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ON THE TOEPLITZ ALGEBRAS OF RIGHT-ANGLED AND FINITE-TYPE ARTIN GROUPS

JOHN CRISP and MARCELO LACA

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Abstract

The graph product of a family of groups lies somewhere between their direct and free products, with the graph determining which pairs of groups commute. We show that the graph product of quasi-lattice ordered groups is quasi-lattice ordered, and, when the underlying groups are amenable, that it satisfies Nica's amenability condition for quasi-lattice orders. The associated Toeplitz algebras have a universal property, and their representations are faithful if the generating isometries satisfy a joint properness condition. When applied to right-angled Artin groups this yields a uniqueness theorem for the C^*-algebra generated by a collection of isometries such that any two of them either $*$-commute or else have orthogonal ranges. The analogous result fails to hold for the nonabelian Artin groups of finite type considered by Brieskorn and Saito, and Deligne.

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1. Introduction

Several celebrated results in C^*-algebra theory assert that the C^*-algebra generated by a semigroup of isometries does not depend on the specific isometries, provided they satisfy a properness condition. The situations described by these results are of considerable interest, stemming from the fact that the algebraic structure given by the semigroup operation determines a unique C^*-norm on the $*$-algebra generated by the isometries. As examples we have Coburn's theorem on the C^*-algebra generated by a

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single isometry [4], Douglas' theorem on the C^*-algebra of a one parameter semigroup of isometries [11], and the generalization by Murphy to the Toeplitz C^*-algebra of a totally ordered group [17]; in all these cases the properness condition simply says that the isometries are not unitary.

Moving away from total orders on abelian groups, Nica [18] considered a class of partially ordered groups (G, P) he called *quasi-lattice ordered*. Inspired by what happens with the left regular (Toeplitz) representation of the positive cone P, he isolated a key covariance condition, which is automatic for total orders, and defined a universal C^*-algebra $C^*(G, P)$ whose representations are given by the covariant isometric representations of P. He proved that the uniqueness of the C^*-algebra generated by a covariant isometric representation depends on an amenability property of the quasi-lattice order that is strictly weaker than amenability of the underlying group. Indeed, he showed that Cuntz's result [7] on the uniqueness of the C^*-algebra $\mathcal{T}\mathcal{O}_n$ generated by n isometries with orthogonal ranges can be seen as an amenability result for the canonical quasi-lattice order on the free group on n generators. In this case the covariance condition requires that the generating isometries have orthogonal ranges, and the properness condition says that the sum of these ranges is not the whole Hilbert space.

In [16] Laca and Raeburn associated a semigroup dynamical system to each quasi-lattice order and showed that the corresponding crossed product is canonically isomorphic to the universal C^*-algebra $C^*(G, P)$. This approach led to two main advances. The first one was the generalisation to all quasi-lattice orders of some key estimates of Cuntz [6], which provides a convenient framework in which to study faithfulness of representations and uniqueness properties. The second one was a direct proof of the amenability of the quasi-lattice orders on a large class of (nonamenable) free product groups, which widened the range of application of the uniqueness results.

Direct products and free products of groups are both special cases of the more general construction of a graph product of groups (see Section 2 below), and in this paper we address the natural questions of whether graph products support quasi-lattice orders, and under which conditions these quasi-lattice orders are amenable in the sense of Nica [18]. Our main technical results are Theorem 10, which shows that graph products of quasi-lattice ordered groups are indeed quasi-lattice ordered, and Theorem 20, which gives a sufficient condition for their amenability. Combining this with the results of [16] in Theorem 21, we characterize faithful representations and give a uniqueness result for the Toeplitz algebras of graph products.

An interesting class of examples is that of graph products of copies of $(\mathbb{Z}, \mathbb{N})$, otherwise known as graph groups or right-angled Artin groups. It follows from our main results that, in the language of Nica [18], they are all amenable quasi-lattice orders, giving a unified statement of the amenability of the canonical quasi-lattice orders on all free groups and all free abelian groups, as well as providing many new

examples of amenable quasi-lattice orders. The corresponding Toeplitz C^*-algebras are thus universal and unique. We state this main result in terms of generators and relations in Theorem 24, which contains, as extreme cases, Cuntz's theorem (in which the generating isometries have mutually orthogonal ranges) and a multivariable version of Coburn's theorem, (in which the generating isometries $*$-commute, that is, they commute with each other and with each other's adjoints). See [20] for results related to this last situation.

Other interesting quasi-lattice orders are provided by the family of Artin groups of finite type, with the embedded Artin monoid as positive cone [1, 10]. These examples, which include the braid groups, are *lattice groups*, because every pair of elements has a least common upper bound. In Section 6 we prove, using an argument essentially due to Nica, that if a group is lattice ordered and amenable as a quasi-lattice order, then the group itself has to be amenable. Thus, in contrast to what happens with graph products, only the Artin groups of finite type that are amenable (and hence abelian) give rise to amenable quasi-lattice orders. The nonabelian Artin groups of finite type appear then as an important class of groups having canonical non-amenable quasi-lattice orders. As a consequence, the C^*-algebra generated by a covariant isometric representation depends, in general, on the specific representation, Theorem 30.

2. Graph products of groups

Graph products were defined in the thesis of Green [13], and have been subsequently studied by various other authors. We refer the reader to [14] and the references therein for further background.

Let Γ denote a graph with vertex set Λ, and edge set $E(\Gamma) \subseteq \{\{I, J\} : I, J \in \Lambda \text{ and } I \neq J\}$; the edges of Γ have no orientation and there is no edge joining a point to itself. We say that vertices I and J are *adjacent* in Γ if there is an edge $\{I, J\} \in E(\Gamma)$. Note that a vertex is never adjacent to itself. Given a family $\{G_I\}_{I\in\Lambda}$ of groups, we define the *graph product* $\Gamma_{I\in\Lambda} G_I$ to be the quotient of the free product $*_\Lambda G_I$ by the smallest normal subgroup containing the elements $x_1x_2x_1^{-1}x_2^{-1}$ for all pairs $x_1 \in G_I$, $x_2 \in G_J$ where I and J are adjacent in Γ. When the G_I are all copies of $\mathbb{Z}$, the graph product is called a *graph group*, or *right-angled Artin group*. We shall not need to assume that Γ is finite.

Suppose, for the rest of this section, that we are given a graph Γ, as above, and groups $\{G_I\}_{I\in\Lambda}$, and let $G = \Gamma_{I\in\Lambda} G_I$ denote the graph product. We may take as a generating set for G the set

$$\mathbb{G} = \coprod_{I\in\Lambda} G_I \setminus \{1\}.$$

Given $x \in \mathbb{G}$ we write $I(x)$ for the unique vertex I such that $x \in G_I$. We say that x

belongs to $I(x)$.

By an *expression* X for an element $x \in G$ we mean a word in the generators $\mathbb{G}$ representing x. Given an expression $X = x_1 x_2 \cdots x_l$, the elements x_i are called *syllables* of X and the number l is called the *length* of X, written $l = \ell(X)$. We say that $I \in \Lambda$ is a *vertex of* X if $I = I(x_i)$ for x_i a syllable of X.

Given an expression $X = x_1 x_2 \cdots x_l$ for $x \in G$, the graph product relations allow one to modify X to obtain a different expression for x by replacing a subexpression $x_i x_{i+1}$ with $x_{i+1} x_i$ if $I(x_i)$ is adjacent to $I(x_{i+1})$. In the terminology of [14], such a substitution is called a *shuffle*, and we shall say that two expressions are *shuffle equivalent* if one may be obtained from the other via a finite sequence of shuffles. If the expression X contains a subexpression of the form $x_i x_{i+1}$, with $I(x_i) = I(x_{i+1})$, then we may give a shorter expression for x by an *amalgamation*, that is by deleting $x_i x_{i+1}$ in the case that $x_{i+1} = x_i^{-1}$ or otherwise by replacing the two syllables $x_i x_{i+1}$ with the single syllable $\hat{x}_i \in \mathbb{G}$ such that $\hat{x}_i = x_i x_{i+1}$.

We say that an expression is *reduced* if it is not shuffle equivalent to any expression which admits an amalgamation.

LEMMA 1. *Given an expression $X = x_1 x_2 \cdots x_l$, the following are equivalent*:

(i) *X is reduced*;

(ii) *for all $i < j$ such that $I(x_i) = I(x_j)$ there exists k such that $i < k < j$ and $I(x_k)$ is not adjacent to $I(x_i)$.*

PROOF. That (ii) is a consequence of (i) is obvious. We note that the truth of condition (ii) is invariant under shuffles, while an amalgamation can never be performed on an expression satisfying (ii). Thus (ii) implies (i). □

Given an expression $X = x_1 x_2 \cdots x_l$ for $x \in G$, one may easily produce a reduced expression for x via a finite process of shuffling and amalgamating. If X fails condition (ii) of Lemma 1, that is, there exists $i < j$ such that $I(x_i) = I(x_j)$ and $I(x_i)$ commutes with $I(x_k)$ for all $i < k < j$, then an obvious sequence of shuffles will allow the syllables x_i and x_j to be amalgamated, reducing the length of X. One may continue reducing the length in this manner until condition (ii) is satisfied. The following theorem, due to Green [13], reduces the solution of the word problem in a graph product of groups to the solution of the word problem in each of the component groups (see [14] for an equivalent formulation).

THEOREM 2 (Green [13]). *Any two reduced expressions for the same element of G are shuffle equivalent.*

As a simple consequence of Theorem 2 we may define the *length* $\ell(x)$ of an element $x \in G$ to be the length of any reduced expression X for x, note that $\ell(x)$ is also the minimal length of any expression for x. If X is a reduced expression for x, we are

also justified in referring to a syllable or a vertex of X as a *syllable of* x or a *vertex of* x, respectively.

Let $X = x_1x_2\cdots x_l$ be a reduced expression, and x_i any syllable of X. If $I(x_i)$ is adjacent to $I(x_j)$ for all $j < i$ then we say that x_i is an *initial syllable* of X, and that $I(x_i)$ is an *initial vertex* of X. We write $\Delta(X)$ for the set of all initial vertices of X. The following facts are easily checked.

LEMMA 3. *Let $X = x_1x_2\cdots x_l$ be a reduced expression.*

(i) *If x_i is an initial syllable of X, then X is shuffle equivalent to the expression*

$$x_ix_1\cdots x_{i-1}x_{i+1}\cdots x_l.$$

(ii) *The initial vertices of X are pairwise adjacent.*

(iii) *For each $I \in \Delta(X)$ there exists a unique initial syllable of X belonging to I. We define the function $\delta_X : \Delta(X) \to \mathbb{G}$ such that $\delta_X(I)$ is equal to the initial syllable of X belonging to I.*

(iv) *If X' is another expression which is shuffle equivalent to X, then $\Delta(X') = \Delta(X)$ and $\delta_{X'} = \delta_X$.*

By virtue of Lemma 3 (iv) and Theorem 2 we may make the following definitions.

DEFINITION 4. Let $x \in G$. We define the set of *initial vertices* of x to be the set $\Delta(x) = \Delta(X)$ where X is any reduced expression for x. We also define, for each $I \in \Lambda$, the element x_I to be the initial syllable of x that belongs to I if I is an initial vertex of x, and 1 if not; that is,

$$x_I = \begin{cases} \delta_X(I) & \text{if } I \in \Delta(x); \\ 1 & \text{if } I \notin \Delta(x). \end{cases}$$

Given an expression $X = x_1x_2\cdots x_l$ let $\mathrm{rev}(X)$ denote the expression $x_lx_{l-1}\cdots x_1$. This allows us to define the set of *final vertices* of $x \in G$ to be the set $\Delta^r(x) = \Delta(\mathrm{rev}(X))$, where X is any reduced expression for x. Similarly also, for each $I \in \Lambda$, we define the element

$$x_I^r = \begin{cases} \delta_{\mathrm{rev}(X)}(I) & \text{if } I \in \Delta^r(x); \\ 1 & \text{if } I \notin \Delta^r(x), \end{cases}$$

where X is any reduced expression for x.

Observe that, by using Lemma 3 (i) and (ii), one may always find, for any given element $x \in G$, a reduced expression which begins with the product, in any order, of the x_I for $I \in \Delta(x)$. Equally, one may always find a reduced expression for x which ends with the product, in any order, of the x_I^r for $I \in \Delta^r(x)$.

LEMMA 5. *Given $x, y \in G$, let $D = \Delta^r(x) \cap \Delta(y)$, and suppose that $z_I = x_I^r y_I$ is nontrivial for each $I \in D$. We define an expression $Z = \prod_{I\in D} z_I$ (in any order). Then, if $X\big(\prod_{I\in D} x_I^r\big)$ and $\big(\prod_{I\in D} y_I\big)Y$ are reduced expressions for x and y respectively, we have that XZY is a reduced expression for xy.*

PROOF. Since each element z_I is nontrivial, it is clear that both expressions XZ and ZY are reduced since they are formally equivalent to the given reduced expressions for x and y respectively. (Two expressions $x_1x_2\cdots x_l$ and $y_1y_2\cdots y_k$ are *formally equivalent* if $k = l$ and $I(x_i) = I(y_i)$ for all $1 \le i \le l$.)

Suppose that $XZY = w_1w_2\cdots w_l$ is not reduced. Then, by Lemma 1, one may find $i < j$ such that $I(w_i) = I(w_j)$ and $I(w_i)$ is adjacent to $I(w_k)$ for all k with $i < k < j$. But w_i and w_j are not both in XZ nor both in ZY, since these are reduced expressions. Thus w_i must be from the subexpression X and w_j from Y. But now it follows that w_i is a final syllable of XZ and w_j an initial syllable of ZY. That is, $I(w_i) = I(w_j) = J$ for some $J \in D$. But this contradicts the fact that there is a syllable $w_k = z_J$ lying between w_i and w_j. □

3. Quasi-lattice orders and their graph products

Let G be a group and let P be a submonoid (subsemigroup containing the identity) of G such that $P \cap P^{-1} = \{1\}$. Then we may define a left-invariant partial order on G by $x \le y$ whenever $x^{-1}y \in P$. Note that $x \in P$ if and only if $1 \le x$. We observe, indeed, that every left-invariant partial order on G arises in this fashion. We say that (G, P) is a *partially ordered group* with *positive cone* P.

DEFINITION 6. A partially ordered group (G, P) is *quasi-lattice ordered* if every finite set in G with an upper bound in G has a (necessarily unique) least upper bound in G. Equivalently, every pair x, y of elements of G with a common upper bound in G has a least upper bound, which we denote by $x \vee y$. If x and y have no common upper bound in G, then we write $x \vee y = \infty$ for convenience.

Given the group G with positive cone P we may equally define a right-invariant partial order on G by $x \le_r y$ whenever $yx^{-1} \in P$. If x and y have a greatest lower bound for $\le_r$, we denote it by $x \wedge_r y$. Clearly one has

$$x \le_r y \text{ if and only if } y^{-1} \le x^{-1}, \quad \text{and} \quad x \wedge_r y = \big(x^{-1} \vee y^{-1}\big)^{-1}. \tag{1}$$

LEMMA 7. *For a partially ordered group (G, P) the following statements are equivalent.*

(i) *(G, P) is a quasi-lattice order.*

(ii) *Every finite set in G with a common upper bound in P has a least upper bound in P.*

(iii) *Every element x of G having an upper bound in P has a least upper bound in P.*

(iv) *If $x \in PP^{-1}$, then there exist a pair of elements $a, b \in P$ with $x = ab^{-1}$ and such that for every $u, v \in P$ with $ab^{-1} = uv^{-1}$, one has $a \leq u$ and $b \leq v$. (The pair a, b is clearly unique.)*

(v) *Every pair u, v of elements in P has a greatest lower bound $u \wedge_r v$ with respect to the right-invariant partial order on G.*

(vi) *If $x \in PP^{-1}$, then there exist a pair of elements $a, b \in P$ with $x = ab^{-1}$ and such that $a \wedge_r b = 1$.*

Assuming that (i)–(vi) *hold, and given $x \in PP^{-1}$, there is in fact a unique pair $a, b \in P$ satisfying statement* (vi)*, being precisely the pair a, b of statement* (iv)*.*

PROOF. We prove (i) implies (ii) implies (iii) implies (i) first. Clearly (ii) follows from (i) by observing that a (least) upper bound in P for a finite set F is the same thing as a (least) upper bound for $F \cup \{1\}$. Statement (iii) is obviously a special case of (ii). Finally, (iii) states that the condition of Definition 6 holds for all pairs $x, 1$, and (i) may be recovered from this by left invariance of the partial order.

(iii) implies (iv): Suppose $x \in PP^{-1}$. Then x has an upper bound in P, and by (iii) we can take a to be the least upper bound in P, and write $x = ab^{-1}$ for $b \in P$. If $x = uv^{-1}$ for $u, v \in P$, then u is an upper bound for x so we have $a \leq u$, and hence also $b \leq v$ (since $b^{-1}v = a^{-1}u \in P$).

(iv) implies (v): Given $u, v \in P$, let $x = uv^{-1}$. Then the pair $a, b \in P$ of (iv) determines an element $w_0 := a^{-1}u = b^{-1}v$ which one checks to be the greatest right lower bound $u \wedge_r v$. For if w is a common right lower bound then $u = cw$ and $v = dw$, for some $c, d \in P$, and one has $cd^{-1} = uv^{-1} = ab^{-1}$. But, by (iv), both $a \leq c$ and $b \leq d$, so that $w \leq_r w_0$.

(v) implies (vi): Given $x \in PP^{-1}$, choose $u, v \in P$ such that $x = uv^{-1}$. Using (v) we may write $u = a(u \wedge_r v)$ and $v = b(u \wedge_r v)$ for some $a, b \in P$. Clearly $uv^{-1} = ab^{-1}$ and $a \wedge_r b = 1$.

(vi) implies (iii): Let $x \in G$ have an upper bound u in P. Then $x \in PP^{-1}$, and by (vi) we can write $x = ab^{-1}$ for $a, b \in P$ with $a \wedge_r b = 1$, or rather $a^{-1} \vee b^{-1} = 1$, using (1). Now, for any $u \in P$ that is an upper bound for x, we have $1 \leq u$ and $ab^{-1} \leq u$. Thus $a = a(a^{-1} \vee b^{-1}) = 1 \vee ab^{-1} \leq u$. Therefore a is the least upper bound of x in P.

Finally suppose that (i)–(vi) hold and let $x = uv^{-1}$ for $u, v \in P$. Then, as in the proof of (iv) implies (v), we have $u \wedge_r v = a^{-1}u = b^{-1}v$ where $a, b \in P$ is the unique pair of statement (iv). Thus the pair u, v satisfies the condition of statement (vi), namely $u \vee_r v = 1$, if and only if $u = a$ and $v = b$. □

REMARK 8. We make the following remarks concerning Lemma 7.

(1) Our definition of quasi-lattice order differs slightly from the one originally given

in [18, Section 2.1], which appears here as statement (ii). Nica had also given an equivalent form of (ii) consisting of two conditions: QL1, which is statement (iii), and QL2, which is the statement of Definition 6 for all pairs $x, y \in P$. By Lemma 7 the various definitions are equivalent; in particular QL2 is not needed as it follows from QL1.

(2) In [18] and [16] least upper bounds are always assumed to be in P, but no such assumption is made here. This causes a slight discrepancy in notation: in this paper, the least upper bound in P of x and y would be written $x \vee y \vee 1$.

(3) While statements (v) and (vi) may appear to be conditions only on the monoid P, in fact they are not, because one must have that $w \leq_r u \wedge_r v$ not only for every common right lower bound w in P but also for every common right lower bound w in G.

Suppose, now, that $G = \Gamma_{I\in\Lambda} G_I$ is a graph product in which each group G_I, for $I \in \Lambda$, is partially ordered with positive cone P_I. We say that a reduced expression $X = x_1 x_2 \cdots x_l$ is *positive* if $x_i \in P_{I(x_i)}$ for all $i = 1, 2, \ldots, l$. Note that this property is invariant under shuffle equivalence. We say that an element $x \in G$ is *positive* if it has a positive reduced expression. It follows, by Theorem 2, that every reduced expression for a positive element is positive. Let P denote the submonoid of G consisting of all positive elements. This is just the submonoid generated by the union of the P_I for $I \in \Lambda$. Moreover, two positive elements are equal in P if and only if their reduced expressions are shuffle equivalent. Thus P may be presented as the *monoid graph product* $\Gamma_{I\in\Lambda} P_I$, that is the monoid obtained from the free product $*_{I\in\Lambda} P_I$ by introducing the relations $xy = yx$ for all $x \in P_I$ and $y \in P_J$ with I adjacent to J in Λ.

It is easily seen that $P \cap P^{-1} = \{1\}$ and hence that (G, P) is a partially ordered group. We refer to (G, P) as the *graph product over* Γ *of the partially ordered groups* $\{(G_I, P_I)\}_{I\in\Lambda}$, and write $(G, P) = \Gamma_{I\in\Lambda}(G_I, P_I)$. Note that each (G_I, P_I) is a partially ordered subgroup of (G, P). That is, the inclusion map is order preserving.

LEMMA 9. *Suppose that $(G, P) = \Gamma_{I\in\Lambda}(G_I, P_I)$ is a graph product of quasi-lattice ordered groups and denote by $\Delta^r(x, y)$ the intersection $\Delta^r(x) \cap \Delta^r(y)$ of the final vertex sets of elements x and y in G. For any pair $u, v \in P$, there exist $a, b \in P$ satisfying the following conditions.*

(i) *$ab^{-1} = uv^{-1}$, with $a \leq u$, $b \leq v$, and $a_I^r \wedge_r b_I^r = 1$ for all $I \in \Delta^r(a, b)$.*
(*Note that by Lemma 7 (v) $\wedge_r$ is defined in each quasi-lattice order (G_I, P_I).*)

(ii) *Writing reduced expressions $A \cdot (\prod_{\Delta^r(a,b)} a_I^r)$ and $\big(\prod_{\Delta^r(a,b)} (b_I^r)^{-1}\big) B$ for a and b^{-1} respectively, one has that*

$$A\left[\prod_{\Delta^r(a,b)} (a_I^r (b_I^r)^{-1})\right] B \tag{2}$$

is a reduced expression for ab^{-1}.

PROOF. (i): We proceed by induction on $\ell(u)+\ell(v)$, the case where $u=v=1$ being trivially true. Given $u, v \in P$ we have

$$u = \hat{u} \cdot \prod_{I \in \Delta^r(u,v)} (u_I^r \wedge_r v_I^r) \quad \text{and} \quad v = \hat{v} \cdot \prod_{I \in \Delta^r(u,v)} (u_I^r \wedge_r v_I^r),$$

where $\hat{u}, \hat{v} \in P$ and $\hat{u}\hat{v}^{-1} = uv^{-1}$. Now if both $u_I^r(u_I^r \wedge_r v_I^r)^{-1}$ and $v_I^r(u_I^r \wedge_r v_I^r)^{-1}$ are nontrivial for each $I \in \Delta^r(a,b)$, then these are precisely the final syllables of $\hat{u}$ and $\hat{v}$ respectively (in particular, $\Delta^r(\hat{u},\hat{v}) = \Delta^r(u,v)$). Putting $a=\hat{u}, b=\hat{v}$ satisfies the claim in this case. Otherwise, some $u_I^r \wedge_r v_I^r$ equals either u_I^r or v_I^r for some $I \in \Delta^r(a,b)$. In this case $\ell(\hat{u})+\ell(\hat{v}) < \ell(u)+\ell(v)$, and the result follows by induction.

(ii): This is a straightforward application of Lemma 5, the condition $a_I^r \wedge_r b_I^r = 1$ ensuring that the syllables $(a_I^r(b_I^r)^{-1})$ are nontrivial. □

THEOREM 10. *A graph product $(G,P) = \Gamma_{I\in\Lambda}(G_I, P_I)$ of quasi-lattice ordered groups is a quasi-lattice ordered group.*

PROOF. We prove that (G,P) satisfies condition (iv) of Lemma 7.

Given $x \in PP^{-1}$, Lemma 9 implies that x has a reduced expression $X = x_1x_2\cdots x_m$ in which each syllable x_i lies in $P_{I(x_i)}P_{I(x_i)}^{-1}$. (More specifically, X may take the form of (2), with the strictly positive syllables appearing first and strictly negative syllables appearing last). Since each (G_I, P_I) is quasi-lattice ordered we may, in view of Lemma 7 (vi), write each x_i uniquely as $a_ib_i^{-1}$ with $a_i \wedge_r b_i = 1$. Define elements $\mathbf{a}, \mathbf{b} \in P$ by $\mathbf{a} = a_1a_2\cdots a_m$ and $\mathbf{b} = b_mb_{m-1}\cdots b_1$. Since any shuffle of the syllables $x_1, x_2, \ldots, x_m$ induces a legal shuffle of the nontrivial a_i's (respectively the nontrivial b_i's) it follows by Theorem 2 that the pair of elements $\mathbf{a}, \mathbf{b}$ is uniquely determined by x (independently of the choice of reduced expression).

Given any pair $u, v \in P$ with $uv^{-1} = x$, let $a, b \in P$ be any pair as in Lemma 9. The unique pair $\mathbf{a}, \mathbf{b}$ which we have just defined may be computed from any reduced expression for x, in particular from expression (2) of Lemma 9 (ii). It follows that $\mathbf{a} = a$ and $\mathbf{b} = b$. This shows that $x = \mathbf{a}\mathbf{b}^{-1}$ and that $\mathbf{a} \le u$ and $\mathbf{b} \le v$ for every $u, v \in P$ such that $x = uv^{-1}$, as required by Lemma 7 (iv). □

LEMMA 11. *Suppose that (G,P) is a graph product of partially ordered groups (G_I, P_I) for $I \in \Lambda$. Let $x, z \in P$ be such that $1 \le x \le z$. Then, for each $I \in \Lambda$,*

(i) *$x_I \le z_I$, and*
(ii) *writing $x = x_Ix'$, either $x_I = z_I$ or I is adjacent to every vertex of x'.*

PROOF. We have $z = xw$, where x, z and w are all positive. Let $D = \Delta^r(x) \cap \Delta(w)$ and take reduced expressions $X\big(\prod_{J\in D} x_J^r\big)$ for x and $\big(\prod_{J\in D} w_J\big)W$ for w. Note that

$x_J^r w_J$ is nontrivial, for each $J \in D$, since it is a product of nontrivial positive elements. Thus, by Lemma 5, we have the following (necessarily positive) reduced expression for z:

$$Z = X\left(\prod_{J\in D}(x_J^r w_J)\right)W.$$

If $z_I = 1$ then the lemma is trivially true. Thus we suppose $z_I \neq 1$ and ask where the initial syllable z_I might appear in the above expression.

Case 1. If the initial syllable z_I falls in the subexpression X, then it is also an initial syllable of the reduced expression given for x. Thus $x_I = z_I$, satisfying (i) and (ii).

Case 2. If $I \in D$ and the initial syllable z_I happens to be the syllable $(x_I^r w_I)$, then every vertex of X and every $J \in D$ other than I itself is adjacent to I. It follows that x_I^r is also an initial syllable of x. That is $x_I = x_I^r$, and x' has a reduced expression

$$X\left(\prod_{J\in D; J\neq I} x_J^r\right).$$

So $z_I = x_I w_I$ giving part (i), and I is adjacent to every vertex of x' giving part (ii).

Case 3. If the initial syllable z_I falls in the subexpression W, then every vertex of x is adjacent to I and, in particular, $x_I = 1$. Both parts of the lemma are again satisfied. □

DEFINITION 12. We consider a graph product $(G, P) = \Gamma_{J\in\Lambda}(G_J, P_J)$ of quasi-lattice orders, and choose $I \in \Lambda$. Given elements $x, y \in P$, write $x = x_I x'$ and $y = y_I y'$. We say that the elements $x, y \in P$ are *I-adjacent* if the following three conditions hold:

(a) x_I and y_I have a common upper bound;
(b) either $x_I = x_I \vee y_I$ or I is adjacent to every vertex of x';
(c) either $y_I = x_I \vee y_I$ or I is adjacent to every vertex of y'.

This definition allows us to give an inductive algorithm for deciding whether two elements $x, y \in P$ have a common upper bound, and for computing $x \vee y$ when it exists. Notice first that if $x \in G_I$ and $y \in G_J$, then the least common upper bound of x and y in (G, P) is given by

$$x \vee y = \begin{cases} x \vee_I y & \text{if } I = J; \\ xy & I \text{ and } J \text{ are adjacent;} \\ \infty & \text{if } I \text{ and } J \text{ are not adjacent.} \end{cases}$$

PROPOSITION 13. *Suppose that* $(G, P) = \Gamma_{J\in\Lambda}(G_J, P_J)$ *is a graph product of quasi-lattice ordered groups. Let* $x, y \in P$ *and, for an arbitrary choice of* $I \in \Lambda$, *write* $x = x_I x'$ *and* $y = y_I y'$. *Then we have the following.*

(i) *The elements* $x, y \in P$ *have a common upper bound if and only if they are* I*-adjacent and* $x' \vee y' \neq \infty$.

(ii) *Suppose that the elements* $x, y \in P$ *do have a common upper bound. Then*

$$x \vee y = (x_I \vee y_I) \cdot (x' \vee y').$$

Note that conditions (b) *and* (c) *of Definition* 12 *apply to this expression.*

PROOF. Suppose initially that the elements x and y are I-adjacent and, by condition (a) of Definition 12, write $x_I \vee y_I = x_I u = y_I v$ for some $u, v \in P_I$. Then, by condition (b), either $u = 1$ or I is adjacent to every vertex of x', in which case x' has no I component. In either case one has that $ux' = x'u = u \vee x'$. Similarly, by condition (c), we have $vy' = y'v = v \vee y'$. Therefore, supposing in addition that $x' \vee y' \neq \infty$ we have:

$$\begin{aligned}(3)\qquad (x_I \vee y_I)(x' \vee y') &= x_I u x' \vee y_I v y' \\ &= x_I(u \vee x') \vee y_I(v \vee y') \\ &= (x_I \vee y_I) \vee x \vee y = x \vee y.\end{aligned}$$

It follows that x and y have a common upper bound and $x \vee y$ is given by the above equation.

On the other hand, suppose that $z = x \vee y \neq \infty$. We first observe that x and y are I-adjacent. Clearly $x_I \vee y_I \leq z$, justifying (a), and conditions (b) and (c) follow directly from Lemma 11 (ii). Finally, since $x \vee y \neq \infty$, the equalities of (3) may be read in reverse order to show that $x' \vee y' \neq \infty$. □

REMARK 14. To see that Proposition 13 leads to an effective algorithm, we note that by always choosing $I \in \Delta(x) \cup \Delta(y)$ we ensure that $\ell(x') + \ell(y')$ is always strictly less than $\ell(x) + \ell(y)$.

4. Amenability for graph products of quasi-lattice orders

Let (G, P) be a quasi-lattice ordered group. Recall from [18] that an isometric representation $V : P \to \mathrm{Isom}(\mathscr{H})$ on a Hilbert space $\mathscr{H}$ is *covariant* if it is compatible with the quasi-lattice structure in the sense that

$$V_x V_x^* V_y V_y^* = V_{x\vee y} V_{x\vee y}^* \quad \text{for } x, y \in P.$$

The notation is meant to include the convention $V_\infty = 0$, so in particular covariance implies $V_x V_x^* V_y V_y^* = 0$ when x and y have no common upper bound.

The main example of such a representation is the Toeplitz representation $T : P \to \mathrm{Isom}(\ell^2(P))$, defined by $T_x \varepsilon_y := \varepsilon_{xy}$, where ε_x denotes the typical orthonormal basis vector of $\ell^2(P)$. The C^*-algebra generated by the T_x is called the *Toeplitz C^*-algebra of the quasi-lattice ordered group* (G, P) and is denoted $\mathcal{T}(G, P)$. Nica also considered the C^*-algebra $C^*(G, P)$, universal for covariant isometric representations of P and made the following definition.

DEFINITION 15. When the canonical homomorphism of $C^*(G, P)$ to $\mathcal{T}(G, P)$ is injective we say that (G, P) is an *amenable quasi-lattice order*.

There is a semigroup C^*-dynamical system (B_P, P, α) canonically associated to P, in which B_P is the C^*-subalgebra of $\ell^\infty(P)$ generated by the characteristic functions 1_y of the semi-infinite intervals $[y, \infty)$ for $y \in P$; the endomorphism α_x corresponding to $x \in P$ is defined by $\alpha_x(1_y) = 1_{xy}$. Covariant isometric representations of P are in one to one correspondence with covariant representations of the semigroup dynamical system (B_P, P, α) and this leads to the realisation of $C^*(G, P)$ as a semigroup crossed product, see [16, Section 2] for the details. There is a canonical conditional expectation from $B_P \rtimes_\alpha P$ onto B_P, which is faithful if and only if (G, P) is amenable [18, Section 4.3]. This property, taken as the definition of amenability of (G, P) in [16], is instrumental in the direct proof of amenability for free product orders, which we aim to generalise in this section.

We will need the following reformulation of Proposition 6.6 of [16].

PROPOSITION 16 (Laca, Raeburn [16]). *Suppose $\phi : (G, P) \to (\mathcal{G}, \mathcal{P})$ is an order preserving homomorphism of quasi-lattice ordered groups such that, whenever $x, y \in P$ have a common upper bound in P,*

(a) *$\phi(x) = \phi(y)$ only if $x = y$, and*
(b) *$\phi(x) \vee \phi(y) = \phi(x \vee y)$.*

If $\mathcal{G}$ is an amenable group, then (G, P) is an amenable quasi-lattice order.

REMARK 17. Proposition 6.6 of [16] should have been stated like this. The reason is that the proof indicated there, modelled on that of [16, Proposition 4.2], requires that the conditional expectation for the coaction of $\mathcal{G}$ on $C^*(G, P)$ be faithful, which is true if $\mathcal{G}$ is amenable by [19]. We do not know whether Proposition 6.6 of [16] is correct as originally stated; however all that is required for the other results in [16] is the version stated above.

Suppose now that $(G, P) = \Gamma_{I \in \Lambda}(G_I, P_I)$ is a graph product of quasi-lattice

ordered groups. We define the group homomorphism

$$\phi : (G, P) \longrightarrow \bigoplus_{I\in\Lambda}(G_I, P_I)$$

such that each factor (G_I, P_I) of (G, P) is mapped to the corresponding factor in the direct sum $\bigoplus_{I\in\Lambda}(G_I, P_I)$ via the identity on G_I. In what follows we shall, for simplicity, write $\phi(u)$ as u whenever $u \in G_I$ for some $I \in \Lambda$.

We view the direct product $\bigoplus_{I\in\Lambda}(G_I, P_I)$ as a graph product (over the full graph on Λ). Let $x \in G$ and let X be any reduced expression for x. Then, choosing a vertex $I \in \Lambda$, we observe that $\phi(x)_I$ is simply the product of all those syllables of X which belong to the vertex I, taken in the order in which they appear. In particular, if $x = x_I x'$ then $\phi(x)_I = x_I \phi(x')_I$. On the other hand, $\phi(x)_J = \phi(x')_J$ for all $J \neq I$.

LEMMA 18. *Let ϕ be the map defined above. Suppose that $x, y \in P$ satisfy $x \vee y \neq \infty$ and, for an arbitrary choice of $I \in \Lambda$, write $x = x_I x'$ and $y = y_I y'$. Then*

$$\phi(x)_I \vee \phi(y)_I = (x_I \vee y_I) \cdot (\phi(x')_I \vee \phi(y')_I)\,. \tag{4}$$

PROOF. We refer to the final claim of Proposition 13, which states that either $y_I \leq x_I = x_I \vee y_I$ or I is adjacent to every vertex of x'. The latter condition implies, in particular, that $\phi(x)_I = x_I$ and $\phi(x')_I = 1$. A similar statement also holds with respect to y'. Thus we have the following four cases to consider:

Case 1. $\phi(x)_I = x_I$ and $\phi(y)_I = y_I$: Since in this case $\phi(x')_I$ and $\phi(y')_I$ are both trivial, (4) is self-evident.

Case 2. $\phi(x)_I = x_I$ and $x_I \leq y_I = x_I \vee y_I$: In this case $\phi(x)_I \leq y_I \leq \phi(y)_I$. Also $\phi(x')_I = 1$ and $x_I \vee y_I = y_I$. Thus (4) reduces to $\phi(y)_I = y_I \phi(y')_I$.

Case 3. $y_I \leq x_I = x_I \vee y_I$ and $\phi(y)_I = y_I$: This case is similar to Case 2.

Case 4. $x_I = y_I = x_I \vee y_I$: (4) follows in this case by left invariance of the quasi-lattice order. □

PROPOSITION 19. *Let $(G, P) = \Gamma_{I\in\Lambda}(G_I, P_I)$ be a graph product of quasi-lattice ordered groups. Then the map $\phi : (G, P) \to \bigoplus_{I\in\Lambda}(G_I, P_I)$ (defined by the identity on each factor) is an order preserving homomorphism such that, whenever $x, y \in P$ have a common upper bound in P, the following hold*:

(a) $\phi(x) = \phi(y)$ *only if* $x = y$, *and*
(b) $\phi(x) \vee \phi(y) = \phi(x \vee y)$.

PROOF. The induced map is clearly an order preserving homomorphism. Suppose throughout that $x, y \in P$ have a common upper bound.

We first prove condition (a) assuming that (b) holds: Observe first of all that if $u \in P$ then $u_I \leq \phi(u)_I$ for every I. So $\phi(u) = 1$ implies that $u = 1$. Suppose that $\phi(x) = \phi(y)$. Then, by condition (b), we have

$$\phi(x \vee y) = \phi(x) \vee \phi(y) = \phi(x) = \phi(y).$$

Writing $x \vee y = xu = yv$ for $u, v \in P$, it follows that $\phi(u) = \phi(v) = 1$. But by the preceding observation we must then have $u = v = 1$ and hence $x = y$.

We now prove condition (b): Choose $I \in \Lambda$ such that either $x_I \neq 1$ or $y_I \neq 1$, and write $x = x_I x'$ and $y = y_I y'$. By Proposition 13 we may write $x \vee y = (x_I \vee y_I)(x' \vee y')$, and hence

$$\phi(x \vee y) = (x_I \vee y_I)\phi(x' \vee y').$$

By induction on $\ell(x) + \ell(y)$ we have that $\phi(x' \vee y') = \phi(x') \vee \phi(y')$. Thus it remains to show that

$$\phi(x) \vee \phi(y) = (x_I \vee y_I) \cdot (\phi(x') \vee \phi(y')). \tag{5}$$

Note that in the direct product $\bigoplus_{J \in \Lambda}(G_J, P_J)$ every element ξ may be written $\xi = \prod_{J \in \Lambda} \xi_J$. Combining this with Proposition 13 we have:

$$\phi(x) \vee \phi(y) = (\phi(x)_I \vee \phi(y)_I) \prod_{J \in \Lambda; J \neq I} (\phi(x)_J \vee \phi(y)_J). \tag{6}$$

Let (6′) denote the equation similarly obtained for the pair x' and y'. We recall that $\phi(x)_J = \phi(x')_J$ and $\phi(y)_J = \phi(y')_J$ for all $J \neq I$. Thus, (5) follows from (6) and (6′) via Lemma 18. □

We can now extend Theorem 6.7 of [16] to graph products of amenable groups.

THEOREM 20. *Any graph product of a family of quasi-lattice orders in which the underlying groups are amenable is an amenable quasi-lattice order* (*that is, the Toeplitz representation is faithful*).

PROOF. Let (G, P) denote a graph product of the family $(G_I, P_I)_{I \in \Lambda}$ of quasi-lattice orders, and let $(\mathscr{G}, \mathscr{P})$ denote their direct product. By Proposition 19, the map $\phi : (G, P) \to (\mathscr{G}, \mathscr{P})$ induced by the identity on each factor satisfies the hypothesis of Proposition 16. If each G_I is amenable as a group, then so is $\mathscr{G}$. It then follows, by Proposition 16, that (G, P) is an amenable quasi-lattice order. □

As an application of Theorem 20, we obtain the following characterisation of faithfulness of representations of $C^*(G, P)$, and the consequent result about uniqueness of the C^*-algebra generated by a covariant isometric representation of P. Monoid representations $V : P \to \mathrm{Isom}(\mathscr{H})$ and $W : Q \to \mathrm{Isom}(\mathscr{H})$ are said to *$*$-commute* if every V_x or V_x^* for $x \in P$ commutes with every W_y or W_y^* for $y \in Q$, and to be *orthogonal* to one another if $V_x^* W_y = 0$ for all $x \in P$ and $y \in Q$.

THEOREM 21. *Let (G, P) be the graph product of a family $(G_I, P_I)_{I\in\Lambda}$ of quasi-lattice ordered groups.*

(i) *If $\{V_I : P_I \to \mathrm{Isom}(\mathscr{H})\}_{I\in\Lambda}$ is a family of covariant isometric representations such that V_I $*$-commutes with V_J when I and J are adjacent in Γ and V_I is orthogonal to V_J when I and J are not adjacent in Γ, then there is a (unique) isometric covariant representation $V : P \to \mathrm{Isom}(\mathscr{H})$ such that $V|_{P_I} = V_I$. All covariant representations of P arise this way.*

Suppose that each G_I, for $I \in \Lambda$, is an amenable group. Then we also have the following:

(ii) *The representation of the universal algebra $C^*(G, P)$ associated to V is faithful if and only if*

$$\prod_{x\in F}(1 - V_x V_x^*) \neq 0 \quad \text{for every finite subset } F \subset \bigcup_{I\in\Lambda}(P_I \setminus \{1\}). \tag{7}$$

(iii) *If $\{V_I\}$ and $\{W_I\}$ are two families of covariant isometric representations as in part* (i) *satisfying* (7)*, then the canonical map $V_I(x) \mapsto W_I(x)$ extends to an isomorphism of $C^*(V_x : x \in P)$ onto $C^*(W_x : x \in P)$.*

PROOF. The isometries satisfy the commuting relations which define P as a graph product of monoids. Therefore the maps $s \in P_I \mapsto V_I(s)$ extend to an isometric representation V of the monoid $P = \Gamma_{I\in\Lambda} P_I$.

We need to show that this representation is covariant, that is, we need to show that $V_{x\vee y} V_{x\vee y}^* = V_x V_x^* V_y V_y^*$ for all x and y in P.

We proceed by induction on $\ell(x) + \ell(y)$. Choose $I \in \Delta(x) \cup \Delta(y)$ and write $x = x_I x'$ and $y = y_I y'$. Since x_I and y_I cannot both be trivial, this ensures that $\ell(x') + \ell(y') < \ell(x) + \ell(y)$.

Suppose first that $x \vee y < \infty$. By Proposition 13 and the induction hypothesis, we have

$$V_{x\vee y} V_{x\vee y}^* = V_z V_{x'} V_{x'}^* V_{y'} V_{y'}^* V_z^*, \tag{8}$$

where $z = x_I \vee y_I \in P_I$. Moreover, x and y are I-adjacent, and from Definition 12 (b) and (c) we have four cases to consider:

Case 1. The vertex I is adjacent to all the vertices of x' and of y'. Then V_I $*$-commutes with $V_{x'}$ and with $V_{y'}$, so that (8) becomes

$$\begin{aligned} V_{x\vee y}V_{x\vee y}^* &= V_{x'}V_{x'}^*V_zV_z^*V_{y'}V_{y'}^* \\ &= V_{x'}V_{x'}^*V_{x_I}V_{x_I}^*V_{y_I}V_{y_I}^*V_{y'}V_{y'}^* \quad \text{by covariance of } V_I \\ &= V_{x_I}V_{x'}V_{x'}^*V_{x_I}^*V_{y_I}V_{y'}V_{y'}^*V_{y_I}^* = V_xV_x^*V_yV_y^*. \end{aligned}$$

Case 2. $z = x_I$ and I is adjacent to all vertices of y'. Write $z = y_I v$, with $v \in P_I$; then $V_{y'}V_{y'}^*V_z^* = V_v^*V_{y'}V_{y'}^*V_{y_I}^* = V_z^*V_yV_y^*$. Thus (8) becomes

$$V_{x\vee y}V_{x\vee y}^* = V_zV_{x'}V_{x'}^*V_z^*V_yV_y^* = V_xV_x^*V_yV_y^*.$$

Case 3. $z = y_I$ and I is adjacent to all vertices of x'. (This is analogous to Case 2.)

Case 4. $z = x_I = y_I$. Inserting $V_z^*V_z$ in the middle of the right-hand side of (8) we get $V_{x\vee y}V_{x\vee y}^* = V_xV_x^*V_yV_y^*$.

Suppose now that $x \vee y = \infty$. Then by convention $V_{x\vee y}V_{x\vee y}^* = 0$ and it suffices to show that $V_x^*V_y = 0$. Clearly if $x_I \vee y_I = \infty$ then, by covariance of V_I, we have $V_{x_I}^*V_{y_I} = 0$, so $V_x^*V_y = 0$. Thus, we may suppose that $x_I \vee y_I \neq \infty$ and hence that

$$V_{x_I}^*V_{y_I} = V_uV_v^* \quad \text{where } x_I \vee y_I = x_Iu = y_Iv.$$

Thus

$$V_x^*V_y = V_{x'}^*V_uV_v^*V_{y'}.$$

Let $A = a_1a_2\cdots a_k$ be a reduced expression for y'. Either condition (c) of Definition 12 holds or $v \neq 1$ and A has a syllable with vertex not adjacent to I. Suppose the latter, and let a_i be the first syllable of A for which $I(a_i)$ is not adjacent to I. Note that $I(a_i) \neq I$ because y' cannot have initial vertex I. Write $\alpha = a_1a_2\cdots a_{i-1}$ and $\beta = a_{i+1}a_{i+2}\cdots a_k$. We now have

$$V_v^*V_{y'} = V_v^*V_\alpha V_{a_i}V_\beta = V_\alpha V_v^*V_{a_i}V_\beta = 0$$

using the fact that $V_v^*V_{a_i} = 0$ by orthogonality.

We may thus suppose that condition (c) of Definition 12 holds, in which case $V_v^*V_{y'} = V_{y'}V_v^*$. By a similar argument we may suppose also that condition (b) holds and that $V_{x'}^*V_u = V_uV_{x'}^*$. We have already assumed that $x_I \vee y_I \neq \infty$ (part (a) of Definition 12). All these conditions together imply that x and y are I-adjacent (Definition 12) and that

$$V_x^*V_y = V_uV_{x'}^*V_{y'}V_v^*.$$

By Proposition 13 (i) we now have $x' \vee y' = \infty$ (since $x \vee y = \infty$ and x is I-adjacent to y), and applying the induction hypothesis to x' and y' completes the proof that V is covariant.

To prove (ii) observe first that from Theorem 20 and [16, Theorem 3.7], it follows that a representation is faithful if and only if

$$\prod_{x\in F}(1 - V_x V_x^*) \neq 0 \quad \text{for every finite subset } F \subset P.$$

It is equivalent to consider only products of the form stated in (7) because replacing each $x \in F$ by one of its initial syllables has the effect of replacing each factor $(1 - V_x V_x^*)$ by a smaller one. Part (iii) follows from Theorem 20 and Corollary 3.8 and Corollary 3.9 of [16]. □

REMARK 22. In some cases condition (7) is automatically satisfied by all covariant representations, in which case the Toeplitz C^*-algebra is simple, and purely infinite by [15, Theorem 5.4]. The best known example of this is $\mathcal{O}_\infty$ [6]. See [16, Corollary 5.2 and Corollary 5.3] and [8, Theorem 2.4] for more examples involving free products.

5. The C^*-algebra of a right-angled Artin semigroup of isometries

Let Λ be a set (usually taken to be finite, although we shall not make this restriction here). A matrix $M = (m_{s,t})_{s,t\in\Lambda}$ is a *Coxeter matrix* if $m_{s,t} = m_{t,s} \in \{2, 3, \dots, \infty\}$ for $s \neq t$ and $m_{s,s} = 1$. Denote by $\langle st\rangle^m$ the word $sts\cdots$, beginning with s and having length m, in which the letters s and t alternate.

The *Artin group* A_M associated to M is the group with presentation

$$\langle \Lambda \mid \langle st\rangle^{m_{s,t}} = \langle ts\rangle^{m_{s,t}} \text{ for each } s, t \in \Lambda\rangle,$$

in which a relation of the form $\langle st\rangle^\infty = \langle ts\rangle^\infty$ is to be interpreted as vacuous. The *Artin monoid* A_M^+ is defined via the same presentation, taken in the category of monoids (semigroups with unit), see [1]. We may view A_M as a partially ordered group with positive cone P_M generated by Λ. The cone P_M is in general a quotient of A_M^+ via the obvious map, although in many cases of interest this map is known to be injective (see [2] for the most recent results).

Adding the relations $s^2 = 1$ for $s \in \Lambda$ to the above ones yields a somewhat unusual presentation of the more familiar Coxeter group W_M associated with M, which is usually presented via the relations $(st)^{m_{s,t}} = 1$.

DEFINITION 23. If the Coxeter group W_M is finite, the associated Artin group A_M is said to be of *finite type*, [1].

A Coxeter group and its associated Artin group A_M are said to be *right-angled* if every nondiagonal entry of the Coxeter matrix M is either 2 or ∞; see for example [3]. (By abuse we also refer to the matrix as right-angled.)

The terminology is motivated by noting that the right-angled Coxeter groups are those linear reflection groups whose reflecting hyperplanes are mutually orthogonal or parallel.

Every right-angled Coxeter matrix M over Λ determines a graph Γ with vertex set Λ having an edge joining s and t when $m_{s,t} = 2$. The only relations in the presentation of A_M say that two generators commute if they are joined by an edge, hence A_M is precisely the graph product $\Gamma_{I\in\Lambda}\mathbb{Z}$ of copies of $\mathbb{Z}$. In this connection, right-angled Artin groups are also referred to as *graph groups*, see [14].

By virtue of Theorem 2 the right-angled Artin monoid A_M^+ may be identified with the positive cone of the corresponding Artin group A_M, and (A_M, A_M^+) is a quasi-lattice order by Theorem 10. Applying Theorem 20 we see that this quasi-lattice order is amenable, and hence the Toeplitz representation of $C^*(A_M, A_M^+)$ is faithful. As with Coburn's and Cuntz's theorems, it is more appealing to formulate the result in terms of the generators themselves; indeed, notice that assertion (iii) below does not contain any explicit reference to quasi-lattice orders or Artin groups.

THEOREM 24. *Let Γ be a graph with set of vertices Λ and suppose $\{V_s : s \in \Lambda\}$ is a collection of isometries on Hilbert space such that for every pair of distinct vertices s and t one has*

$$V_sV_t = V_tV_s \quad \text{and} \quad V_s^*V_t = V_tV_s^* \quad \text{if } s \text{ and } t \text{ are adjacent in } \Gamma, \text{ and}$$
$$V_s^*V_t = 0 \quad \text{if } s \text{ and } t \text{ are not adjacent in } \Gamma.$$

Let $A_M = \Gamma_{s\in\Lambda}\mathbb{Z}\langle s\rangle$ be the right-angled Artin group associated to Γ. Then

(i) *the maps $s \to V_s$, for $s \in \Lambda$, extend to a covariant isometric representation V of the right-angled Artin semigroup A_M^+,*

(ii) *the corresponding representation of $C^*(A_M, A_M^+)$ is faithful if and only if*

$$\prod_{s\in F}(I - V_sV_s^*) \neq 0 \quad \text{for every finite } F \subset \Lambda, \quad \text{and} \tag{9}$$

(iii) *if $\{W_s : s \in \Lambda\}$ is another collection of isometries satisfying the same relations and condition (9), then the map $V_s \mapsto W_s$ extends to a C^*-algebra isomorphism of $C^*(V_s : s \in \Lambda)$ to $C^*(W_s : s \in \Lambda)$.*

PROOF. Since the generators V_s satisfy the stated relations, the collection of isometric representations $\{n \in \mathbb{N} \mapsto V_s^n\}_{s\in\Lambda}$ satisfies the hypothesis of Theorem 21 (i) and so extends to an isometric representation of the semigroup A_M^+, giving (i). The amenability hypothesis is satisfied because in this case each factor in the graph product is isomorphic to $\mathbb{Z}$, so (ii) and (iii) follow directly from Theorem 21. Notice that the

necessary and sufficient condition for faithfulness in (ii) is equivalent to the one obtained in Theorem 21 because every projection of the form (7) majorates a projection of the form

$$\prod_{s\in F}(I - V_s V_s^*) \quad \text{for some finite } F \subset \Lambda,$$

to see this it suffices to replace each syllable $x = s^n \in \mathbb{N}\langle s\rangle$ by the corresponding generator s. □

REMARK 25. If the set Λ of generators is finite, the projection $\prod_{s\in\Lambda}(I - T_s T_s^*)$ belongs to the Toeplitz algebra $\mathcal{T}(A_M, A_M^+)$. Since every nontrivial element in A_M^+ is bounded below by a generator, it follows from [18, Proposition 6.3] that the compact operators on $\ell^2(P)$ are contained in $\mathcal{T}(A_M, A_M^+)$ as the ideal generated by this projection. It is easy to see that (9) holds if and only if it holds for $F = \Lambda$, so this ideal is minimal.

6. Non-amenability of lattice ordered groups

Next we concentrate on partially ordered groups in which least common upper bounds always exist.

DEFINITION 26. A partially ordered group (G, P) is *lattice ordered* if every pair of elements has a least common upper bound. By left invariance, an equivalent condition is that every element $x \in G$ have a least upper bound in P.

Lattice orders are special cases of quasi-lattice orders; in fact we have the following characterisation, see for example [18].

LEMMA 27. *The following are equivalent for a partially ordered group* (G, P):

(i) (G, P) *is lattice ordered.*
(ii) (G, P) *is quasi-lattice ordered and* $G = PP^{-1}$.
(iii) (G, P) *is quasi-lattice ordered,* P *generates* G (*in which case we say* (G, P) *is connected*), *and* $aP \cap bP \neq \emptyset$ *for all* $a, b \in P$.

PROOF. (i) implies (ii): Suppose (G, P) is lattice ordered and let $x \in G$; then $x \leq x \vee 1$ so that both $a := x \vee 1$ and $b := x^{-1}a$ are in P. Clearly $x = ab^{-1}$.

(ii) implies (iii): If $G = PP^{-1}$ then obviously P generates G. Given a pair $a, b \in P$, write $a^{-1}b$ as xy^{-1} with $x, y \in P$. Then $ax = by \in aP \cap bP$.

(iii) implies (i): Suppose (iii) holds. Take any $x \in G$ and (using the connectedness) write $x = a_1 a_2 \cdots a_k$ with each $a_i \in P$ or P^{-1}. Since any element $a^{-1}b$ for $a, b \in P$

may be rewritten cd^{-1} for $c, d \in P$ (by finding $ac = bd \in aP \cap bP$), any such expression for x may ultimately be simplified to the form $x = uv^{-1}$ with $u, v \in P$. Thus x has an upper bound $u \in P$ and, since (G, P) is a quasi-lattice order, it must have a least upper bound in P. So (i) holds. □

For a quasi-lattice order (G, P), we know that if G is amenable then (G, P) is amenable, by [18, Section 4.5], see also [16, Lemma 6.5]. It turns out that for lattice ordered groups the converse is also true; the proof follows the argument outlined in Remark 2 of [18, Section 5.1].

PROPOSITION 28. *If (G, P) is lattice ordered and amenable (in the sense of Nica, Definition 15) then G is an amenable group.*

PROOF. Suppose (G, P) is an amenable lattice order, and denote the left regular representation of P on $\ell^2(P)$ by W. As observed by Nica, the map $x \in P \mapsto 1 \in \mathbb{C}$ is a (one-dimensional) covariant isometric representation, so the map $W_x \mapsto 1$ extends to give a one-dimensional representation of $\mathcal{T}(G, P)$, and an easy argument shows that

$$\left\| \sum_{s \in P} \lambda_s W_s \right\| = \sum_{s \in P} \lambda_s \tag{10}$$

for every finitely (or countably) supported nonnegative function λ in $\ell^1(P)$.

Notice that the sum $\sum_s \lambda_s W_s$ is the operator of left-convolution by $\lambda \in \ell^1(P)$ on $\ell^2(P)$, and that (10) implies that $\left\| \sum_{s \in P} \lambda_s W_s \right\| = 1$ for every probability density λ. Since the support of λ can be chosen to contain an arbitrary finite subset of P including the identity, we have that condition (e) of [9, Theorem 1] holds, with $p = 2$, $U = 1 \in P$, and ϕ a probability density whose support contains $\xi \cup \{1\}$. By [9, Theorem 1], the semigroup P has a left-invariant mean. (Day's Theorem is about right-amenability and right-convolutions, but there is no difficulty in transforming it into a theorem for left-amenability and left-convolutions.)

Finally, the group $G = PP^{-1}$ is amenable, by Corollary 3.6 of [21]. □

By Proposition 5.5 and Theorem 5.6 of [1], if A_M is of finite type then the Artin semigroup A_M^+ embeds as a subsemigroup of A_M and the pair (A_M, A_M^+) is a lattice ordered group (see also [10]). We wish to apply Proposition 28 to characterize amenability of these lattice orders; the first observation is that most Artin groups are not amenable.

PROPOSITION 29. *Let M be a Coxeter matrix over a finite set Λ. The Artin group A_M is amenable (as a group) if and only if it is the free abelian group on Λ, that is, $m_{s,t} = 2$ for all $s, t \in \Lambda$, $s \neq t$.*

PROOF. In [5] it is shown that the set of elements $\mathcal{Q} = \{Q_s = s^2 : s \in \Lambda\}$ generates a subgroup H_M of A_M with presentation

$$\langle \mathcal{Q} \mid Q_s Q_t = Q_t Q_s \text{ if } m_{s,t} = 2 \rangle. \tag{11}$$

That is, H_M is a right-angled Artin group. In particular, H_M contains at least one free subgroup of rank 2, and hence is not amenable, unless of course $m_{s,t} = 2$ for all $s \neq t \in \Lambda$. In the latter case A_M is free abelian and therefore amenable. □

As a consequence of Proposition 29, the analog of Theorem 21 (ii) fails for non-abelian Artin groups of finite type, instead we have the following non-uniqueness result.

THEOREM 30. *Let A_M be a nonabelian Artin group of finite type. Suppose $\{V_s : s \in \Lambda\}$ is a collection of isometries satisfying the Artin relations relative to M:*

$$\langle V_s V_t \rangle^{m_{s,t}} = \langle V_t V_s \rangle^{m_{s,t}} \quad s, t \in \Lambda. \tag{12}$$

Then the map $s \mapsto V_s$ extends to an isometric representation, denoted also by V, of the Artin semigroup A_M^+. The representation V is covariant provided that

$$V_s V_s^* V_t V_t^* = V_{s\vee t} V_{s\vee t}^* \quad s, t \in \Lambda. \tag{13}$$

The Toeplitz representation T on $\ell^2(A_M^+)$ satisfies (12) and (13) and, moreover, the projection $\prod_\Lambda (1 - T_s T_s^*)$ does not vanish. However, the Toeplitz representation of $C^*(A_M, A_M^+)$ is not faithful; in particular, the C^*-algebra generated by a collection $\{V_s : s \in \Lambda\}$ as above is not canonically unique, even if we assume $\prod_\Lambda (1 - V_s V_s^*) \neq 0$.

PROOF. The map can be extended to an isometric representation because the given isometries satisfy the relations (12), which constitute a presentation of A_M^+.

Suppose that (13) holds. We need to show that the covariance condition holds for every pair $x, y \in A_M^+$. We use the length homomorphism $l : A_M^+ \to \mathbb{N}$ such that $l(s) = 1$ for each generator $s \in \Lambda$. Choose x, y such that the covariance condition is not satisfied and such that $l(x \vee y)$ is minimised. Amongst the possible pairs, choose one so that $l(x)$ is the smallest possible.

Case 1. Suppose $l(x) = 1$, so that x is actually a generator s. Write $y = tz$ for t a generator, so that $V_y V_y^* = V_t (V_t^* V_t) V_z V_y^* = V_t V_t^* V_y V_y^*$. Then, by (13),

$$V_s V_s^* V_y V_y^* = V_s V_s^* V_t V_t^* V_y V_y^* = V_{s\vee t} V_{s\vee t}^* V_y V_y^*.$$

Now $s \vee t$ and y have a common left factor t; writing $s \vee t = tu$ and $y = tz$ we have

$$V_{s\vee t} V_{s\vee t}^* V_y V_y^* = V_t V_u V_u^* V_z V_z^* V_t^*.$$

Since $s \vee y = s \vee t \vee y = t(u \vee z)$, we have that $l(u \vee z) < l(s \vee y)$ and covariance follows by applying the induction hypothesis to $V_u V_u^* V_z V_z^*$ in this expression.

Case 2. Suppose now $l(x) > 1$, and write $x = su$ for s a generator. Put $s \vee y = sz$, so that $x \vee y = x \vee s \vee y = s(u \vee z)$. Since $l(s \vee y) \leq l(x \vee y)$ and $l(s) < l(x)$, the induction hypothesis implies that $V_s V_s^* V_y V_y^* = V_s V_z V_z^* V_s^*$. Applying V_s^* on the left and using the fact that $V_s^* V_s = 1$ this equation becomes $V_s^* V_y V_y^* = V_z V_z^* V_s^*$. We then have: $V_x V_x^* V_y V_y^* = V_s V_u V_u^* V_s^* V_y V_y^* = V_s V_u V_u^* V_z V_z^* V_s^*$. Since $x \vee y = s(u \vee z)$ we have $l(u \vee z) < l(x \vee y)$, and covariance follows in this case, as before, by applying the induction hypothesis to $V_u V_u^* V_z V_z^*$.

It is clear that the Toeplitz representation satisfies the stated relations, and the projection $\prod_\Lambda (1 - T_s T_s^*)$ does not vanish at $\delta_1 \in \ell^2(A_M^+)$. The quasi-lattice ordered group (A_M, A_M^+) is not amenable by Proposition 28 and Proposition 29 and the last assertion follows, see [16, Corollary 3.9]. □

REMARK 31. Since Theorem 30 shows that the Toeplitz algebra $\mathcal{T}(A_M, A_M^+)$ of a nonamenable finite type Artin group A_M is not universal for covariant isometric representations, it is generally hard to decide whether a given collection of isometries satisfying the Artin relations and the covariance condition actually generates a representation of $\mathcal{T}(A_M, A_M^+)$. In any case, it follows from Theorem 6.7 and Corollary 6.8 of [12] that a given a representation of $\mathcal{T}(A_M, A_M^+)$ is faithful if and only if the generating family of isometries is proper, in the sense that $\prod_\Lambda (1 - V_s V_s^*) \neq 0$.

Our results about Toeplitz algebras cover the Artin groups (A_M, A_M^+) that are presently known to be quasi-lattice ordered, namely the finite type Artin groups from [1, 10] and the right-angled Artin groups dealt with by Theorem 10. It would be interesting to formulate and decide questions of amenability and uniqueness in the remaining cases. It is known that, for Λ finite, the monoid A_M^+ always has a quasi-lattice structure (see [1, 10]), but even when it is known that A_M^+ embeds canonically in A_M this is not enough to show that (A_M, A_M^+) is quasi-lattice ordered, which is essential for our techniques.

Note added in proof. At the time this paper was written the best known results on the 'injectivity of the Artin monoid' were contained in [2]. Since then, the proof that A_M^+ embeds canonically in A_M for any Coxeter matrix M has been announced by Luis Paris in a preprint entitled 'Artin monoids inject in their groups'.

References

[1] E. Brieskorn and K. Saito, 'Artin-Gruppen und Coxeter-Gruppen', *Invent. Math.* **17** (1972), 245–271.

[2] R. Charney, 'Injectivity of the positive monoid for some infinite type Artin groups', in: *Geometric group theory down under. Proceedings of a special year in geometric group theory, Canberra, Australia, 1996*, (eds. J. Cossey, *et al.*), (W. de Gruyter, Berlin, 1999) pp. 103–118.
[3] I. M. Chiswell, 'Right-angled Coxeter groups', in: *Low dimensional topology and Kleinian groups* (ed. D. B. A. Epstein), London Math. Soc. Lecture Note Ser. 112, (Cambridge University Press, Cambridge, 1986) pp. 297–304.
[4] L. A. Coburn, 'The C^*-algebra generated by an isometry I', *Bull. Amer. Math. Soc.* **73** (1967), 722–726.
[5] J. Crisp and L. Paris, 'The solution to a conjecture of Tits on the subgroup generated by the squares of the generators of an Artin group', *Invent. Math.* **145** (2001), 19–36.
[6] J. Cuntz, 'Simple C^*-algebras generated by isometries', *Comm. Math. Phys.* **57** (1977), 173–185.
[7] ———, 'K-theory for certain C^*-algebras', *Ann. of Math. (2)* **113** (1981), 181–197.
[8] K. R. Davidson and G. Popescu, 'Noncommutative disc algebras for semigroups', *Canad. J. Math.* **50** (1998), 290–311.
[9] M. M. Day, 'Amenable semigroups', *Illinois J. Math.* **8** (1964), 100–111.
[10] P. Deligne, 'Les immeubles des groupes de tresses généralisés', *Invent. Math.* **17** (1972), 273–302.
[11] R. G. Douglas, 'On the C^*-algebra of a one-parameter semigroup of isometries', *Acta Math.* **128** (1972), 143–152.
[12] R. Exel, M. Laca and J. C. Quigg, 'Partial dynamical systems and C^*-algebras generated by partial isometries', *J. Operator Theory*, to appear.
[13] E. R. Green, *Graph products of groups* (PhD Thesis, The University of Leeds, 1990).
[14] S. Hermiller and J. Meier, 'Algorithms and geometry for graph products of groups', *J. Algebra* **171** (1995), 230–257.
[15] M. Laca, 'Purely infinite simple Toeplitz algebras', *J. Operator Theory* **41** (1999), 421–435.
[16] M. Laca and I. Raeburn, 'Semigroup crossed products and the Toeplitz algebras of nonabelian groups', *J. Funct. Anal.* **139** (1996), 415–440.
[17] G. J. Murphy, 'Ordered groups and Toeplitz algebras', *J. Operator Theory* **18** (1987), 303–326.
[18] A. Nica, 'C^*-algebras generated by isometries and Wiener-Hopf operators', *J. Operator Theory* **27** (1992), 17–52.
[19] J. C. Quigg, 'Full and reduced C^*-coactions', *Math. Proc. Cambridge Philos. Soc.* **116** (1994), 435–450.
[20] H. Salas, 'Semigroups of isometries with commuting range projections', *J. Operator Theory* **14** (1985), 311–346.
[21] C. Wilde and K. Witz, 'Invariant means and the Stone-Cech compactification', *Pacific J. Math.* **21** (1967), 577–586.

Laboratoire de Topologie
Université de Bourgogne
UMR 5584 du CNRS
B.P. 47 870
21078 Dijon Cedex
France
e-mail: crisp@topolog.u-bourgogne.fr

Department of Mathematics
The University of Newcastle
NSW 2308
Australia
Current address: Mathematisches Institut
Westfälische Wilhelms-Universität
Einsteinstr. 62
48149 Münster
Germany
e-mail: laca@math.uni-muenster.de