J. Austral. Math. Soc. **72** (2002), 153–160

ON GRAPH C^*-ALGEBRAS

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(Received 19 June 2000; revised 2 February 2001)

Communicated by G. Willis

Abstract

Certain C^*-algebras on generators and relations are associated to directed graphs. For a finite graph Γ, C^*-algebra $\mathscr{O}_\Gamma$ is canonically isomorphic to Cuntz-Krieger algebra corresponding to the adjacency matrix of Γ. It is shown that if a countably infinite graph Γ is strongly connected, $\mathscr{O}_\Gamma$ is simple and purely infinite.

2000 *Mathematics subject classification*: primary 46L05, 46L35.

1. Introduction and notation

Let Γ be a countable directed graph. Denote vertices of Γ by $U, V, W \in \mathscr{V}(\Gamma)$ and edges by $u, v, w \in \mathscr{E}(\Gamma)$. If $v \in \mathscr{E}(\Gamma)$ is connecting U and V, call U *the source* of v and V *the range* of v, and write

$$s(v) = U, \quad r(v) = V.$$

Let H be an infinite-dimensional Hilbert space. To every edge $v \in \mathscr{E}(\Gamma)$ we associate a non-zero partial isometry s_v, acting on H, with the following properties:

(i) $s_v s_v^* s_w s_w^* = \delta_{v,w} s_v s_v^*$, for all $v, w \in \mathscr{E}(\Gamma)$;
(ii) $s_v^* s_v s_w^* s_w = \delta_{r(v),r(w)} s_v^* s_v$, for all $v, w \in \mathscr{E}(\Gamma)$;
(iii) $s_v^* s_v s_u s_u^* = \delta_{r(v),s(u)} s_u s_u^*$, for all $u, v \in \mathscr{E}(\Gamma)$;
(iv) $s_v^* s_v = \sum_{r(v)=s(w)} s_w s_w^*$, if the set $\{w \in \mathscr{E}(\Gamma); s(w) = r(v)\}$ is finite.

DEFINITION 1. With the notation as above, we set

$$\mathscr{O}_{\Gamma,\{s_v\}} = C^*(s_v;\ v \in \mathscr{E}(\Gamma))$$

and call $\mathcal{O}_{\Gamma,\{s_v\}}$ *a Cuntz-Krieger algebra associated to* Γ *and the family* $\{s_v\}$. The corresponding universal C^*-algebra will be denoted $\mathcal{O}_\Gamma$.

REMARK 1. In general, $\mathcal{O}_{\Gamma,\{s_v\}}$ defined above depends on the choice of generating partial isometries. Note also that arguments from [5] show that $\mathcal{O}_\Gamma$ exists, for any Γ.

DEFINITION 2. Let Γ be a directed graph. We call Γ *infinite* if the set $\mathcal{E}(\Gamma)$ is infinite, and *row finite* if, for each vertex, the number of outgoing edges is finite. The *adjacency matrix* of Γ is defined as

$$A_\Gamma(u, v) = \begin{cases} 1, & r(u) = s(v) \\ 0, & r(u) \neq s(v), \end{cases}$$

for all pairs of edges (u, v) in $\mathcal{E}(\Gamma)$.

DEFINITION 3. Let A be a non-negative, $n \times n$-matrix. Call A *irreducible* if for each pair of indices (i, j) from $\{1, \ldots, n\}$, there is $k \in \mathbb{N}$ such that $A^k(i, j) \neq 0$. A directed graph Γ is called *strongly connected* (or *transitive*) if for all pairs of vertices (U, V), there exists a path $v_1 \cdots v_k$ such that $s(v_1) = U$ and $r(v_k) = V$.

If Γ is finite, strongly connected and every loop in Γ has an exit—in other words, if A_Γ is an irreducible, non-permutation matrix, $\mathcal{O}_\Gamma$ is canonically isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{A_\Gamma}$ (see [4]). In particular, $\mathcal{O}_\Gamma$ is simple and purely infinite, and $\mathcal{O}_{\Gamma,\{s_v\}}$ does not depend on the choice of generators.

In this note, we give a simple proof of an analogous theorem for infinite graphs (the theorem is proved at the end of the paper—see Theorem 2.6):

THEOREM 1.1. *Let Γ be a countably infinite, strongly connected graph. Then $\mathcal{O}_\Gamma$ is simple and purely infinite.*

We should point out that the above theorem has been proved by Laca and Exel (see [6, 16.2 and 14.1]). Using the presentation of $\mathcal{O}_\Gamma$ as the crossed product algebra for a partial dynamical system, they extended to the infinite case some of the main results known to hold in the finite case—including the above criterion for $\mathcal{O}_\Gamma$ to be simple and purely infinite.

An important special case of Theorem 2.6 is when Γ is assumed to be row finite (in which case it suffices to use only relations (i) and (ii)—the usual Cuntz-Krieger relations). This situation has been studied by Kumjian, Pask and Raeburn (see [14, Corollary 3.10.]). Using a groupoid approach, they carry out a detailed analysis of how the distribution of loops affects the structure of $\mathcal{O}_\Gamma$, for any row-finite graph Γ.

In contrast to that, the method used here is just an adaptation of the original proof of Cuntz. Namely, in analogy with $\mathcal{O}_\infty$, we use the fact that

$$\mathcal{O}_\Gamma = \varinjlim \mathcal{E}_{A_n},$$

where we show that each $\mathcal{E}_{A_n}$ is a universal algebra on generators and relations, canonically isomorphic to an extension of some Cuntz-Krieger algebra by a direct sum of a finite number of copies of compact operators. We then modify the proof of [1, Theorem 3.4] to this slightly more general setup. Finally, it is clear that algebras $\mathcal{O}_\Gamma$ satisfy the UCT, so are within the range of Kirchberg's classification (see [12, 13]).

Let us also mention that results similar to Theorem 2.6 appear in [10, 11], albeit in a different setting, and that $\mathcal{O}_\Gamma$ can be realized as a Pimsner algebra $\mathcal{O}_X$, for a suitable choice of bimodule X (see [16]).

2. Preliminaries and results

Let Γ be a countably infinite, directed graph. Unless stated otherwise, it is always assumed that Γ is strongly connected. Relabel the edges of Γ as $v_1, v_2, \dots,$ write $A_\Gamma(i, j)$ for $A_\Gamma(v_i, v_j)$ and denote the partial isometry s_{v_i} by s_i. Also, let A_n stand for the upper-left hand corner of the matrix A_Γ, and

$$\mathcal{M}_{A_n} = \{\mu = s_{i_1} \cdots s_{i_k};\ i_j \in \{1, \dots, n\} \text{ and } A_n(i_j, i_{j+1}) = 1\}.$$

REMARK 2. It is easy to see that, with Γ as above, there exists an increasing filtration $(\Gamma_i)_{i\in\mathbb{N}}$ of Γ, where each Γ_i is a finite, strongly connected graph. Furthermore, we will assume that the edges of Γ (hence, the generators of $\mathcal{O}_\Gamma$) are labelled in a way compatible with this filtration.

LEMMA 2.1. *Let Γ be a countably infinite directed graph. Then Γ is strongly connected if and only if there exists a strictly increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ such that each A_{n_k} is an irreducible, non-permutation matrix.*

PROOF. If Γ is infinite and strongly connected, there exists a vertex with at least two outgoing edges. Together with the above ordering, that gives the sequence (A_{n_k}). The other direction is obvious. □

REMARK 3. Assume that Γ is as above and denote $\mathcal{E}_{A_n,\{s_i\}} = C^*\{s_1, \dots, s_n\}$, $p_i = s_i s_i^*$, $q_i = s_i^* s_i$ and $r_i = s_i^* s_i - \sum_{j=1}^n A_n(i, j) s_j s_j^*$. Since the projections q_i and q_j are either equal or orthogonal, the same holds for r_i and r_j, $i, j = 1, \dots, n$, so denote by $m_1, \dots, m_k$ distinct projections among $r_1, \dots, r_n$, for some $k \le n$. Set $I^{(j)} = C^*\{s_\mu m_j s_\nu^*; \mu, \nu \in \mathscr{M}_{A_n}\}$, for every j, and

$$I_n = I^{(1)} \oplus \cdots \oplus I^{(k)}.$$

We then have $I^{(j)} \cong \mathscr{K}$, for all j, where $\mathscr{K}$ stands for the compact operators on a separable Hilbert space (see [1, Proposition 3.1]). Furthermore, if A_n is assumed to be irreducible and non-permutation,

$$\mathscr{E}_{A_n,\{s_i\}}/I_n \cong \mathscr{O}_{A_n}. \tag{1}$$

The following result is analogous to [3, Lemma 3.1]:

PROPOSITION 2.2. *Let Γ be a countably infinite, strongly connected directed graph, and let s_i, $i \in \mathbb{N}$, be a set of generators of $\mathscr{O}_{\Gamma,\{s_i\}}$. Then, for any $m \in \mathbb{N}$, the algebra $\mathscr{E}_{A_m,\{s_i\}} = C^*\{s_1, \dots, s_m\}$ does not depend on the choice of generators.*

PROOF. Suppose that $s_i \in B(H)$. As above, we set $r_i = s_i^* s_i - \sum_{j=1}^m A(i,j) s_j s_j^*$. Let $m_0 > m$ be such that for each non-zero r_i, $i = 1, \dots, m$, there is $j(i) \in \{m+1, \dots, m_0\}$ such that

$$r_i s_{j(i)} s_{j(i)}^* = s_{j(i)} s_{j(i)}^*,$$

and let $n > m_0$ be such that A_n is irreducible and non-permutation. Note that Lemma 2.1 (and Remark 2) imply that such m_0 and n exist. We want to construct partial isometries $t_{m+1}, \dots, t_n \in B(H)$ such that $C^*(s_1, \dots, s_m, t_{m+1}, \dots, t_n)$ is canonically isomorphic to $\mathscr{O}_{A_n}$.

Let I be a subset of $\{m+1, \dots, n\}$ defined by: $j \in I$ if and only if there is $1 \le i \le m$ such that $A(i,j) = 1$ and $A(i,k) = 0$, for $k = j+1, \dots, n$. For $j \in I$, let $\tilde{p}_j = r_i - \sum_{k=m+1}^{j} A(i,k) s_k s_k^*$. Note that $\tilde{p}_j = s_j s_j^* + s_i^* s_i - \sum_{k=1}^{n} A(i,k) s_k s_k^*$, and that $\tilde{p}_j$ does not depend on the choice of i in the above formula. For j not in I, set $\tilde{p}_j = s_j s_j^*$. Define projections $\tilde{q}_j$, for $j = m+1, \dots, n$, by

$$\tilde{q}_j = \sum_{k=1}^{m} A(j,k) s_k s_k^* + \sum_{k=m+1}^{m} A(j,k) \tilde{p}_k.$$

Since Γ is strongly connected, every $s_i s_i^*$ is an infinite-dimensional projection, so the same holds for $\tilde{p}_j$ and $\tilde{q}_j$. For $j = m+1, \dots, n$, let t_j be any partial isometry such that $t_j t_j^* = \tilde{p}_j$ and $t_j^* t_j = \tilde{q}_j$. Then

$$s_i^* s_i = \sum_{j=1}^{m} A(i,j) s_j s_j^* + \sum_{j=m+1}^{n} A(i,j) t_j t_j^*, \quad i = 1, \dots, m,$$

and

$$t_i^* t_i = \sum_{j=1}^{m} A(i,j) s_j s_j^* + \sum_{j=m+1}^{n} A(i,j) t_j t_j^*, \quad i = m+1, \ldots, n,$$

hence, $\mathscr{A} = C^*(s_1, \ldots, s_m, t_{m+1}, \ldots, t_n) \cong \mathscr{O}_{A_n}$ canonically.

If $s_i' \in B(H)$, $i = 1, 2, 3, \ldots$, is another set of generators for $\mathscr{O}_\Gamma$, the above procedure gives $t_{m+1}', \ldots, t_n'$ such that $\mathscr{A}' = C^*\{s_1', \ldots, s_m', t_{m+1}', \ldots, t_n'\} \cong \mathscr{O}_{A_n}$, and the map

$$s_i \mapsto s_i', \quad i = 1, \ldots, m, \qquad t_j \mapsto t_j', \quad j = m+1, \ldots, n$$

extends to an isomorphism from $\mathscr{A}$ into $\mathscr{A}'$, mapping $\mathscr{E}_{A_m,\{s_i\}}$ onto $\mathscr{E}_{A_m,\{s_i'\}}$. $\square$

Corollary 2.3. *Let Γ be as in Proposition 2.2. Then $\mathscr{O}_\Gamma$ does not depend on the choice of generators.*

Remark 4. It is clear that Proposition 2.2 and Corollary 2.3 will remain true as long as one can construct a canonical embedding of $\mathscr{E}_{A_{n_k}}$ into some Cuntz-Krieger algebra that does not depend on the choice of generators. This has already been argued by Cuntz and Krieger in [4, Remark 2.15]. Note also that $\mathscr{E}_{A_{n_k}}$ can be described as a C^*-algebra associated to some inverse semigroup (see, for example, [9]).

The following result, due to Cuntz (see [3, Proposition 1.6]), describes simple purely infinite C^*-algebras. We use this in the proof of Theorem 2.6:

Proposition 2.4. *Let the C^*-algebra $\mathscr{A}$ satisfy*:

(i) $\mathscr{A} \neq 0, \mathbb{C}$.

(ii) *For every $\varepsilon > 0$ and every positive $a, b \in \mathscr{A}$, there is $c \in \mathscr{A}$ such that $\|b - cac^*\| < \varepsilon$.*

Then $\mathscr{A}$ is simple and purely infinite.

Lemma 2.5. *Let A_n be irreducible. Then there exists a non-unitary isometry $v \in \mathscr{E}_{A_n}$, such that*

$$\lim_{k\to\infty} (v^*)^k x v^k = 0, \quad \textit{for all } x \in I_n.$$

Proof. In the case of $\mathscr{O}_\infty$, this has been proved in [1, Proposition 3.1, Remark 2]. Since we do not necessarily have an isometry among the generators of $\mathscr{E}_{A_n}$, we have to construct one. Let p_i and m_i be as in Remark 3. Note that

$$\text{for all } j \text{ there is } i \text{ such that } s_i m_j = s_i s_i^* s_i m_j \neq 0,$$

and denote that s_i by $\tilde{t}_j$. Also,

$$\text{for all } j \text{ there is } i \text{ such that } s_i p_j = s_i s_i^* s_i p_j \neq 0.$$

Denote that s_i by t_j, and set

$$v = \sum_{i=1}^{n} t_i p_i + \sum_{j=1}^{k} \tilde{t}_j m_j.$$

We immediately get $v^* v = 1$ and $vv^* < 1$, so v is a proper isometry. Let $s_\mu = s_{i_1} s_{i_2} \cdots s_{i_p}$, and note that $v^* s_\mu \neq 0$ implies

$$\begin{aligned} v^* s_\mu &= \left(\sum_{i=1}^{n} p_i t_i^* s_{i_1} + \sum_{j=1}^{k} m_j \tilde{t}_j^* s_{i_1} \right) s_{i_2} \cdots s_{i_p} \\ &= (p_{j_1} + \cdots + p_{j_m} + m_{k_1} + \cdots + m_{k_l}) p_{i_2} (s_{i_2} \cdots s_{i_p}) = s_{i_2} \cdots s_{i_p}. \end{aligned}$$

It remains to be shown that $v^* m_j v = 0$, $j = 1, \ldots, k$. Since $p_i m_j = 0$, we get

$$v^* m_l = \sum_{i=1}^{n} p_i t_i^* (t_i t_i^*) m_l + \sum_{j=1}^{k} m_j \tilde{t}_j^* (\tilde{t}_j \tilde{t}_j^*) m_l = 0, \quad l = 1, \ldots, k. \qquad \square$$

DEFINITION 4. Let α be the action of $\mathbb{T} = \{z \in \mathbb{C};\ |z| = 1\}$ on $\mathcal{O}_\Gamma$, given on generators by $\alpha_t(s_v) = ts_v$, $v \in \mathcal{E}_\Gamma$, and set

$$P(x) = \int_{\mathbb{T}} \alpha_t(x)\, dt, \quad x \in \mathcal{O}_\Gamma \tag{2}$$

(see [1]).

Now we are ready to prove the result announced in the Introduction. The proof closely follows the proof of [1, Theorem 3.4]:

THEOREM 2.6. *Let Γ be a countably infinite, directed, strongly connected graph. Then $\mathcal{O}_\Gamma$ is simple and purely infinite.*

PROOF. Let positive elements $a, b \in \mathcal{O}_\Gamma$ and $0 < \varepsilon < 1/4$ be given. Let $a = zz^*$, for some $z \in \mathcal{O}_\Gamma$, and let $y \in \mathcal{E}_{A_m}$, for some $m \in \mathbb{N}$, be a finite linear combination of words in s_i, s_i^* such that $\|b - y\| < \varepsilon$. We can assume that $\|P(y)\| = 1$ and $\|z\| = 1$.

From Lemma 2.1 there is $n > m$ such that A_n is irreducible and non-permutation. With $t_1, \ldots, t_n$ as in Proposition 2.2, we consider C^*-algebras $\mathcal{A}_1 = C^*\{s_1, \ldots, s_m, t_{m+1}, \ldots, t_n\}$, $\mathcal{E}_{A_n}$, and $\mathcal{A}_2 = \mathcal{E}_{A_n}/I_n$. Denote by π the quotient map $\mathcal{E}_{A_n} \to \mathcal{E}_{A_n}/I_n$. It follows from (1) and Proposition 2.2 that the map

$$s_i \mapsto \pi(s_i), \quad i = 1, \ldots, m, \qquad t_j \mapsto \pi(s_j), \quad j = m+1, \ldots, n$$

extends to an isomorphism from $\mathscr{A}_1$ into $\mathscr{A}_2$. Let $P_1(x)$ and $P_2(x)$ stand for $P(x)$ (see (2) above), computed in $\mathscr{A}_1$ and $\mathscr{A}_2$, respectively. Since y is a word in s_i, s_i^*, with i only in $\{1, \dots, m\}$, $P_1(y) = P(y)$. Together with the above isomorphism, that gives

$$\|P_2(\pi(y))\| = \|P_1(y)\| = \|P(y)\| = 1.$$

From [1, Remark 1.13], there is $\hat{w} \in \mathscr{A}_2$ such that $\|\hat{w}\| \le 1+\varepsilon$, and $\hat{w}\pi(y)\hat{w}^* = 1$. Lifting from the quotient gives

$$wyw^* = 1 + I_n$$

in $\mathscr{E}_{A_n}$, with $\|w\| \le 1 + 2\varepsilon$. Then, from Lemma 2.5, there is $v \in \mathscr{E}_{A_n}$ and $k \in \mathbb{N}$, such that

$$\|(v^*)^k wyw^* v^k - 1\| < \varepsilon.$$

Hence, we get

$$\|z(v^*)^k wbw^* v^k z^* - a\| < 4\varepsilon,$$

which completes the proof. □

REMARK 5. If a directed graph Γ is row finite and strongly connected, [15, Theorem 4.2.4] gives the K-theory of $\mathscr{O}_\Gamma$:

$$K_0(\mathscr{O}_\Gamma) \cong \tilde{\mathbb{Z}}^\infty / \operatorname{Im}(1 - A_\Gamma^t)\tilde{\mathbb{Z}}^\infty \quad \text{and} \quad K_1(\mathscr{O}_\Gamma) \cong \operatorname{Ker}(1 - A_\Gamma^t)\tilde{\mathbb{Z}}^\infty$$

(see [2, 15]). In case of general Γ, see [7]. Finally, note that the K-theory of $\mathscr{O}_\Gamma$ can be computed in the same way as that of $\mathscr{O}_\infty$ (see [3]). That has been done in [8].

Acknowledgement

This research was partially supported by an ORS Award. The author would like to thank E. Beggs and D. E. Evans for useful discussions, and the referee for several very helpful remarks. An initial draft of this note was presented at the Summer School in Operator Algebras, Odense, August 1996.

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