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NILPOTENT GROUPS ARE NOT DUALIZABLE

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Abstract

It is shown that no finite group containing a non-abelian nilpotent subgroup is dualizable. This is in contrast to the known result that every finite abelian group is dualizable (as part of the Pontryagin duality for all abelian groups) and to the result of the authors in a companion article that every finite group with cyclic Sylow subgroups is dualizable.

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1. Introduction

In [3] and [4] a strong natural duality is proved for groups of the form $\mathbf{Z}_n \rtimes \mathbf{Z}_m$, where $(n, m) = 1$. In this paper we show that a finite nonabelian nilpotent group cannot admit a natural duality. In fact, for every finite group $\mathbf{H}$ having at least one nonabelian Sylow subgroup (which is then nilpotent of class at least 2), we focus our attention on a p-subgroup $\mathbf{G}$ of $\mathbf{H}$ of nilpotence class 2, and use $\mathbf{G}$ to prove that the original group $\mathbf{H}$ is not dualizable.

For the benefit of readers not familiar with the theory of natural dualities, we begin with a brief review of what is meant by '*admitting a* (*natural*) *duality*' and refer to the text of Clark and Davey [1] for a detailed account.

Let $\mathbf{A}$ be a finite algebra and let $\tilde{A} = \langle A; F, P, R, \tau \rangle$ be a topological structure on the same underlying set A, where

(a) each $f \in F$ is a homomorphism $f : \mathbf{A}^n \to \mathbf{A}$ for some $n \in \mathbb{N} \cup \{0\}$,

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(b) each $p \in P$ is a homomorphism $p : \mathrm{dom}(p) \to \mathbf{A}$ where $\mathrm{dom}(p)$ is a subalgebra of $\mathbf{A}^n$ for some $n \in \mathbb{N}$,

(c) each $r \in R$ is (the universe of) a subalgebra of $\mathbf{A}^n$ for some $n \in \mathbb{N}$,

(d) τ is the discrete topology.

Whenever (a), (b) and (c) hold, we say that the operations in F, the partial operations in P and the relations in R are *algebraic over* $\mathbf{A}$. These compatibility conditions between the structure on $\mathbf{A}$ and the structure on $\tilde{A}$ guarantee that there is a naturally defined dual adjunction between the quasivariety $\mathscr{A} := \mathbb{ISP}\mathbf{A}$ generated by $\mathbf{A}$ and the topological quasivariety $\mathscr{X}_{\tilde{\mathscr{A}}} := \mathbb{ISP}\tilde{\mathscr{A}}$ generated by $\tilde{A}$; if there is no chance of confusion, we will write $\mathscr{X}$ for $\mathscr{X}_{\tilde{\mathscr{A}}}$. For all $\mathbf{B} \in \mathscr{A}$ the homset $D(\mathbf{B}) := \mathscr{A}(\mathbf{B}, \mathbf{A})$ of all homomorphisms from $\mathbf{B}$ to $\mathbf{A}$ is a closed substructure of the direct power $\tilde{A}^B$ and for all $\tilde{X} \in \mathscr{X}$ the homset $E(\tilde{X}) := \mathscr{X}(\tilde{\mathscr{X}}, \tilde{\mathscr{A}})$ is a subalgebra of the direct power $\mathbf{A}^X$. It follows easily that the contravariant hom-functors $\mathscr{A}(-, \mathbf{A}) : \mathscr{A} \to \mathscr{S}$ and $\mathscr{X}(-, \tilde{\mathscr{A}}) : \mathscr{X} \to \mathscr{S}$, where $\mathscr{S}$ is the category of sets, lift to contravariant functors $D : \mathscr{A} \to \mathscr{X}$ and $E : \mathscr{X} \to \mathscr{A}$.

For each $\mathbf{B} \in \mathscr{A}$ there is a natural embedding $e_{\mathbf{B}}$ of $\mathbf{B}$ into $ED(\mathbf{B})$ given by evaluation: for each $b \in B$ and each $x \in D(\mathbf{B}) = \mathscr{A}(\mathbf{B}, \mathbf{A})$ define $e_{\mathbf{B}}(b)(x) := x(b)$. Similarly, for each $\tilde{X} \in \mathscr{X}$ there is an embedding $\varepsilon_{\tilde{X}}$ of $\tilde{X}$ into $DE(\tilde{X})$. A simple calculation shows that $e : \mathrm{id}_{\mathscr{A}} \to ED$ and $\varepsilon : \mathrm{id}_{\mathscr{X}} \to DE$ are natural transformations. If $e_{\mathbf{B}}$ is an isomorphism for all $\mathbf{B} \in \mathscr{A}$, we say that $\tilde{A}$ yields a (*natural*) *duality* on $\mathscr{A}$. If there is some choice of F, P and R such that $\tilde{A}$ yields a duality on $\mathscr{A}$, then we say that $\mathbf{A}$ (or $\mathscr{A}$) *admits a natural duality* or, briefly, is *dualizable*.

We wish to prove that for no choice of F, P and R does $\tilde{H}$ yield a duality on $\mathscr{H}$, the quasivariety generated by the finite group $\mathbf{H}$. For this, it is enough to show that there is no duality when $F = P = \emptyset$ and R consists of all subgroups of all finite powers of $\mathbf{H}$, the so-called *brute force duality*; see [1]. In order to prove that there is no brute force duality, we need to find a (necessarily infinite) group $\mathbf{D} \in \mathscr{H}$ such that $e_{\mathbf{D}}$ is not onto $ED(\mathbf{D})$. We will use what is known as the *ghost element* method. We will choose $\mathbf{D}$ to be a proper subgroup of $\mathbf{G}^{\mathbb{Z}}$ and choose a particular element $\underline{w} = (w_i)_{i \in \mathbb{Z}} \in \mathbf{G}^{\mathbb{Z}} - \mathbf{D}$. We will then construct an element Φ of $ED(\mathbf{D})$ which will not be an evaluation map for any element of $\mathbf{D}$ because it will act as if it were an evaluation map at the ghost element; that is, for every $i \in \mathbb{Z}$, $\Phi(\pi_i) = w_i = \pi_i(\underline{w})$.

More precisely, we will find a sequence $\{\underline{v}_n\}$ of elements in $\mathbf{D}$ such that the sequence converges to $\underline{w}$. Here convergence is pointwise (that is, componentwise) and in each component a sequence is convergent if and only if it is eventually constant and converges to its eventual constant. Then for $\mu \in D(\mathbf{D})$ we define $\Phi(\mu)$ to be $\lim_{n\to\infty} \mu(\underline{v}_n)$. We need to prove four things about Φ: (1) Φ is well defined; (2) Φ 'acts like' evaluation at $\underline{w}$; (3) Φ is structure preserving, and (4) Φ is continuous.

The first and second will be easy. The third will also be easy since being structure preserving is a local property. That is, if on every finite subset F of $D(\mathbf{D})$ there is an

element of $ED(\mathbf{D})$ which agrees with Φ on F, then Φ is structure preserving. But this will follow from the fact that Φ is a limit of evaluation functions (sequentially, at the $\underline{v}_n$).

The last, continuity, will be difficult. We recall that the topology on $D(\mathbf{D})$ is boolean (in the vernacular, a zero-dimensional compact Hausdorff space). That is, $D(\mathbf{D})$ has a basis of clopen sets consisting of sets of the form

$$\{\phi \mid \phi(\underline{d}) = h_{\underline{d}} \text{ for all } \underline{d} \in F\},$$

where F is a finite subset of $\mathbf{D}$, and $h_{\underline{d}} \in H$. Thus for every $\alpha \in D(\mathbf{D})$, we must find a finite subset F_α of $\mathbf{D}$ such that if $\beta \in D(\mathbf{D})$ with $\alpha(\underline{d}) = \beta(\underline{d})$ for all $\underline{d} \in F_\alpha$, then $\Phi(\alpha) = \Phi(\beta)$. But compactness tells us that finitely many of these clopen sets cover $D(\mathbf{D})$; taking F to be the union of the finitely many F_α we see that Φ will be continuous if and only if there is a finite subset F of $\mathbf{D}$ such that if $\mu, \nu \in D(\mathbf{D})$ and $\mu(\underline{d}) = \nu(\underline{d})$ for all $\underline{d} \in F$, then $\Phi(\mu) = \Phi(\nu)$.

2. The group D

LEMMA 2.1. *For each finite nonabelian p-group* $\mathbf{P}$ *there is a nonabelian subgroup* $\mathbf{G} \leq \mathbf{P}$ *and* $a, b \in \mathbf{G}$ *such that*:

(i) $\mathbf{G} = \langle a, b\rangle$;
(ii) *all proper subgroups of* $\mathbf{G}$ *are abelian*;
(iii) *in* $\mathbf{G}$, *commutators are in the center* $Z(\mathbf{G})$, *that is,* $\mathbf{G}$ *is of nilpotence class* 2;
(iv) *the commutator is an alternating bilinear form*;
(v) $Z(\mathbf{G}) = \langle a^p, b^p, [a, b]\rangle$, *and so* $g^p \in Z(\mathbf{G})$ *for every* $g \in G$;
(vi) *in* $\mathbf{G}$, *the centralizer of* g, $C_{\mathbf{G}}(g)$, *is* $\langle g, Z(\mathbf{G})\rangle$, *provided* $g \notin Z(\mathbf{G})$.

PROOF. Let $\mathbf{G}$ be a minimal nonabelian subgroup of $\mathbf{P}$; thus, every pair of noncommuting elements generates $\mathbf{G}$. Hence, (i) and (ii) hold. Let $c = [a, b]$; if $c \notin Z(\mathbf{G})$, then we can replace one of a and b with c in condition (i). As $\mathbf{G}$ is nilpotent, iterating this procedure eventually leads us to $[a, b] \in Z(\mathbf{G})$. This implies that for every i and j, $b^j a^i = a^i b^j [a, b]^{-ij}$. Hence, every commutator in $\mathbf{G}$ is a power of $[a, b]$ and so is in $Z(\mathbf{G})$. Thus, (iii) follows. From (iii), the bilinearity of the commutator is a standard exercise; thus, (iv) holds. If $a^p \notin Z(\mathbf{G})$, then replace a with a^p; iteration yields $a^p \in Z(\mathbf{G})$, so that $\langle a^p, b^p, [a, b]\rangle \leq Z(\mathbf{G})$. If this inequality were strict, then for some $0 \leq i, j < p$ with $i + j \geq 1$ we would have $a^i b^j \in Z(\mathbf{G})$. But then this element and one of a and b would generate $\mathbf{G}$, making $\mathbf{G}$ abelian. In turn this implies that (v) holds. Thus, $|\mathbf{G}/Z(\mathbf{G})| = p^2$, and (vi) follows. □

We prove that a finite group $\mathbf{H}$ having a subgroup $\mathbf{G}$ with the properties in Lemma 2.1 is not dualizable. Of course, every finite group having at least one nonabelian Sylow

subgroup contains such a group $\mathbf{G}$. Item (iii) implies that $[x^i, y^j] = [x, y]^{ij}$, a fact the reader should keep in mind when verifying the computations in this section.

Choose a positive integer, t, whose value will be fixed later in this paper. For $i \in \mathbb{Z}$ let $\underline{d}_i \in \mathbf{G}^{\mathbb{Z}}$ be $(\dots, 1, 1, a, b^{-1}, 1, \dots, 1, b, a^{-1}, 1, 1, \dots)$ with $(\underline{d}_i)_i = b^{-1}$ and $(\underline{d}_i)_{i+t} = b$; thus, there are $(t-1)$ 1's between b^{-1} and b. The identity element of $\mathbf{G}^{\mathbb{Z}}$ is denoted $\underline{1}$.

DEFINITION 2.2. $\mathbf{D} := \langle \underline{d}_i \mid i \in \mathbb{Z} \rangle$.

DEFINITION 2.3. Our ghost vector is $\underline{w} := (\dots, 1, 1, c^{-1}, 1, 1, \dots)$ where $c := [a, b]$ and where $(\underline{w})_0 = c^{-1}$.

For integers $i < j$, let $\underline{v}_{i,j}$ be defined by

$$(\underline{v}_{i,j})_m = \begin{cases} c^{-1}, & \text{if } m = i; \\ c, & \text{if } m = j; \\ 1, & \text{otherwise.} \end{cases}$$

Let $\mathbf{V} := \langle \{\underline{v}_{i,j} \mid i < j\} \rangle$, and note that $\underline{w} = \lim_{n\to\infty} \underline{v}_{0,n}$.

We want to show that $\underline{w} \in \mathbf{G}^{\mathbb{Z}} - \mathbf{D}$. For this we need to describe $\mathbf{D}'$, the commutator subgroup of $\mathbf{D}$. Because of the bilinearity of the commutator, $\mathbf{D}'$ is generated by the set of commutators of any generating set of $\mathbf{D}$. It is clear that $[\underline{d}_j, \underline{d}_i] = 1$ for $|i - j| \notin \{1, t+1\}$. Recalling that the commutator is an alternating form, we have only two computations to perform. They yield $[\underline{d}_j, \underline{d}_{j-1}] = \underline{v}_{j-1,i+t}$ and $[\underline{d}_j, \underline{d}_{j+t+1}] = \underline{v}_{j+t,i+t+1}$. Since for $i < j < k$ we have $\underline{v}_{i,j}\underline{v}_{j,k} = \underline{v}_{i,k}$, then $\mathbf{D}' = \mathbf{V}$.

LEMMA 2.4. $\underline{w} \in \mathbf{G}^{\mathbb{Z}} - \mathbf{D}$.

PROOF. We need to show that $\underline{w} \notin \mathbf{D}$. Suppose that $\underline{w} \in \mathbf{D}$. Then we can write $\underline{w} = \underline{d}_{i_1}^{j_1} \cdots \underline{d}_{i_k}^{j_k} \underline{d}'$ where $\underline{d}' \in \mathbf{D}'$ and $i_1 < \cdots < i_k$. We will prove that $p \mid j_m$ for $1 \le m \le k$. As $w_{i_1-1} \in \langle c \rangle$, we must have $a^{j_1} \in \langle c \rangle \le Z(\mathbf{G})$; hence, $p \mid j_1$. Suppose that $p \mid j_n$ for $n < m$. Then as $i_1 < \cdots < i_k$ and $w_{i_m-1} \in \langle c \rangle$, our inductive assumption ensures that $a^{j-m} \in Z(\mathbf{G})$, so that $p \mid j_m$. By induction, $p \mid j_m$ for $1 \le m \le k$. Thus, $\underline{w}$ and each $\underline{d}_{i_m}^{j_m}$ lie in $(Z(\mathbf{G}))^{\mathbb{Z}}$, an abelian group. Notice that each $\underline{d}_{i_m}^{j_m}$ and each generator of $\mathbf{D}'$ has the property that the product of their components is 1. Hence, so must $\underline{w}$, a contradiction; hence, $\underline{w} \notin \mathbf{D}$. □

3. Homomorphisms from D to H

Let $\mu \in \mathrm{Hom}(\mathbf{D}, \mathbf{H})$. Recall that our ghost element $\underline{w}$ is the limit of the $\underline{v}_{0,n}$:

$$\underline{w} = \lim_{n\to\infty} \underline{v}_{0,n}.$$

Thus, for every μ we will prove that there is an n such that $\mu(\underline{v}_{0,i}) = \mu(\underline{v}_{0,i+1})$ for $i \geq n$. We will then define $\Phi(\mu)$ to be this *eventual value* of $\mu(\underline{v}_{0,n})$:

$$\Phi(\mu) := \lim_{n\to\infty} \mu(\underline{v}_{0,n}).$$

LEMMA 3.1. *Let $\mu \in \mathrm{Hom}(\mathbf{D}, \mathbf{H})$. Then there is an n such that $\mu(\underline{v}_{0,i}) = \mu(\underline{v}_{0,i+1})$ for $i \geq n$.*

PROOF. Let $\mu(\underline{d}_{i_1}), \ldots, \mu(\underline{d}_{i_k})$ generate $\mu(\mathbf{D}) \leq \mathbf{H}$, where $i_j < i_{j+1}$; of course, we may assume $k \leq |H|$. Recalling that $[\underline{d}_i, \underline{d}_j] = 1$ save for $|i - j| \in \{1, t+1\}$, let $J := \bigcup_{j=1}^{k} \{i_j - t - 1, i_j - 1, i_j + 1, i_j + t + 1\}$, and note that $|J| \leq 4|H|$. Suppose $i \notin J$; then $[\underline{d}_i, \underline{d}_{j_j}] = 1$ for all j, and so $\mu(\underline{d}_i) \in Z(\mu(\mathbf{D}))$. Using the computations $[\underline{d}_i, \underline{d}_{i-1}] = \underline{v}_{i-1,i+t}$ and $[\underline{d}_i, \underline{d}_{i+t+1}] = \underline{v}_{i+t,i+t+1}$, we see that

$$\underline{v}_{0,n} = \prod_{i=1}^{n} \underline{v}_{i-1,i} = \prod_{i=1}^{n} [\underline{d}_{i-t-1}, \underline{d}_i].$$

This implies that if n is sufficiently large, then $\mu(\underline{v}_{0,i}) = \mu(\underline{v}_{0,i+1})$ for all $i \geq n$. □

Thus, $\Phi(\mu)$ is well defined, proving the first of the four properties we need to prove about Φ. Notice that for every $i \in \mathbb{Z}$, $\Phi(\pi_i) = w_i$, so that Φ acts like evaluation at $\underline{w}$, proving the second property. It is easy to see that Φ is structure preserving: for any finite subset F of $\mathrm{Hom}(\mathbf{D}, \mathbf{H})$, we can find a large enough n such that Φ agrees with the evaluation map at $\underline{v}_{0,n}$ at each member of F; since all evaluation maps are structure preserving, so is Φ. This proves the third property. Thus, we are left with the hard part, showing that Φ is continuous.

Necessarily, the n from Lemma 3.1 depends on μ (consider the case where $\mathbf{H}$ contains a copy of $\mathbf{G}^2$ and for all $k \geq 1$, $\mu_k(\underline{d}) = (d_0, d_k)$). Potentially, this can disrupt the continuity of Φ. We counter this threat by showing that we can choose a large enough N (depending on $\mathbf{H}$ but not on μ) and choose t large enough (again, depending on $\mathbf{H}$ but not on μ) so that we can determine the eventual value of $\mu(\underline{v}_{0,n})$ by looking only at $\mu(\underline{v}_{0,i})$ for $1 \leq i \leq N$.

DEFINITION 3.2. We define an interval I of $\mathbb{Z}$ to be a *gap* if $|I| \geq 6t + 6$ and $\mu(\underline{v}_{i-1,i}) = 1$ for all $i \in I$. We permit gaps to be infinite.

Note that we have not yet decided how big t should be. Recall from the proof of Lemma 3.1 that as $\mu(\underline{d}_i) \in Z(\mu(\mathbf{D}))$ when $i \notin J$, then we have $\mu(\underline{v}_{i-1,i}) = 1$ except for at most $4|H|$ indices i. Thus, by choosing $N > 30|H|(t+1)$, the interval $1 \leq i \leq N$ contains a gap. Obviously, there are at most $4|H| + 1$ maximal gaps. We will then prove that if $i < j$ and each is in a gap, then $\mu(\underline{v}_{i,j}) = 1$. We will do this by choosing t to be sufficiently large.

Next, we make what seems to be a strange definition. It sets the stage for proving a key lemma using a Ramsey-like argument.

DEFINITION 3.3. Define the positive integer M by

$$M := \max |\{(g_i, h_i)\}|,$$

where $g_i, h_i \in H$, $[g_i, h_i] \neq 1$, $[g_i, g_j] = [g_i, h_j] = [h_i, h_j] = 1$ for $i \neq j$. As $|H|$ is finite and the conditions preclude repetitions among the g_i, M is finite.

For instance, if $\mathbf{H} = \mathbf{G}^k$, then it is easy to find $2k$ elements satisfying the above conditions.

LEMMA 3.4. *Let I_1 and I_3 be two gaps such that I_1 is to the left of I_3, with I_2 the non-empty interval between them. Let $i \in I_1$ and $j \in I_3$. Then $\mu(\underline{v}_{i,j}) = 1$.*

PROOF. Suppose the hypotheses of the lemma hold but that $\mu(\underline{v}_{i,j}) = e \neq 1$. Notice that e is independent of the choices of $i \in I_1$ and $j \in I_3$ due to the defining property of a gap and that for $i < j < k$ we have $\underline{v}_{i,j}\underline{v}_{j,k} = \underline{v}_{i,k}$.

For a positive integer s, define $\underline{g}_j := \prod_{k=0}^{s} \underline{d}_{j+k(t+2)}$ and note that

$$(\underline{g}_j)_m = \begin{cases} a, & \text{if } m = j-1; \\ b^{-1}, & \text{if } m = j + k(t+2) \text{ for } 0 \leq k \leq s; \\ b, & \text{if } m = j - 2 + k(t+2) \text{ for } 1 \leq k \leq s+1; \\ a^{-1}, & \text{if } m = j - 1 + (s+1)(t+2); \\ 1, & \text{otherwise.} \end{cases}$$

This means that if $[\underline{g}_i, \underline{g}_j] \neq 1$, then $i - j \equiv \pm 1 \pmod{t+2}$. Thus, $[\underline{g}_j, \underline{g}_{j-1}] = \underline{v}_{j-1, j-2+(s+1)(t+2)}$ so that if $j - 1 \in I_1$ and $j - 2 + (s+1)(t+2) \in I_3$, then $[\mu(\underline{g}_j), \mu(\underline{g}_{j-1})] = e$. Of course, we can choose j and s so that $j - 1 \in I_1$ and $j - 2 + (s+1)(t+2) \in I_3$. We can do much better. Choose t to be a multiple of 8; because each of I_1 and I_3 has size at least $6t + 6$, we can find $t/8 + 1$ values of j $(4j_0, 4(j_0 + 1), \ldots, 4(j_0 + t/8))$ and a value of s such that $j - 1 \in I_1$ and $j - 2 + (s+1)(t+2) \in I_3$.

We are now ready to define g_i and h_i. Set $g_i := \mu(\underline{g}_{4(j_0+i)})$ and $h_i := \mu(\underline{g}_{4(j_0+i)-1})$ for $0 \leq i \leq t/8$. We have $[g_i, h_i] = e$ for all i. Also, $[g_i, g_j] = 1$, since $4(j_0 + i) - 4(j_0 + j)$ is even modulo $t + 2$. Similarly, $[h_i, h_j] = 1$. Finally, $[g_i, h_j] = 1$ for $i \neq j$ since $4(j_0 + i) - 4(j_0 + j) + 1 = 4(i - j) + 1$; as $0 < |i - j| < t/8$, we cannot have $i - j \equiv \pm 1 \pmod{t+2}$. If we now take $t = 8(M + 1)$, we contradict the definition of M, and so have proved the lemma. Notice that our choice of t is independent of μ. □

COROLLARY 3.5. *Let $\mu \in \mathrm{Hom}(\mathbf{D}, \mathbf{H})$. There is a unique $e_\mu \in H$ such that $\mu(\underline{v}_{0,j}) = e_\mu$ for all but finitely many $j \geq 1$. Moreover, $\mu(\underline{v}_{0,j}) = e_\mu$ holds for j in any gap, and for $N > 30|H|(t+1)$, the interval $1 \leq j \leq N$ contains a gap. Thus, e_μ can be determined by examining $\mu(\underline{v}_{0,j})$ on $1 \leq j \leq N$.*

4. The theorem

THEOREM 4.1. *Let $\mathbf{H}$ be a finite group having at least one nonabelian Sylow subgroup; then $\mathbf{H}$ is not dualizable.*

PROOF. Choose $\mathbf{G}$ to be a minimal non-abelian p-subgroup of $\mathbf{H}$. Take N and e_μ as given in Corollary 3.5. Let $\Phi : D(\mathbf{D}) \to \tilde{H}$ be defined by $\Phi(\mu) := \lim_{n\to\infty} \mu(\underline{v}_{0,n})$. Notice that $\Phi(\mu) = e_\mu$ and that if $\mu(\mathbf{D})$ is abelian, then $e_\mu = 1$. By the results of the last section, we need only prove that Φ is continuous. We choose F to be $\{\underline{v}_{0,i} \mid 1 \leq i \leq N\}$, and suppose that $\mu|_F = \nu|_F$. If $\nu(\mathbf{D})$ is abelian, then as $[1, N]$ contains a large interval for μ and $\mu(\underline{v}_{0,i}) = \nu(\underline{v}_{0,i}) = 1$ for $1 \leq i \leq N$, we must have $\Phi(\mu) = \Phi(\nu) = 1$; similarly if $\mu(\mathbf{D})$ is abelian. If neither image is abelian, then $\mu(\underline{v}_{0,i}) = \nu(\underline{v}_{0,i})$ for $1 \leq i \leq N$ implies that an interval in $[1, N]$ is large for ν if and only if it is large for μ, and so again $\Phi(\mu) = \Phi(\nu)$. Thus, Φ is continuous and the theorem is proved. □

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