b in $\mathbf{B}/\{g_b\}$ is a singleton. Thus the coheight of b does not increase when we factor $\mathbf{B}$ over a set of homomorphisms including g_b. The significance of this is found in the next two lemmas.

LEMMA 2.7. *For b a branch element of* $\mathbf{B}$, $\operatorname{rank}(g_b) \leq 1$.

PROOF. Since g_b is a wrapping on all but one component, by Lemma 2.3 and Lemma 2.5, we may assume without loss of generality that $\mathbf{B}$ is a connected component. Assume $\mathbf{D} \leq \mathbf{M}^{n+k}$, $g'_b = g_b \circ \sigma^{-1}$ lifts to $\mathbf{D}$, and the following diagram commutes.



Let $Y = \{\pi_j \mid j \in J\}$ be a set of projections that separates $\mathbf{B}'$. By Lemma 2.2 we may assume $|Y| \leq |\mathbf{B}'| - 1$. The natural map embeds $\mathbf{B}'$ in $\mathbf{C}/Y$ as Y separates $\mathbf{B}'$. We will use Lemma 2.6 show g'_b lifts to $\mathbf{C}/Y$.

Recall $g'_b(\sigma(b)) = g_b(b)$ lies in an essential component of $\mathbf{B}$ and was chosen such that $\operatorname{coheight}_{\mathbf{M}}(g_b(b))$ is maximal over all essential components of $\mathbf{B}$, which are exactly the essential components of $\mathbf{B}'$. For all branch elements $a \in \mathbf{B}'$, either $\operatorname{coheight}_{\mathbf{M}}(g'_b(a)) = \infty$ or for some finite $s \geq 0$, $f^s(\sigma^{-1}(a)) = b$. We only need to consider the latter case.

$$\begin{aligned}\operatorname{coheight}_{\mathbf{C}/Y}([a]_Y) &\leq \min\{\operatorname{coheight}_{\mathbf{M}}(\pi_j(a)) \mid j \in J\} \leq \operatorname{coheight}_{\mathbf{M}}(g_b(b)) \\ &= \operatorname{coheight}_{\mathbf{M}}(g'_b(\sigma(b))) = \operatorname{coheight}_{\mathbf{M}}(g'_b(a)) + s.\end{aligned}$$

The first inequality holds as an element has a pre-image only if each coordinate has a pre-image. The second inequality holds because for all i, the projection $\pi_i(a)$ is in an essential component of $\mathbf{B}'$. Finally, since $f^s(\sigma^{-1}(a)) = b$, the last equality holds by the definition of g_b. Thus, by Lemma 2.6, g'_b lifts to $\mathbf{C}/Y$. □

LEMMA 2.8. *Let $\mathbf{B} \leq \mathbf{M}^n$ and $h : \mathbf{B} \to \mathbf{M}$ be a homomorphism; then* $\operatorname{rank}(h) \leq 2$.

PROOF. By Lemma 2.5, we may assume that $\mathbf{B}$ is connected. Assume $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}' \leq \mathbf{C} \leq \mathbf{D} \leq \mathbf{M}^{n+k}$ and $h' = h \circ \sigma^{-1}$ lifts to $\mathbf{D} \leq \mathbf{M}^{n+k}$ as in the commuting diagram in Figure 1. By Lemma 2.2 we may choose a set of projections Y_1 from $\mathbf{D}$ to $\mathbf{M}$ that separates $\mathbf{B}'$ such that Y_1 has size at most $|\mathbf{B}| - 1$. Let

$$Y_2 = \{g_b : \mathbf{D} \to \mathbf{M} \mid b \text{ is a branch element of } \mathbf{B}'\}.$$

Define $Y = Y_1 \cup Y_2$ and let $N = (|\mathbf{B}| - 1) + |\mathbf{B}|$. Then $|Y| \le N$. For $\alpha \in Y$, $\operatorname{rank}(\alpha|_{\mathbf{C}}) \le 1$ as either $\alpha|_{\mathbf{C}}$ is a projection or is $g_b|_{\mathbf{C}} : \mathbf{C} \to \mathbf{M}$ for some $b \in \mathbf{B}' \le \mathbf{C}$. Since Y separates points of $\mathbf{B}'$, the latter embeds naturally in $\mathbf{C}/Y$.

For every branch element $b \in \mathbf{B}'$, the inclusion of g_b in Y forces $[b]_Y = \{b\}$. This means $\operatorname{coheight}_{\mathbf{C}/Y}([b]_Y) = \operatorname{coheight}_{\mathbf{C}}(b)$. Since h' lifts to $\mathbf{C}$, $\operatorname{coheight}_{\mathbf{C}}(b) \le \operatorname{coheight}_{\mathbf{M}}(h'(b))$. Thus $\operatorname{coheight}_{\mathbf{C}/Y}([b]_Y) \le \operatorname{coheight}_{\mathbf{M}}(h'(b))$. By Lemma 2.6, h' lifts to $\mathbf{C}/Y$. ☐

THEOREM 2.9. *Finite mono-unary algebras are strongly dualizable.*

PROOF. In [7], Pitkethly shows that finite mono-unary algebras are dualizable. In [8], Willard shows that dualizable algebras with finite rank are strongly dualizable. ☐

3. Examples

Consider the mono-unary algebra, $\mathbf{M}$, with four elements $\{0, a, b, c\}$, where $f(c) = a$, $f(a) = f(b) = f(0) = 0$. This algebra, found by R. Willard, was previously the only known algebra with rank 2. In fact, there are still no known algebras with finite rank larger than 2. Here we illustrate that $\mathbf{M}$ has rank 2 and construct an alter ego that provides a strong duality. This can, in fact, be done for any finite, mono-unary algebra with core a single element.

Let $B = \{0, a\}$. Consider the homomorphism $h : \mathbf{B} \to \mathbf{M}$ given by $h(a) = b$ and $h(0) = 0$. Let $N \ge 1$. Let $\mathbf{D} = \mathbf{M}^{N+1} \setminus \{\hat{c}\}$, where $\hat{c}$ is the element $(c, \dots, c)$ and let $\mathbf{B} \Rrightarrow_\sigma \mathbf{B}' \le \mathbf{D}$. Then h', the natural extension of h to $\mathbf{B}'$, lifts to $\mathbf{D}$. Neither $\hat{a} = (a, \dots, a)$ has a pre-image in $\mathbf{D}$ nor does $h'(\hat{a})$ have a pre-image in $\mathbf{M}$. Let Y be a collection of N or fewer projections, then $[\hat{a}]_Y$ has a pre-image in $\mathbf{D}/Y$. Thus h' does not lift to $\mathbf{D}/Y$. (See the example illustrated in Figure 3.) In order for h' to lift to $\mathbf{D}/Y$, Y must have either $k + 1$ projections or a non-projection. So $\operatorname{rank}(h) \not\le 1$. By Lemma 2.8 $\operatorname{rank}(h) = 2$. A similar argument works for any connected mono-unary algebra that has two branches of different heights. Thus there are many mono-unary algebras of rank 2.

A relation on $\mathbf{M}$ is *algebraic* if it is a subalgebra of a finite power of $\mathbf{M}$ viewed as a relation. A (partial) operation h is *algebraic* if it is a homomorphism $h : \mathbf{B} \to \mathbf{M}$, where $\mathbf{B}$ is a (subalgebra of a) finite power of $\mathbf{M}$. Let $\mathscr{P}$ be a set of algebraic operations and algebraic partial operations on $\mathbf{M}$. The set of algebraic operations and partial operations on $\mathbf{M}$ generated by $\mathscr{P}$ using projections, composition, and restriction of domain is called the *closure* of $\mathscr{P}$. If the closure of $\mathscr{P}$ is all finitary algebraic and partial algebraic operations on $\mathbf{M}$ then we say $\mathscr{P}$ *generates the finitary algebraic operations on* $\mathbf{M}$. In order to show that the alter ego, $\mathbb{M} = \langle M; \mathscr{P}, \tau \rangle$, where τ is the discrete topology, strongly dualizes $\mathbf{M}$ it is sufficient to know that

M is strongly dualizable and to then show that $\mathscr{P}$ generates the finitary algebraic operations on **M** (see [1], page 282). By Theorem 2.9, **M** is strongly dualizable. We now explicitly construct a set $\mathscr{P}$ of algebraic operations and partial operations of arity at most 2 that generates the finitary algebraic operations on **M**. That is, we construct an alter ego for **M** that strongly dualizes **M**.

The partial order defined by $0 \leq a \leq c$ and $0 \leq b$ induces a semilattice meet operation, $\wedge$, which is a homomorphism from $\mathbf{M}^2$ to **M**. The join operation, $\vee$, defined by the linear ordering $0 \leq b \leq a \leq c$ is also a homomorphism from $\mathbf{M}^2$ to **M**. Let $\mathbf{B} \leq \mathbf{M}^n$. Every homomorphism $h : \mathbf{B} \to \mathbf{M}$ is defined by its behaviour on the branches of $\mathbf{M}^n$ so it will be sufficient to have in $\mathscr{P}$ homomorphisms that behave in a fixed way on a particular branch and are 0 elsewhere. We then may use the join operation to build h. We now define these branch homomorphisms on $\mathbf{M}^2$.

For $\bar{v} \in \mathbf{M}^2 \setminus \{(0,0)\}$, define $g^a_{\bar{v}} : \mathbf{M}^2 \to \mathbf{M}$ by

$$g^a_{\bar{v}}(x) = \begin{cases} c & \text{if } x \in f^{-1}(\bar{v}); \\ a & \text{if } x = \bar{v}; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\underline{0}$ be the constant valued homomorphism from **M** to **M** with value 0 and define the homomorphism $\phi : \langle\{0, a, b\}, f\rangle \to \mathbf{M}$ by $\phi(a) = b$ and $\phi(0) = \phi(b) = 0$. We now show that the set of homomorphisms

$$\mathscr{P} = \{\wedge, \vee, \underline{0}, \phi\} \cup \{g^a_{\bar{v}} \mid \bar{v} \in \mathbf{M}^2 \setminus \{(0,0)\}\}$$

generates the algebraic operations of **M**. It will follow that the alter ego $\mathbb{M} = \langle M; \mathscr{P}, \tau\rangle$, where τ is the discrete topology, strongly dualizes **M**.

For a branch element $\bar{u} \in \mathbf{M}^n$ we construct, in the closure of $\mathscr{P}$, all of the algebraic operations and maximal algebraic partial operations that are nonzero on the branch of $\bar{u}$ and are zero elsewhere. For each $s \in M$, define

$$I_s = \{i \mid 1 \leq i \leq n, \pi_i(\bar{u}) = s\}.$$

Since $I_a \cup I_b \cup I_c$ is nonempty, we can define $G^a_{\bar{u}} : \mathbf{M}^n \to \mathbf{M}$ in the closure of $\mathscr{P}$ by

$$G^a_{\bar{u}}(x) = \bigwedge\{\, g^a_{(s,t)}(\pi_i(x), \pi_j(x)) \mid i \in I_s,\ j \in I_t \text{ and } (s,t) \in M^2 \setminus \{(0,0)\}\,\},$$

and we have

$$G^a_{\bar{u}}(x) = \begin{cases} c & \text{if } x \in f^{-1}(\bar{u}); \\ a & \text{if } x = \bar{u}; \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in \mathbf{M}^n$. If I_c is nonempty, define $G^c_{\bar{u}} : \mathbf{M}^n \to \mathbf{M}$ by $G^c_{\bar{u}} = G^a_{f(\bar{u})}$. Then, for all $x \in \mathbf{M}^n$,

$$G^c_{\bar{u}}(x) = \begin{cases} c & \text{if } x \in f^{-1}(f(\bar{u})); \\ a & \text{if } x = f(\bar{u}); \\ 0 & \text{otherwise.} \end{cases}$$

We now define $B_{\bar{u}} = M^n \setminus f^{-1}(\bar{u})$ and $G^b_{\bar{u}} : \mathbf{B}_{\bar{u}} \to \mathbf{M}$ by $G^b_{\bar{u}} = \phi \circ G^a_{\bar{u}}|_{\mathbf{B}_{\bar{u}}}$, so for all $x \in \mathbf{B}_{\bar{u}}$

$$G^b_{\bar{u}}(x) = \begin{cases} b & \text{if } x = \bar{u}; \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for all $\bar{u} \in \mathbf{M}^n$ define $G^0_{\bar{u}} : \mathbf{M}^n \to \mathbf{M}$ by $G^0_{\bar{u}} = \underline{0}$.

LEMMA 3.1. *Let $h : \mathbf{B} \to \mathbf{M}$ be a homomorphism, where $\mathbf{B} \leq \mathbf{M}^n$. Then $h = \bigvee_{\bar{u} \in \mathbf{B}} G^{h(\bar{u})}_{\bar{u}}|_{\mathbf{B}}$, and therefore h is in the closure of $\mathscr{P}$.*

PROOF. Since $G^c_{\bar{u}} = G^a_{f(\bar{u})}$, for all $\bar{u}, \bar{w} \neq \hat{0}$ in $\mathbf{B}$ either $G^{h(\bar{u})}_{\bar{u}} = G^{h(\bar{w})}_{\bar{w}}$ or $\bar{u}$ and $\bar{w}$ are in different branches. In the latter case, for all x, at least one of $G^{h(\bar{u})}_{\bar{u}}(x)$ and $G^{h(\bar{w})}_{\bar{w}}(x)$ is 0. Thus for $x \in \mathbf{B}$, we have $\{G^{h(\bar{u})}_{\bar{u}}(x) \mid \bar{u} \in \mathbf{B}\} = \{0, h(x)\}$ and $\bigvee_{\bar{u} \in \mathbf{B}} G^{h(\bar{u})}_{\bar{u}}(x)$ is well-defined and equals $h(x)$. $\square$

Thus every algebraic operation and partial operation on $\mathbf{M}$ is in the closure of $\mathscr{P}$ so $\mathbb{M}$ strongly dualizes $\mathbf{M}$.

Define the algebra $\mathbf{L}$ to have universe $\{0, a, b, c, d\}$ with a unary operation f defined as follows $f(c) = a$, $f(a) = 0$, $f(d) = b$, $f(b) = 0$, and $f(0) = 0$.

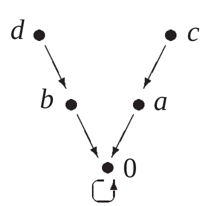


FIGURE 4. The algebra $\mathbf{L}$.

Since the height of every maximal branch of $\mathbf{L}$ is 2, $\mathbf{L}$ is injective in the quasivariety it generates. That is, for every pair of algebras $\mathbf{B} \leq \mathbf{C}$ in $\mathbf{ISP}(\mathbf{L})$, every homomorphism $h : \mathbf{B} \to \mathbf{L}$ can be lifted to $\mathbf{C}$. As injective algebras have rank 1, we have $\text{rank}(\mathbf{L}) = 1$.

Note that $\mathbf{L} \in \mathbf{SP}(\mathbf{M})$ and $\mathbf{M} \in \mathbf{SP}(\mathbf{L})$ so they generate the same quasivariety. Thus a quasivariety can have generators with different ranks. This contrasts with duality and strong duality, as two algebras that generate the same quasivariety are either both (strongly) dualizable or both not (strongly) dualizable. See [4] and [3].

In [5] we give an example of a bi-unary algebra with infinite rank that is dualizable but not fully dualizable. Hence one remaining open problem is to determine if an algebra with finite rank greater than 2 exists.

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