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# VECTOR VALUED MEAN-PERIODIC FUNCTIONS ON GROUPS

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**Abstract**

Let $G$ be a locally compact Hausdorff abelian group and $X$ be a complex Banach space. Let $C(G, X)$ denote the space of all continuous functions $f : G \to X$, with the topology of uniform convergence on compact sets. Let $X'$ denote the dual of $X$ with the weak* topology. Let $M_c(G, X')$ denote the space of all $X'$-valued compactly supported regular measures of finite variation on $G$. For a function $f \in C(G, X)$ and $\mu \in M_c(G, X')$, we define the notion of convolution $f \star \mu$. A function $f \in C(G, X)$ is called mean-periodic if there exists a non-trivial measure $\mu \in M_c(G, X')$ such that $f \star \mu = 0$. For $\mu \in M_c(G, X')$, let $MP(\mu) = \{f \in C(G, X) : f \star \mu = 0\}$ and let $MP(G, X) = \bigcup_\mu MP(\mu)$. In this paper we analyse the following questions: Is $MP(G, X) \neq \emptyset$? Is $MP(G, X) \neq C(G, X)$? Is $MP(G, X)$ dense in $C(G, X)$? Is $MP(\mu)$ generated by 'exponential monomials' in it? We answer these questions for the groups $G = \mathbb{R}$, the real line, and $G = \mathbb{T}$, the circle group. Problems of spectral analysis and spectral synthesis for $C(\mathbb{R}, X)$ and $C(\mathbb{T}, X)$ are also analysed.

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## 1. Introduction

The notion of mean-periodic functions was introduced in 1935 by Delsarte [5]. It is well known that every solution of a constant coefficient homogeneous ordinary differential equation is a finite linear combination of solutions of the type $t^k e^{\iota\lambda t}$, where $\lambda \in \mathbb{C}$, and $k \in \mathbb{Z}_+$. Delsarte was interested in knowing whether this result is still true for convolution equation of the following type

$$(1) \qquad \int_{\mathbb{R}} f(s-t)k(t)\,dt = 0, \quad \forall\, s \in \mathbb{R},$$

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where $k$ is a continuous function which is zero out side some interval. For $\tau > 0$, periodic continuous functions of period $\tau$ are solutions of the convolution equation

$$\frac{1}{\tau}\int_{s-\tau/2}^{s+\tau/2} f(t)\,dt = 0, \quad \forall\, s \in \mathbb{R}. \tag{2}$$

For this reason Delsarte called the continuous functions which are solutions of equation (1) as *mean-periodic*. In [35], Schwartz observed that the mean-periodicity of a continuous function does not depend upon the function $k$, and he extended Delsarte's definition as follows:

DEFINITION 1.1. A continuous function $f : \mathbb{R} \to \mathbb{C}$ is said to be *mean-periodic* if there exists a non-trivial regular measure $\mu$ of compact support and finite variation such that $(f \star \mu)(s) = \int_{\mathbb{R}} f(s-t)\,d\mu(t) = 0, \forall\, s \in \mathbb{R}$.

Schwartz also gave an intrinsic characterization of mean-periodic functions. Let $C(\mathbb{R})$ denote the vector space of complex valued continuous functions on $\mathbb{R}$ with the topology of uniform convergence on compact sets (u.c.c.). Let $M_c(\mathbb{R})$ denote the space of all regular measures of compact support and finite variation on $\mathbb{R}$. For $f \in C(\mathbb{R})$, let $\tau(f)$ denote the closed translation invariant subspace of $C(\mathbb{R})$ generated by $f$. Schwartz in [35] showed that $f \in C(\mathbb{R})$ is mean-periodic if and only if $\tau(f) \neq C(\mathbb{R})$. Further, if $f \star \mu = 0$ for some non-zero $\mu \in M_c(\mathbb{R})$, then $f$ is a limit of finite linear combination of exponential monomials $t^k e^{\iota\lambda t}$ which satisfy $t^k e^{\iota\lambda t} \star \mu = 0$. More generally, convolution equation of the type

$$f \star \mu = g, \tag{3}$$

where $\mu \in M_c(\mathbb{R})$ and $g \in C(\mathbb{R})$ are given, can be analysed as in the case of ordinary differential equations. If $p$ is a particular solution of the equation (3), then every other solution is of the form $h + p$, where $h$ is a solution of the homogeneous equation $f \star \mu = 0$. In general, equation (3) need not have any solution in $C(\mathbb{R})$. For instance, let $\mu$ be such that $d\mu(t) = \phi(t)\,dt$, where $\phi \in C_c^\infty(\mathbb{R})$, space of all infinitely differentiable functions on $\mathbb{R}$, and $g$ is a nowhere differentiable continuous function on $\mathbb{R}$. Some particular cases of (3) were analysed in [31, 32]. In general, no necessary and sufficient conditions for the existence of solutions of equation (3) are known. A variant of the above problem is the following: Consider the following convolution equation

$$f_1 \star \mu_1 = -f_2 \star \mu_2, \tag{4}$$

where $\mu_1, \mu_2 \in M_c(\mathbb{R})$ are given. Equation (4) can be written as a convolution equation for vector valued functions: let $\underline{f} = (f_1, f_2) : \mathbb{R} \to \mathbb{C}^2$ and $\underline{\mu} = (\mu_1, \mu_2) : \mathscr{B}_{\mathbb{R}} \to \mathbb{C}^2$.

Then equation (4) is a homogeneous equation $\underline{f} \star \underline{\mu} = 0$. This leads to consideration of vector valued mean-periodic functions, the main content of this paper. We consider such equations in a more general setting and analyse their solutions.

Let $G$ be a locally compact abelian group. Let $X$ be a complex Banach space and $X'$ denote the weak*-dual of $X$. We denote by $\mathscr{B}_G$ the $\sigma$-algebra of Borel subsets of $G$. We recall some results on integration of functions $f : G \to X$ with respect to $X'$-valued measures on $\mathscr{B}_G$, denoted by $M(G, X')$. For details one may refer Schmets [34]. Let $\mu \in M(G, X')$ and for every $x$, let $\mu_x$ denote the scalar measure on $\mathscr{B}_G$ defined by $\mu_x(E) := \langle x, \mu(E)\rangle$ for every $E \in \mathscr{B}_G$. The measure $\mu$ is said to be *regular* if $\mu_x$ is regular for every $x \in X$. For $E \in \mathscr{B}_G$, if $E = \bigcup_{i=1}^n E_i$ for some $E_1, E_2, \ldots, E_n \in \mathscr{B}_G$ such that $E_i \bigcap E_j = \emptyset$ for $i \neq j$, we call $\{E_1, E_2, \ldots, E_n\}$ a *measurable partition* of $E$. Let $\mathscr{P}(E)$ denote the set of all measurable partitions of $E$. Let

$$V_\mu(E) := \sup\left\{\sum_{i=1}^n \|\mu(E_i)\| : \{E_1, E_2, \ldots, E_n\} \in \mathscr{P}(E)\right\}.$$

The scalar measure $V_\mu$ is called the *variation* of $\mu$. We say $\mu$ has finite variation if $V_\mu(E) < +\infty$ for every $E \in \mathscr{B}_G$. Let $M(G, X')$ denote the set of all regular Borel measures $\mu$ on $G$ such that $\mu$ has finite variation. For $\mu \in M(G, X')$ the smallest closed set $S$ with $\mu(E) = 0$ for every $E \in \mathscr{B}_G$ with $E \cap S = \emptyset$ is called the support of $\mu$. We write $S = \mathrm{supp}(\mu)$ if $S$ is the support of $\mu$. Let $M_c(G, X')$ denote the set of all $\mu \in M(G, X')$ such that support of $\mu$ is compact. Let $C(G, X)$ denote the space of all $X$-valued continuous functions on $G$ with the topology of uniform convergence on compact sets. Let $f \in C(G, X)$ and $\mu \in M_c(G, X')$ with $\mathrm{supp}(\mu) \subseteq K$, a compact set. Then there exists a sequence $\mathscr{P}_k(K) := \{B^k_{k_1}, B^k_{k_2}, \ldots, B^k_{k_n}\}$ of measurable partitions of $K$ with the following property : for arbitrary choice of $t_i \in B_{k_i}$, the sequence $\left\{\sum_{i=1}^n \langle f(t_i),\ \mu(B^k_{k_i})\rangle\right\}_{k\geq 1}$ is convergent and is independent of the choice of $t_i's$. This limit is called the *integral* of $f$ with respect to $\mu$ and is denoted by $\int f\, d\mu$. For $f \in C(G, X)$ and $\mu \in M_c(G, X')$ the scalar valued function

$$(f \star \mu)(g) := \int_G f(g - h)\, d\mu(h), \quad \forall\, g \in G$$

is called the *convolution* of $f$ with $\mu$, that is, $(f \star \mu)(g) = \mu(f_g) = \langle \mu,\ f_g\rangle$, where $f_g(h) = f(g + h)$ and $\langle \mu,\ f\rangle = \mu(f) = \int_G f(-g)\, d\mu(g)$ is the duality pairing of $M_c(G, X')$ with $C(G, X)$.

DEFINITION 1.2. We say $f \in C(G, X)$ is *mean-periodic* if there exists a non-trivial $\mu \in M_c(G, X')$ such that $(f \star \mu)(g) = \int_G f(g - h)\, d\mu(h) = 0, \forall\, g \in G$.

The aim of this paper is to answer the following questions: let $MP(G, X)$ denote the space of all $X$-valued mean-periodic functions on $G$.

- Is $MP(G, X) \neq \emptyset$? That is, when does there exist non-zero mean-periodic functions?
- Is $MP(G, X) \neq C(G, X)$? That is, do there exist continuous functions which are not mean-periodic?
- Is $MP(G, X)$ dense in $C(G, X)$? That is, how large is $MP(G, X)$ as a subspace of $C(G, X)$?

We answer these questions for the particular cases $G = \mathbb{R}$, in Section 2 and $G = \mathbb{T}$, circle group, in Section 3. Analysis of such questions for more general groups remain open.

The problem of analysing mean-periodic functions is also related to the problem of 'spectral analysis' and 'spectral synthesis'. In order to carry-out the analysis, we define next vector valued exponential monomials and exponential polynomials.

An *additive function* on a locally compact abelian group is a complex valued continuous function $a$ on $G$ such that $a(g_1 + g_2) = a(g_1) + a(g_2)$ for all $g_1$ and $g_2$ in $G$. A *polynomial* on $G$ is a function of the form $p(a_1, a_2, \dots, a_m)$, where $p$ is a polynomial in $m$ variables and $a_1, a_2, \dots, a_m$ are additive functions on $G$. A *monomial* on $G$ is a function of the form $p(a_1, a_2, \dots, a_m)$, where $p$ is a monomial in $m$ variables and $a_1, a_2, \dots, a_m$ are additive functions on $G$. An *exponential* on $G$ is a non-zero continuous complex valued function $\omega$ such that $\omega(g_1 + g_2) = \omega(g_1)\omega(g_2)$ for all $g_1$ and $g_2$ in $G$. An *exponential monomial* is a point-wise product of a monomial and an exponential. An *exponential polynomial* is a point-wise product of a polynomial and an exponential. The set of all exponentials is denoted by $\Omega$. Note that $\Omega \subset C(G)$.

We define exponential polynomials in $C(G, X)$ as follows:

DEFINITION 1.3. (i) We call $f \in C(G, X)$ an $X$-valued *exponential* if for every $g \in G$, $f(g) = \omega(g)x$ for some $\omega \in \Omega$ and $x \in X$.

(ii) We call $f \in C(G, X)$ an $X$-valued *exponential monomial* if for every $g \in G$, $f(g) = p(g)\omega(g)x$ for some $x \in X$, $p$ a monomial in $C(G)$ and $\omega$ an exponential in $C(G)$.

(iii) We call $f \in C(G, X)$ an $X$-valued *exponential polynomial* if for every $g \in G$, $f(g) = p(g)\omega(g)x$ for some $x \in X$, $p$ a polynomial in $C(G)$ and $\omega$ an exponential in $C(G)$.

EXAMPLE 1. (1) Let $f \in C(\mathbb{R}, X)$. Then $f$ is an exponential if and only if for every $t \in \mathbb{R}$, $f(t) = e^{\iota\lambda t}x$ for some $\lambda \in \mathbb{C}$ and $x \in X$. $f$ is an exponential monomial if and only if for every $t \in \mathbb{R}$, $f(t) = t^k e^{\iota\lambda t}x$ for some $\lambda \in \mathbb{C}$, $k \in \mathbb{N}$ and $x \in X$. Finally, $f$ is an exponential polynomial if and only if for every $t \in \mathbb{R}$, $f(t) = p(t)e^{\iota\lambda t}x$ for some $\lambda \in \mathbb{C}$, polynomial $p(t)$ and $x \in X$. Thus the exponentials, exponential monomials and exponential polynomials are the scalar multiples of the ones defined by Schwartz [35].

(2) A function $f \in C(\mathbb{T}, X)$ is an exponential if and only if for every $t \in \mathbb{R}$, $f(e^{\iota t}) = e^{\iota n t}x$ for some non-negative integer $n$ and $x \in X$.

REMARK. We shall use the following convention: When $X = \mathbb{C}$ we choose the $x \in X$ appearing in the exponential, exponential monomial and exponential polynomial to be the scalar constant 1. The generality is not lost due to this choice, since if a closed translation invariant subspace contains an exponential or exponential monomial or exponential polynomial if and only if it contains their scalar multiples.

DEFINITION 1.4. Let $V$ be a closed translation invariant subspace of $C(G, X)$. We say

(i) *spectral analysis holds for* $V$ if $V$ contains an exponential;

(ii) *spectral synthesis holds for* $V$ if the linear span of the set of all exponential monomials in $V$ is dense in $V$;

(iii) if spectral analysis (synthesis) holds for every closed translation invariant subspace $V$ of $C(G, X)$, then we say that *spectral analysis* (*synthesis*) *holds in* $C(G, X)$.

DEFINITION 1.5. Let $V$ be a closed translation invariant subspace of $C(G, X)$ and $f \in C(G, X)$ be mean-periodic. Let $\tau(f)$ denote the closed translation invariant subspace of $C(G, X)$ generated by $f$.

(i) The *spectrum* of $V$ is defined to be the set of all exponential monomials in $V$ and is denoted by $\mathrm{spec}(V)$ or $\sigma(V)$.

(ii) The *spectrum* of $f$ is defined to be $\mathrm{spec}(\tau(f))$ and is denoted by $\mathrm{spec}(f)$ or $\sigma(f)$.

Some of the known results for spectral analysis and spectral synthesis for $G = \mathbb{R}^n$ are as follows: Let $E(\mathbb{R}^n)$ be the space of all infinitely differentiable functions on $\mathbb{R}^n$ in the topology of compact convergence of functions and their derivatives. Then its dual $E(\mathbb{R}^n)'$ is the space of all compactly supported distributions on $\mathbb{R}^n$. Schwartz [35] proved the following theorem:

THEOREM 1.6 ([35]). *In $E(\mathbb{R})$, every closed translation invariant subspace is the closure of finite linear combinations of the exponential monomials in it.*

As a consequence of this theorem, the linear span of exponential monomials in every closed translation invariant subspace $V$ of $C(\mathbb{R})$ is dense in $V$. That is, spectral analysis and spectral synthesis hold in $C(\mathbb{R})$. Using this Schwartz [35] described mean-periodic functions on $\mathbb{R}$.

Let $V$ be the closed translation invariant subspace of $E(\mathbb{R}^n)$ generated by the solutions of the homogeneous constant coefficient partial differential equation $p(D)f = 0$. Malgrange [28] proved that spectral synthesis holds for $V$.

In 1975 Gurevich [17] proved that Theorem 1.6 cannot be extended for $\mathbb{R}^n$, $n > 1$. Though Theorem 1.6 fails for $\mathbb{R}^n$, $n > 1$, spectral analysis and spectral synthesis hold in $C(G)$ for certain groups, for example, for $G = \mathbb{Z}^n$ (see [26]) and for discrete abelian groups (see [12, 13]). Consider the following example from [15].

EXAMPLE 2 ([10, 15]). Define $f_1, f_2 : \mathbb{R}^2 \to \mathbb{C}$ by

$$f_1(x_1, x_2) := 1 \quad \text{and} \quad f_2(x_1, x_2) := x_1 + x_2, \quad \forall\, (x_1, x_2) \in \mathbb{R}^2.$$

Let $V$ be the closed translation invariant subspace of $C(\mathbb{R}^2)$ generated by $f_1$ and $f_2$. Then the spectrum of $V$ is $\{f_1\}$. But the closed linear span of the spectrum of $V$ is a proper subspace of $V$. Thus spectral synthesis fails in $C(\mathbb{R}^2)$ and spectral synthesis fails for $V$ even if $V$ is finite dimensional.

However, for certain closed translation invariant subspaces $V \subset C(\mathbb{R}^2)$ the linear span of all exponential polynomials in $V$ is dense in $V$. These subspaces are described in the following three theorems.

THEOREM 1.7 ([4]). *Let $V$ be a closed translation and rotation invariant subspace of $C(\mathbb{R}^2)$. Then the linear span of exponential polynomials in $V$ is dense in $V$.*

THEOREM 1.8 ([16]). *Let $\mu \in M_c(\mathbb{R}^n)$. Then the linear span of exponential polynomials in $\tau_\mu := \{f \in C(\mathbb{R}^n) : f \star \mu = 0\}$ is dense in $\tau_\mu$.*

THEOREM 1.9 ([14]). *Let $V$ be a finite dimensional translation invariant subspace of $C(\mathbb{R}^n)$. Then every element of $V$ is a finite linear combination of exponential polynomials.*

The following question is raised in [15] and the answer is not known: Let $V$ be closed translation invariant subspace of $C(\mathbb{R}^2)$.

- Does there exist an exponential in $V$?

In Section 4, we answer this question affirmatively when $V$ is either finite dimensional or rotation invariant or $V = \tau_\mu := \{f \in C(\mathbb{R}^2) : f \star \mu = 0\}$ for some $\mu \in M_c(\mathbb{R}^2)$.

Let $V$ be a closed translation invariant subspace of $C(G, X)$. Then the *problems of spectral analysis and synthesis* are the following:

- Is every exponential monomial in $C(G, X)$ mean-periodic?
- Are exponential monomials dense in $C(G, X)$?
- When does there exist an exponential monomial in $V$ ?
- When is the linear span of exponential monomials in $V$ dense $V$?
- Does there exist an exponential monomial solution for the convolution equation $f \star \mu = 0$ for a given $\mu \in M_c(G, X')$?

We analyse these problems for $G = \mathbb{R}$ in Section 2 and $G = \mathbb{T}$ in Section 3.

## 2. Mean-periodic functions on $G = \mathbb{R}$

For $G = \mathbb{R}$ and $X = \mathbb{C}$, it is known (see Schwartz [35]) that $f \in C(\mathbb{R}, \mathbb{C})$ is mean-periodic if and only if $\tau(f)$, the closed translation invariant subspace of $C(\mathbb{R}, \mathbb{C})$ is proper. We first extend this result to $X$, arbitrary Banach space.

THEOREM 2.1. *The following are equivalent*:

(i) $f$ *is mean-periodic*;
(ii) $\tau(f) \neq C(\mathbb{R}, X)$.

PROOF. We use the fact that $C(\mathbb{R}, X)$ is a locally convex space and its dual is $M_c(\mathbb{R}, X')$. To show that (i) implies (ii): let $\mu \in M_c(\mathbb{R}, X')$ be non-trivial such that $f \star \mu = 0$. Then $\mu(g) = 0$ for every $g \in \tau(f)$. Hence $\tau(f) \neq C(\mathbb{R}, X)$, for otherwise $\mu(g) = 0$ for every $g \in C(\mathbb{R}, X)$, which is not possible, since $\mu$ is non-trivial. The implication (ii) implies (i) follows from the Hahn-Banach theorem for locally convex spaces and the fact that $\tau(f)$ is a proper closed translation invariant subspace of $C(\mathbb{R}, X)$. $\square$

We show next that there exist nontrivial $X$-valued mean-periodic functions on $\mathbb{R}$.

PROPOSITION 2.2. $MP(\mathbb{R}, X) \neq \emptyset$.

PROOF. Let $0 \neq x \in X$ and $0 \neq x' \in X'$. Choose $g \in MP(\mathbb{R})$, scalar valued function mean-periodic with respect to some $\mu \in M_c(\mathbb{R})$. Define $\nu : \mathscr{B}_{\mathbb{R}} \to X'$ by $\nu(E) := \mu(E)x'$ and define $f : \mathbb{R} \to \mathbb{C}$ by $f(t) := g(t)x$. Then $\mu$ is a $X'$-valued measure and $f$ is a continuous $X$-valued function with $f \star \nu = (g \star \mu)\langle x, x'\rangle = 0$. Thus $f$ is mean-periodic with respect to $\nu$. $\square$

We prove next that existence of functions which are not mean-periodic is related to the $X$ being separable.

THEOREM 2.3. $MP(\mathbb{R}, X)$ *is a proper subset of* $C(\mathbb{R}, X)$ *if and only if* $X$ *is separable.*

PROOF. Suppose that $X$ is a non-separable complex Banach space and $f \in C(\mathbb{R}, X)$. Since $f$ continuous, $f(\mathbb{R})$ is separable and hence $\overline{[f(\mathbb{R})]}$ is separable. Since, for every $g \in \tau(f)$, $g(\mathbb{R}) \subseteq \overline{[f(\mathbb{R})]}$, $\tau(f) \neq C(\mathbb{R}, X)$. Hence $f$ is mean-periodic.

Conversely, suppose that $X$ is separable. We show that $MP(\mathbb{R}, X) \neq C(\mathbb{R}, X)$. For every $n \in \mathbb{N}$, let

$$f_n(t) := \sum_{j=1}^{\infty} a_{nj} e^{\iota\lambda_{nj} t}, \quad t \in \mathbb{R}, \tag{5}$$

where $\lambda_{nj}$ and $a_{nj}$ satisfy the following conditions:

(i) $0 \neq a_{nj} \in \mathbb{C}$.

(ii) $\lambda_{nj} \in [\alpha, \beta]$ for some $\alpha < \beta$.

(iii) $\{\lambda_{nj} : j \in \mathbb{N}\} \cap \{\lambda_{mj} : j \in \mathbb{N}\} = \emptyset$ for $m \neq n$ and for every $n$, $\{\lambda_{nj}\}_{j=1}^{\infty}$ has a limit $\lambda_n \in \mathbb{R}$.

(iv) The convergence in (5) is uniform on compact sets with each $f_n$ bounded by 1.

(v) $\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |a_{nj}| < \infty$.

Let $\{x_1, x_2, \dots\}$ be a dense subset of $X$. Define $f : \mathbb{R} \to X$ by

$$f(t) := \sum_{n=1}^{\infty} \frac{1}{2^n(1 + \|x_n\|)} f_n(t)x_n, \quad t \in \mathbb{R}. \tag{6}$$

We show that $f$ is not mean-periodic. Since $\{e^{\iota\lambda_{nj}t}\}_{n,j=1}^{\infty}$ is an equicontinuous family, $\{f_n\}_{n=1}^{\infty}$ is an equicontinuous family. Therefore, for $\mu \in M_c(\mathbb{R}, X')$,

$$f \star \mu = \sum_{n=1}^{\infty} \frac{1}{2^n(1 + \|x_n\|)} (f_n x_n) \star \mu = \sum_{n=1}^{\infty} \frac{1}{2^n(1 + \|x_n\|)} f_n \star \mu_{x_n}.$$

Thus $f \star \mu = 0$ if and only if

$$\sum_{n=1}^{\infty} \frac{1}{2^n(1 + \|x_n\|)} (f_n \star \mu_{x_n})(t) = 0, \quad \forall\, t \in \mathbb{R},$$

that is, for every $t \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} \frac{1}{2^n(1 + \|x_n\|)} \sum_{j=1}^{\infty} a_{nj}\hat{\mu}_{x_n}(\lambda_{nj})e^{\iota\lambda_{nj}t} = 0. \tag{7}$$

Let $S_{pq}(t) = \sum_{n=1}^{p} \sum_{j=1}^{q} e^{\iota\lambda_{nj}t} a_{nj}\hat{\mu}_{x_n}(\lambda_{nj})/2^n(1 + \|x_n\|)$. Notice that $S_{pq}$ is almost periodic and its Fourier coefficients $a(S_{pq}; \lambda)$ satisfy the following:

$$a(S_{pq}; \lambda) = \begin{cases} \dfrac{a_{nj}\hat{\mu}_{x_n}(\lambda_{nj})}{2^n(1 + \|x_n\|)} & \text{if } \lambda = \lambda_{nj},\ 1 \leq n \leq p,\ 1 \leq j \leq q; \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Since the convergence in (6) is uniform, the convergence in (7) also is uniform. Therefore $S_{pq}$ converges to 0 uniformly as $p, q \to \infty$. Further, the Fourier coefficients $a(S_{pq}; \lambda)$ converges to 0 as $p, q \to \infty$ ([27]). In view of (8), $a(S_{pq}; \lambda) = 0$ for every $\lambda$. Moreover, $\hat{\mu}_{x_n}(\lambda_{nj}) = 0$ for every $n$ and $j$. Since $\{\lambda_{nj}\}_{j=1}^{\infty}$ has limit point, this implies $\mu_{x_n} = 0$ for all $n$. Therefore, $\mu = 0$. Hence $f$ is not mean-periodic. $\square$

Let $f \in C(\mathbb{R}, X)$ and let $x' \in X'$. Then $x' \circ f \in C(\mathbb{R})$. It is natural to ask the following question: Is $x' \circ f$ mean-periodic for every $x' \neq 0$ if $f$ is mean-periodic? We analyse this in the following theorem.

THEOREM 2.4. *For $f \in C(\mathbb{R}, X)$ and $x', y' \in X'$ with $x' \neq y'$ the following hold*:

(i) *If $x' \circ f$ is mean-periodic, then $f$ is mean periodic.*
(ii) *If $x' \circ f = y' \circ f$, then $f$ is mean-periodic.*
(iii) *If $X = \mathbb{C}^n$, $n > 1$, then $f$ is a finite sum of mean-periodic functions.*
(iv) *There exists $f \in MP(\mathbb{R}, \mathbb{C}^n)$ such that $x' \circ f$ is not mean-periodic for any $x' \in X'$, $x' \neq 0$.*

PROOF. (i) By Theorem 2.1, it suffices to show that $\tau(f) \neq C(\mathbb{R})$. For this, let $g \in C(\mathbb{R})$, $g \neq 0$ be such that $g \notin \tau(x' \circ f)$. Choose $v \in X$ such that $\langle x', v\rangle \neq 0$ and define $h : \mathbb{R} \to X$ by $h(t) = g(t)v/\langle x', v\rangle$. Then $h$ is continuous and $(x' \circ h)(t) = g(t)$. We show that $h$ is not in $\tau(f)$. If possible let, $h \in \tau(f)$. Then there exists $\sum c_i f_{t_i} \to h$, which implies $x'(\sum c_i f_{t_i}) \to x' \circ h = g$, a contradiction.

(ii) Choose $g \in C(\mathbb{R}, X)$ such that $x'(g) \neq y'(g)$. We show that $g \notin \tau(f)$. If possible, let $g \in \tau(f)$. Since $\sum c_i f_{t_i} \to g \Rightarrow x'(\sum c_i f_{t_i}) \to x'(g)$ and $y'(\sum c_i f_{t_i}) \to y'(g)$, and also since $x'(f) = y'(f)$, $x'(\sum c_i f_{t_i}) = y'(\sum c_i f_{t_i})$. This implies $x'(g) = y'(g)$, a contradiction.

(iii) Let $f = (f_1, f_2, \ldots, f_n)$. Obviously $(0, \ldots, 0, f_i, 0, \ldots, 0)$ is mean-periodic for every $i$ with respect to $\mu = (\mu_1, \ldots, \mu_n)$ where $0 \neq \mu_j \in M_c(\mathbb{R})$ are arbitrary and for $j = i$, $\mu_j = 0$. Hence $f$ is a finite sum of mean-periodic functions.

(iv) Choose a non zero, compactly supported complex valued continuous function $g$. Let $f = (g, g, \ldots, g)$. Then $f$ is a $\mathbb{C}^n$-valued continuous function on $\mathbb{R}$. Clearly $f$ is mean-periodic with respect to $\mu = (\nu_1, -\nu_1, 0, \ldots, 0)$, where $0 \neq \nu_1 \in M_c(\mathbb{R})$ is arbitrary but $x' \circ f$ is not mean periodic for any $0 \neq x' \in X'$. $\square$

REMARK. When $X = \mathbb{C}$, $MP(\mathbb{R}, X)$ is a subspace of $C(\mathbb{R}, X)$. It follows from Theorem 2.4 (iii) that sum of mean-periodic functions in $C(\mathbb{R}, X)$ need not be mean-periodic and hence $MP(\mathbb{R}, X)$ in general need not be a vector subspace of $C(\mathbb{R}, X)$. Moreover, the same argument works for separable complex Hilbert spaces.

THEOREM 2.5. *$MP(\mathbb{R}, X)$ is dense in $C(\mathbb{R}, X)$.*

PROOF. Case (i): $X = \mathbb{C}$. It suffices to show that the annihilator of $MP(\mathbb{R})$ is $\{0\}$. Let $\mu \in M_c(\mathbb{R})$ be such that $\mu(MP(\mathbb{R})) = \{0\}$. In particular $\mu(e^{\iota\lambda t}) = \hat{\mu}(\lambda) = 0$ for every $\lambda \in \mathbb{C}$. Hence $\mu = 0$.

Case (ii): Let $X$ be finite dimensional, $X = \mathbb{C}^n$. Consider $C(\mathbb{R}) \times C(\mathbb{R}) \times \cdots \times C(\mathbb{R})$. This is a finite product of locally convex spaces. Hence it is a locally convex space in the product topology. It is easy to see that $C(\mathbb{R}, X)$ is isomorphic to $C(\mathbb{R}) \times \cdots \times C(\mathbb{R})$ as locally convex spaces. Also $MP(\mathbb{R}) \times MP(\mathbb{R}) \times \cdots \times MP(\mathbb{R}) \subseteq MP(\mathbb{R}, X)$ and $MP(\mathbb{R})$ is dense in $C(\mathbb{R})$. Thus it follows that $MP(\mathbb{R}, X)$ is dense in $C(\mathbb{R}, X)$.

Case (iii): $X$ is not finite dimensional. Consider the set $\mathrm{Exp}(\mathbb{R}, X) = \{e^{\iota\lambda t}x : \lambda \in \mathbb{C}, x \in X\}$. We show that the linear span of $\mathrm{Exp}(\mathbb{R}, X)$ is contained in $MP(\mathbb{R}, X)$ and it is dense in $C(\mathbb{R}, X)$. Let $f(t) = e^{\iota\lambda_1 t}x_1, g(t) = e^{\iota\lambda_2 t}x_2 \in \mathrm{Exp}(\mathbb{R}, X)$ and $\alpha, \beta \in \mathbb{C}$. Choose $0 \neq x' \in X'$ such that $x'(x_1) = x'(x_2) = 0$ and $\mu_1, \mu_2 \in M_c(\mathbb{R})$ such that $e^{\iota\lambda_1 t} \star \mu_1 = 0 = e^{\iota\lambda_2 t} \star \mu_2$. Define $\mu(E) = (\mu_1 \star \mu_2)(E)x'$, for every $E \in \mathscr{B}_{\mathbb{R}}$. Then $(\alpha f + \beta g) \star \mu = 0$. To prove the denseness, let $\mu \in M_c(\mathbb{R}, X')$ be such that $\mu$ annihilates the linear span of $\mathrm{Exp}(\mathbb{R}, X)$. Then $\hat{\mu}_x(\lambda) = 0, \forall\, \lambda \in \mathbb{C}, \forall\, x \in X$. It follows that $\mu = 0$. This completes the proof. □

We analyse next the problem of spectral analysis and spectral synthesis in $C(\mathbb{R}, X)$. Let $V$ be a closed translation invariant subspace of $C(\mathbb{R}, X)$. For $X = \mathbb{C}$, Schwartz [35] proved that $V$ contains exponential monomials and the linear span of exponential monomials in $V$ is dense in $V$. It is well known [17] that spectral synthesis fails for $\mathbb{R}^n, n > 1$. Further, it holds for certain locally compact abelian groups, namely for $\mathbb{Z}^n$ due to Lefranc [26] and discrete groups due to Gilbert [16, 15] and Elliott [12, 13]. However, nothing is known for vector valued functions. In this section, we extend Schwartz's result for finite dimensional closed translation invariant subspace of $C(\mathbb{R}, X)$, $X$ an arbitrary Banach space. For this we need the following lemmas.

LEMMA 2.6. *Let $v^1, v^2, \ldots, v^n \in X^n$, $v^i = (v_1^i, v_2^i, \ldots, v_n^i)$, be linearly independent. Then there exist $x_1', x_2', \ldots, x_n' \in X'$ which satisfy*

$$\begin{aligned} x_1'(v_1^1) + x_2'(v_2^1) + \cdots + x_n'(v_n^1) &= 1, \\ x_1'(v_1^2) + x_2'(v_2^2) + \cdots + x_n'(v_n^2) &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\ x_1'(v_1^n) + x_2'(v_2^n) + \cdots + x_n'(v_n^n) &= 0. \end{aligned}$$

PROOF. Let $Y$ be the linear span of $\{v^2, v^3, \ldots, v^n\}$. Then $Y$ being a finite dimensional subspace of $X^n$ is closed. Since $v^1, v^2, \ldots, v^n$ are linearly independent, $v^1 \notin Y$. Thus by Hahn-Banach theorem, there exists $\Lambda \in (X^n)'$ such that $\Lambda(Y) = \{0\}$ and $\Lambda(v^1) = 1$. Clearly $\Lambda$ can be written as $\Lambda = (x_1', x_2', \ldots, x_n')$, where $x_i' \in X'$ satisfy $\Lambda(x_1, x_2, \ldots, x_n) = x_1'(x_1) + x_2'(x_2) + \cdots + x_n'(x_n)$. Therefore,

$$\begin{aligned} x_1'(v_1^1) + x_2'(v_2^1) + \cdots + x_n'(v_n^1) &= \Lambda(v_1^1, 0, \ldots, 0) + \cdots + \Lambda(0, \ldots, 0, v_n^1) \\ &= \Lambda((v_1^1, v_2^1, \ldots, v_n^1)) = 1. \end{aligned}$$

For every $i$, $2 \leq i \leq n$,

$$\begin{aligned} x_1'(v_1^i) + x_2'(v_2^i) + \cdots + x_n'(v_n^i) &= \Lambda(v_1^i, 0, \ldots, 0) + \cdots + \Lambda(0, \ldots, 0, v_n^i) \\ &= \Lambda((v_1^i, v_2^i, \ldots, v_n^i)) = 0. \end{aligned}$$

This completes the proof of the lemma. □

For sets $A$ and $B$, let $\mathscr{F}(A, B)$ denote the set of all functions from $A$ to $B$. For a set $E \subseteq V$, a vector space, let $LS(E)$ denote the linear span of $E$.

LEMMA 2.7. *Let $S$ be any set containing at-least $n$ points and $V$ be a vector space over $\mathbb{C}$. Let $\{f_1, f_2, \ldots, f_n\} \subset \mathscr{F}(S, V)$. Then $\{f_1, f_2, \ldots, f_n\}$ is linearly independent in $\mathscr{F}(S, V)$ if and only if there exists $n$ distinct points $t_1, t_2, \ldots, t_n \in S$ such that $\{f_1, f_2, \ldots, f_n\}$ is linearly independent in $\mathscr{F}(\{t_1, t_2, \ldots, t_n\}, V)$.*

PROOF. We prove the straight implication by induction. Suppose that $\{f_1, f_2, \ldots, f_n\}$ is a linearly independent set in $\mathscr{F}(S, V)$. As $\{f_1\}$ is linearly independent, there exists $t_1 \in S$ such that $f_1(t_1) \neq 0$. Then $\{f_1\}$ is linearly independent on $\{t_1\}$. Thus the lemma is true when $n = 1$. If $f_1(t_1) = \alpha f_2(t_1)$, for some nonzero $\alpha \in \mathbb{C}$, choose $t_2 \in S$ such that $f_1(t_2) \neq \alpha f_2(t_2)$, which is possible, since $f_1, f_2, \ldots, f_n$ are linearly independent on $S$. Then it is easy to check that $\{f_1, f_2\}$ is linearly independent on $\{t_1, t_2\}$. If $f_1(t_1) \neq \alpha f_2(t_1)$ for any non zero scalar and $f_2(t_1) \neq 0$, then choose any $t_2 \neq t_1$. It is easy to see that $\{f_1, f_2\}$ is linearly independent on $\{t_1, t_2\}$. If $f_2(t_1) = 0$, then choose $t_2$ such that $f_2(t_2) \neq 0$. In this case also one can easily verify that $\{f_1, f_2\}$ is linearly independent on $\{t_1, t_2\}$. Assume that $\{f_1, f_2, \ldots, f_{n-1}\}$ is linearly independent on $\{t_1, t_2, \ldots, t_{n-1}\}$. If $\{f_1, f_2, \ldots, f_{n-1}, f_n\}$ is linearly independent on $\{t_1, t_2, \ldots, t_{n-1}\}$ then choose any $t_n$ which is different from $t_1, t_2, \ldots, t_{n-1}$. If $\{f_1, f_2, \ldots, f_{n-1}, f_n\}$ is linearly dependent, then there exist unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ such that $\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_{n-1} f_{n-1} = f_n$ on $\{t_1, t_2, \ldots, t_{n-1}\}$. Since $\{f_1, f_2, \ldots, f_n\}$ is linearly independent on $S$, there exists $t_n \in S$ such that $\alpha_1 f_1(t_n) + \alpha_2 f_2(t_n) + \cdots + \alpha_{n-1} f_{n-1}(t_n) \neq f_n(t_n)$. It follows from this that $\{f_1, f_2, \ldots, f_n\}$ is linearly independent on $\{t_1, t_2, \ldots, t_n\}$. This proves the required claim.

The converse is trivial. $\square$

Using these lemmas we prove that every finite dimensional translation invariant subspace $V$ of $C(\mathbb{R}, X)$ includes an exponential and every element in $V$ is a finite sum of exponential monomials.

THEOREM 2.8. *Let $V$ be an $n$-dimensional translation invariant subspace of $C(\mathbb{R}, X)$. Then the following hold:*

(i) *There exist $\lambda_1, \lambda_2, \ldots, \lambda_q \in \mathbb{C}$ and $m_1, m_2, \ldots, m_q \in \mathbb{N}$ with $m_1 + m_2 + \cdots + m_q = n$, and $w_1, w_2, \ldots, w_q \in X$, not all zero, such that $e^{\iota\lambda_j t} w_j \in V$, for $1 \leq j \leq q$.*

(ii) *There exist $\lambda_1, \lambda_2, \ldots, \lambda_q \in \mathbb{C}$, $m_1, m_2, \ldots, m_q \in \mathbb{N}$ with $m_1 + m_2 + \cdots + m_q = n$ and $x_1, x_2, \ldots, x_n \in X$ such that every $f \in V$ is of the form $f = \sum_{l=1}^{n} g_l x_l$, where each $g_l \in LS\{t^k e^{\iota\lambda_j t} : 0 \leq k \leq m_j - 1,\ 1 \leq j \leq q\}$.*

(iii) *There exist $\lambda_1, \lambda_2, \ldots, \lambda_q \in \mathbb{C}$ and $m_1, m_2, \ldots, m_q \in \mathbb{N}$ with $m_1 + m_2 + \cdots + m_q = n$ such that every $f \in V$ is of the form $f = \sum_{j=1}^{q} \sum_{k=0}^{m_j - 1} \alpha_{kj} t^k e^{\iota\lambda_j t} y_{kj}$, where $\alpha_{kj} \in \mathbb{C}$ and $y_{kj} \in X$ for $0 \leq k \leq m_j - 1,\ 1 \leq j \leq q$.*

PROOF. Fix a basis $\{f_1, f_2, \dots, f_n\}$ of $V$. Since $V$ is translation invariant, $(f_i)_s \in V$ for every $s \in \mathbb{R}$. Therefore there exist unique scalars $\alpha_{ij} \in \mathbb{C}$ such that $(f_i)_s = \sum_{j=1}^{n} \alpha_{ij}(s) f_j$. Let $f$ denote the $n \times 1$ matrix $f = [f_1, f_2, \dots, f_n]^t$ and $A(s)$ denote the $n \times n$ matrix $(\alpha_{ij}(s))$. Then

$$f_s = A(s)[f_1, f_2, \dots, f_n]^t = A(s) f. \tag{9}$$

Now

$$(f_s - f)/s = ((A(s) - A(0))/s)\, f. \tag{10}$$

CLAIM. $s \mapsto A(s)$ is continuous. We give two proofs of this claim.

PROOF 1. By the Lemma 2.7 there exist $n$ distinct points $\{t_1, t_2, \dots, t_n\} \subset \mathbb{R}$ such that $\{f_1, f_2, \dots, f_n\}$ is linearly independent on $\{t_1, t_2, \dots, t_n\}$. In view of (9), $f(s+t_j) = A(s) f(t_j)$ for $j = 1, 2, \dots, n$. That is $(f_i(s+t_j))_{i,j=1}^n = A(s)(f_i(t_j))_{i,j=1}^n$. Let $v^i = (f_i(t_1), f_i(t_2), \dots, f_i(t_n))$, $1 \le i \le n$. Then $\{v^1, v^2, \dots, v^n\}$ is a linearly independent subset of $X^n$. By the Lemma 2.6 there exists $x'_{ij} \in X'$ such that

$$\sum_{k=1}^{n} \langle f_i(t_k), x'_{kj} \rangle = \delta_{ij}, \quad \text{where } \delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus we have

$$(f_i(s+t_j))_{i,j=1}^n (x'_{ij})_{i,j=1}^n = A(s)(f_i(t_j))_{i,j=1}^n (x'_{ij})_{i,j=1}^n = A(s)(\delta_{ij})_{i,j=1}^n = A(s).$$

The entries of the matrix obtained by multiplying the matrices on the left side of the above equation are continuous. This shows that $s \mapsto A(s)$ is continuous from $\mathbb{R}$ to $BL(\mathbb{C}^n)$.

PROOF 2. For every $t \in \mathbb{R}$, define an operator $T_t : V \to V$ by

$$(T_t f)(s) := f(t+s), \quad \forall\, f \in V,\ s \in \mathbb{R}.$$

Then $T_t \in BL(V)$ and satisfies the following properties: For every $s, t \in \mathbb{R}$

(i) $T_s \circ T_t = T_{s+t}$;
(ii) $T_0 = I$;
(iii) $T_s \circ T_t = T_t \circ T_s$.

Let $\{t_1, t_2, \dots, t_n\}$ be as given by Lemma 2.7. Let $\{K_n\}_{n\ge1}$ be compact subsets of $\mathbb{R}$ such that $\bigcup_{m=1}^{\infty} K_m = \mathbb{R}$ with $\{t_1, t_2, \dots, t_n\} \subseteq K_1 \subseteq K_2 \subseteq \cdots$. To show the required claim we have to show that $t \mapsto T_t$ is continuous in $BL(V)$. We shall show first that $t \mapsto T_t$ is continuous point-wise. Let $s_n \to s$ as $n \to \infty$. Now $T_{s_n}(f) = f_{s_n}$ and $T_s(f) = f_s$, for every $f \in V$. Since $f$ is uniformly continuous on compact sets, $f_{s_n} \to f_s$ in $C(\mathbb{R}, X)$. Therefore $T_{s_n} \to T_s$ point-wise. To show that $T_{s_n} \to T_s$ in $BL(V)$, it is sufficient to show that for every $m$, $\|T_{s_n} - T_s\|_{K_m} \to 0$ as $n \to \infty$, where

$\|T_{s_n} - T_s\|_{K_m} = \sup_{\|f\|_{K_m} \le 1} \|T_{s_n}(f) - T_s(f)\|_{K_m}$. Let $\epsilon > 0$. Since $\{f_1, f_2, \ldots, f_n\}$ is a basis of $V$, for every $f \in V$, there exist unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ such that $f = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$. Also since $\{f_1, f_2, \ldots, f_n\}$ is linearly independent on $K_m$, $\{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n : \|\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n\|_{K_m} \le 1\}$ is bounded in $\mathbb{C}^n$, that is, there exists $M > 0$ such that $\|\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n\|_{K_m} \le 1$ implies that $\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\| \le M$. Since $\{f_1, f_2, \ldots, f_n\}$ is equicontinuous, there exists a $\delta > 0$, with $\delta < 1$, such that whenever $t_1, t_2 \in s + K_m + [0, 1]$ with $|t_1 - t_2| < \delta$, $\|f_j(t_1) - f_j(t_2)\| < \epsilon/M$, for every $j = 1, 2, \ldots, n$. Choose $N \in \mathbb{N}$ such that $|s_n - s| < \delta$, whenever $n \ge N$. Then for every $f \in V$ with $\|f\|_{K_m} \le 1$, for every $t \in K_m$, and $n \ge N$, we have

$$\begin{aligned} \|f_{s_n}(t) - f_s(t)\| &= \|f(s_n + t) - f(s + t)\| \\ &= \|(\alpha_1 f_1 + \cdots + \alpha_n f_n)(s_n + t) - (\alpha_1 f_1 + \cdots + \alpha_n f_n)(s + t)\| \\ &\le |\alpha_1| \|f_1(s_n + t) - f_1(s + t)\| + \cdots + |\alpha_n| \|f_n(s_n + t) - f_n(s + t)\| \\ &\le \epsilon. \end{aligned}$$

Thus $\|T_{s_n} - T_s\|_{K_m} \to 0$ as $n \to \infty$ for every $m$ and hence $T_{s_n} \to T_s$ in $BL(V)$ as $n \to \infty$. This completes the second proof of the claim.

Thus $A(s)$ satisfies the following properties:

(i) $s \mapsto A(s)$ is continuous.
(ii) $A(0) = I$.
(iii) $A(s + t) = A(s)A(t) = A(t)A(s)$.

Therefore, $s \mapsto A(s)$ is differentiable (refer [18]) and

$$A(s) = e^{sA'(0)}. \tag{11}$$

By virtue of equations (10) and (11),

$$f' = A'(0) f. \tag{12}$$

This equation can be solved ([21]) and the solution is given by

$$f(t) = e^{tA'(0)}[x_1, x_2, \ldots, x_n]^t.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_q \in \mathbb{C}$ be the eigen values of $A'(0)$ with multiplicities $m_1, m_2, \ldots, m_q$, respectively. Let the Jordan canonical form of $A'(0)$ be given by

$$BA'(0)B^{-1} = \begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_q} \end{bmatrix},$$

where $J_1, \ldots, J_q$ are the Jordan blocks of $A'(0)$, $B$ is an invertible matrix. This gives

$$e^{tA'(0)} = B^{-1} \begin{bmatrix} \boxed{B_1} & & & \\ & \boxed{B_2} & & \\ & & \ddots & \\ & & & \boxed{B_q} \end{bmatrix} B,$$

where each $B_k$ is an $m_k \times m_k$ matrix given by

$$B_k = \begin{bmatrix} e^{\iota\lambda_k t} & te^{\iota\lambda_k t} & \cdots & e^{\iota\lambda_k t} t^{m_k-1}/(m_k-1)! \\ 0 & e^{\iota\lambda_k t} & \cdots & e^{\iota\lambda_k t} t^{m_k-2}/(m_k-2)! \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\iota\lambda_k t} \end{bmatrix}.$$

Thus $f(t) = C[x_1, x_2, \ldots, x_n]^t$, where $C = (c_{ij})$ and each $c_{ij} \in LS\{t^k e^{\iota\lambda_j t} : 0 \leq k \leq m_j - 1, 1 \leq j \leq q\}$, that is, for every $i$, $f_i(t) = \sum_{j=1}^n g_{ij}(t)x_j$, where $g_{ij} \in LS\{t^k e^{\iota\lambda_j t} : 0 \leq k \leq m_j - 1, 1 \leq j \leq q\}$. Hence every element $h$ of $V$ is of the form $h(t) = \sum_{j=1}^n g_j(t)x_j$, where each $g_j \in LS\{t^k e^{\iota\lambda_j t} : 0 \leq k \leq m_j - 1, 1 \leq j \leq q\}$. This proves (ii).

(iii) By the discussion above, each $f_i$ can be expressed as follows:

$$f_i = \sum_{j=1}^{q} \sum_{k=0}^{m_j-1} t^k e^{\iota\lambda_j t} \beta_{kj}^i x_{kj}^i.$$

Every $h \in V$ is of the form

$$\begin{aligned} h = \sum_{i=1}^{n} \alpha_i f_i &= \sum_{i=1}^{n} \sum_{j=1}^{q} \sum_{k=0}^{m_j-1} t^k e^{\iota\lambda_j t} \alpha_i \beta_{kj}^i x_{kj}^i \\ &= \sum_{j=1}^{q} \sum_{k=0}^{m_j-1} t^k e^{\iota\lambda_j t} \left( \sum_{i=1}^{n} \alpha_i \beta_{kj}^i x_{kj}^i \right) = \sum_{j=1}^{q} \sum_{k=0}^{m_j-1} t^k e^{\iota\lambda_j t} y_{kj}. \end{aligned}$$

This proves (iii). For (i), $f_i = \sum_{j=1}^q \sum_{k=0}^{m_j-1} t^k e^{\iota\lambda_j t} y_{kj}^i$. For every $j$ choose largest $k$ such that $y_{kj}^i \neq 0$, let it be $k_j$. We will show that $e^{\iota\lambda_j t} y_{k_j j}^i \in V$. To prove this, let $\mu \in M_c(\mathbb{R}, X')$ be such that $\mu(V) = \{0\}$. Then $f \star \mu = 0$ for every $f \in V$, since $V$ is translation invariant. Hence $f_i \star \mu = 0$, for every $i$. As $f_i \star \mu$ is a finite sum of complex valued exponential monomials and $\hat{\mu}_{y_{k_j j}^i}(\lambda_j)$ is the coefficient of $e^{\iota\lambda_j t}$, $\hat{\mu}_{y_{k_j j}^i}(\lambda_j) = 0$. This implies that $e^{\iota\lambda t} y_{k_j j}^i \in V$. $\square$

COROLLARY 2.9. *Let $f \in C(\mathbb{R}, X)$. Then $\tau(f)$ is finite dimensional if and only if $f$ is a finite linear combination of exponential monomials in $C(\mathbb{R}, X)$.*

PROOF. Suppose that $\tau(f)$ is finite dimensional. Then it follows from the above theorem that $f$ is a finite linear combination of exponential monomials. Conversely, suppose $f$ is a finite linear combination of exponential monomials. Let $f = \sum_{j=1}^{q} \sum_{k=0}^{m_j-1} \alpha_{jk} t^k e^{\iota\lambda_j t} x_{jk}$. Then $\tau(f) \subseteq LS\{t^{k-l} e^{\iota\lambda_j t} x_{jk} : 0 \leq l \leq k, 0 \leq k \leq m_j - 1, 1 \leq j \leq q\}$. Therefore $\tau(f)$ is finite dimensional. $\square$

REMARK. (i) Some authors (see [14, 25]) define exponential polynomials to be functions of the form $\sum_{j=1}^{m} f_j$, where $f_j$ are exponential polynomials defined as in Definition 1.3. With this definition, our result states that every finite dimensional translation invariant subspace $V$ of $C(\mathbb{R}, X)$ is generated by exponential polynomials in $V$.

(ii) Anselone and Korevaar [1] have proved that when $X = \mathbb{C}$, $V \subset C(\mathbb{R})$ is finite dimensional if and only if $V$ is the solution space of a homogeneous constant coefficient ordinary differential equation. This result is not true for arbitrary $X$ which can be seen by the following examples.

EXAMPLE 3. Let $X$ be a separable infinite dimensional complex Hilbert space. Let $\{e_n\}$ be a complete orthonormal basis. Consider the homogeneous ordinary differential equation with constant coefficient.

$$a_0 f + a_1 f' + \cdots + a_n f^{(n)} = 0. \tag{13}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_q$ with multiplicities $m_1, m_2, \ldots, m_q$ be the roots of the characteristic polynomial $p(t)$. Then for every $n \in \mathbb{N}, 0 \leq k \leq m_j, 1 \leq j \leq q, t^k e^{\iota\lambda_j t} e_n$ is a solution of the differential equation (13). Thus the solution space is not finite dimensional.

EXAMPLE 4. Let $X$ be a complex Banach space. Fix $A \in BL(X)$. Consider the following differential equation $du/dt = Au$. Then the solution space $\{u \in C(\mathbb{R}, X) : du/dt = Au\} = \{e^{tA}x : x \in X\}$ is a closed translation invariant subspace of $C(\mathbb{R}, X)$. Further, it is finite dimensional if and only if $X$ is finite dimensional.

Let $\mu \in M_c(\mathbb{R}, X')$. In the case when $X = \mathbb{C}$ it is known [35] that for a given $\mu$ the linear span of exponential monomial solutions of the convolution equation $f \star \mu = 0$ is dense in the space of all solutions. We extend this for $X = \mathbb{C}^n$ as follows:

THEOREM 2.10. *Let $f = (f_1, f_2, \ldots, f_n) \in C(\mathbb{R}, \mathbb{C}^n)$ satisfies the following*:

(i) *$f_j$ is mean-periodic, for every $1 \leq j \leq n$;*

(ii) *$\sigma(f_j) \cap \sigma(f_k) = \emptyset$ for $j \neq k$.*

*Then $\tau(f)$ contains exponential monomials and the linear span of exponential monomials in $\tau(f)$ is dense in $\tau(f)$.*

PROOF. Clearly $\tau(f) \subseteq \tau(f_1) \times \tau(f_2) \times \cdots \times \tau(f_n)$. We show that these two sets are equal. Let $g \in \tau(f_1) \cap \tau(f_2)$. Then $\tau(g) \subseteq \tau(f_1) \cap \tau(f_2)$ and hence by Schwartz's theorem, $\tau(f_1) \cap \tau(f_2) = (0)$. Thus $\tau(f_i) \cap \tau(f_j) = (0)$ for $i \neq j$. Let $\mu \in M_c(\mathbb{R}, X')$ be such that $\mu(\tau(f)) = \{0\}$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$. Since $\tau(f)$ is translation invariant, $f \star \mu = \sum_{j=1}^{n} f_j \star \mu_j = 0$. Let $e^{\iota\lambda t}, te^{\iota\lambda t}, \ldots, t^{m_1-1}e^{\iota\lambda t} \in \tau(f_1)$ and $t^{m_1}e^{\iota\lambda t} \notin \tau(f_1)$. By Hahn-Banach theorem there exists a measure $\nu_1 \in M_c(\mathbb{R})$ such that $\nu_1(\tau(f_l)) = \{0\}$ for every $l \neq 1$ and $\nu_1(e^{\iota\lambda t}) \neq 0$. Therefore $f_l \star \nu_1 = 0$, for $l \neq 1$. Now $f_1 \star \mu_1 \star \nu_1 = (f \star \mu) \star \mu_1 = 0$. Therefore $(\hat{\mu}_1\hat{\nu}_1)(\lambda) = (\hat{\mu}_1\hat{\nu}_1)'(\lambda) = \cdots = (\hat{\mu}_1\hat{\nu}_1)^{(m_1-1)}(\lambda) = 0$. As $\hat{\mu}_1(\lambda)\hat{\nu}_1(\lambda) = 0$ and $\hat{\nu}_1(\lambda) \neq 0$, $\hat{\mu}_1(\lambda) = 0$. Also $(\hat{\mu}_1\hat{\nu}_1)'(\lambda) = 0$ implies $\hat{\mu}_1'(\lambda)(\hat{\nu}_1)(\lambda) + \hat{\mu}_1(\lambda)\hat{\nu}_1'(\lambda) = 0$. This implies $\hat{\mu}_1'(\lambda) = 0$. Similarly we can show that $\hat{\mu}_1''(\lambda) = \cdots = \hat{\mu}_1^{(m_1-1)}(\lambda) = 0$. Thus $\lambda$ is a zero of $\hat{\mu}_1$ with multiplicity at-least $m_1$. This shows that $f_1 \star \mu_1 = 0$. Similarly, $f_j \star \mu_j = 0$ for every $j$. Thus $\mu(\tau(f_1) \times \tau(f_2) \times \cdots \times \tau(f_n)) = 0$. It follows that $\tau(f) = \tau(f_1) \times \tau(f_2) \times \cdots \times \tau(f_n)$. This completes the proof. □

COROLLARY 2.11. *Let $X = \mathbb{C}^n$. Let $f = (f_1, f_2, \ldots, f_n) \in C(\mathbb{R}, X)$ and $\mu \in M_c(\mathbb{R}, X')$. Suppose that each $f_j$ is mean-periodic and $\sigma(f_j) \cap \sigma(f_k) = \emptyset$ for $j \neq k$. If $f \star \mu = 0$, then $f$ is a finite linear combination of exponential monomials solutions.*

PROOF. Since spectral synthesis holds for $\mathbb{R}$, $LS(\sigma(f_j))$ is dense in $\tau(f_j)$, for every $j$. It is easy to see that $\sigma(f_1) \times \sigma(f_2) \times \cdots \times \sigma(f_n) \subset LS(E)$, where $E = \{t^k e^{\iota\lambda t}x : x \neq 0, t^k e^{\iota\lambda t}x \star \mu = 0\}$. Thus $\overline{LS(E)} = \tau(f_1) \times \tau(f_2) \times \cdots \times \tau(f_n)$. The required result follows from the Theorem 2.10. □

EXAMPLE 5. (1) When $G = \mathbb{R}$ and $X = \mathbb{C}$, the notion of mean-periodic functions was introduced by Delsarte in 1935 [5]. In [35] Schwartz gave an intrinsic characterization of mean-periodic functions: $f \in C(\mathbb{R}, \mathbb{C})$ is mean-periodic if and only if $\tau(f)$, the closed translation invariant subspace of $C(\mathbb{R}, \mathbb{C})$ is proper. Clearly, for every $\lambda \in \mathbb{C}$, $f_\lambda(t) = e^{\iota\lambda t}$, $t \in \mathbb{R}$, is mean-periodic, $f \star \mu = 0$ for $\mu = \delta_0 - e^{\iota\lambda}\delta_1$, where $\delta_x$ denote the Dirac measure on $\mathbb{R}$ at $x \in \mathbb{R}$. Schwartz [35] showed that if $f \in C(\mathbb{R}, \mathbb{C})$ is mean-periodic with $f \star \mu = 0$, then $f$ is a limit of finite linear combinations of functions of the type $f_\lambda(t) = t^k e^{\iota\lambda t}$, such that $f_\lambda \star \mu = 0$. In Laird [22] it is shown that if $f \in C(\mathbb{R}, \mathbb{C})$ is mean-periodic and $g$ is an exponential polynomial, that is, $g(t) = p(t)e^{\iota\lambda t}$, where $p(t)$ is a polynomial, then $fg$ is mean-periodic.

(2) Let $G$ be a compact abelian group. Then every character of $G$ is mean-periodic, as observed in Rana [33].

(3) For $X = \mathbb{C}$, mean-periodic functions on various locally compact groups have been analysed by various authors (see [2, 3, 5, 7, 10, 11, 17, 19, 20, 23, 24, 22, 29, 30, 36, 38, 37, 39]).

In general setting, even when $G = \mathbb{R}$ and $X$ is an arbitrary Banach space, nothing seem to be known.

NOTE. The following questions still remain unanswered:

(1) Let $V$ be a closed translation invariant subspace of $C(\mathbb{R}, X)$. Does $V$ always include a monomial exponential? Is $V$ the closed linear span of the monomial exponentials in it?

(2) The problem of finding solutions for $f \star \mu = g$, for a given $\mu$ and $g$, seems to be much more difficult even for the case $G = \mathbb{R}$ and $X = \mathbb{C}$: Some particular situations are analysed in [31] and [32]. Another particular case is given in the next theorem.

THEOREM 2.12. *For a given $\mu \in M_c(\mathbb{R})$ and $g$ a finite sum of exponential polynomials in $C(\mathbb{R})$, there exists $f \in C(\mathbb{R})$ such that $f \star \mu = g$.*

PROOF. First suppose that $g$ is an exponential polynomial. Let $g(t) = e^{\iota\lambda t} \sum_{k=0}^{n} a_k t^k$. Let $Z(\hat{\mu}) = \{\lambda \in \mathbb{C} : \hat{\mu}(\lambda) = 0\}$. We say

(i) $\lambda \in Z(\hat{\mu})$ is of multiplicity 0 if $\hat{\mu}(\lambda) \neq 0$.

(ii) $\lambda \in Z(\hat{\mu})$ of multiplicity $m \in \mathbb{N}$, if $\hat{\mu}(\lambda) = 0$, $\hat{\mu}'(\lambda) = 0, \ldots, \hat{\mu}^{(m-1)}(\lambda) = 0$ and $\hat{\mu}^{(m)}(\lambda) \neq 0$.

Let $m$ be the multiplicity of $\lambda \in Z(\hat{\mu})$. Define $f(t) := \sum_{k=0}^{n} b_k t^{m+k} e^{\iota\lambda t}$, where

$$b_n = \frac{(\iota)^m}{\binom{n+m}{m}\hat{\mu}^{(m)}(\lambda)} a_n, \quad b_{n-1} = \left[ a_{n-1} - b_n \frac{\binom{m+n}{m+1}\hat{\mu}^{(m+1)}(\lambda)}{(\iota)^{m+1}} \right] \frac{(\iota)^m}{\binom{m+n-1}{m}\hat{\mu}^{(m)}(\lambda)}, \ldots,$$

$$b_0 = \left[ a_0 - b_1 \frac{\binom{m+1}{m+1}\hat{\mu}^{(m+1)}(\lambda)}{(\iota)^{m+1}} - b_2 \frac{\binom{m+2}{m+2}\hat{\mu}^{(m+2)}(\lambda)}{(\iota)^{m+2}} - \cdots \right.$$
$$\left. - b_n \frac{\binom{m+n}{m+n}\hat{\mu}^{(m+n)}(\lambda)}{(\iota)^{m+n}} \right] \frac{(\iota)^m}{\binom{m}{m}\hat{\mu}^{(m)}(\lambda)}.$$

A simple computation of $f \star \mu$ gives $f \star \mu = g$. In the general case, suppose that $g = \sum_{j=1}^{p} g_j$, where $g_j(t) = p_j(t) e^{\iota\lambda_j t}$, for every $j$ and $\lambda_k \neq \lambda_j$ for $k \neq j$. Let $f_j$ be the exponential polynomial function corresponding to $g_j$ obtained as in the first case, that is, $f_j \star \mu = g_j$. Then $f = \sum_{j=1}^{p} f_j$ is a solution of the given convolution equation. $\square$

## 3. Mean-periodic functions on $G = \mathbb{T}$

We shall consider integrals of $X$-valued functions with respect to scalar measures in the sense of Bochner integral, and the integrals of scalar valued continuous functions with respect to $X'$-valued measures in the sense similar to that of Bochner discussed in the last section.

DEFINITION 3.1. Let $f \in C(\mathbb{T}, X)$ and $\mu \in M(\mathbb{T}, X')$. For every $n \in \mathbb{Z}$,

$$\hat{f}(n) := \int_{\mathbb{T}} z^{-n} f(z)\, dz \quad \text{and} \quad \hat{\mu}(n) := \int_{\mathbb{T}} z^{-n} d\mu(z)$$

are called the *nth-Fourier coefficient* of $f$ and $\mu$, respectively.

For $f \in C(\mathbb{T}, X)$, let $\tau(f)$ denote the closed translation invariant subspace generated by $f$.

PROPOSITION 3.2. *$f \in C(\mathbb{T}, X)$ is mean-periodic if and only if $\tau(f) \neq C(\mathbb{T}, X)$.*

PROOF. Follows from the fact that the dual of $C(\mathbb{T}, X)$ is $M(\mathbb{T}, X')$. □

LEMMA 3.3. *For $f \in C(\mathbb{T}, X)$ and $\mu \in M(\mathbb{T}, X')$, the following hold*:

(i) *$f \star \mu$ is a uniformly continuous function on $\mathbb{T}$*;
(ii) $(f \star \mu)\hat{} = \langle \hat{f}(n), \hat{\mu}(n)\rangle$.

PROOF. (i) Follows from the facts that $f$ is uniformly continuous, $\mu$ has finite variation and that $|(f \star \mu)(z) - (f \star \mu)(w)| \leq \int_{\mathbb{T}} \| f(z\overline{s}) - f(w\overline{s})\| \, dV_\mu(s)$.

(ii) Since $\mathbb{T}$ is compact, $f$ is uniformly continuous on $\mathbb{T}$. Let $\epsilon_k > 0$ be such that $\epsilon_k \to 0$ as $k \to \infty$. Since the metric on $\mathbb{T}$ is invariant under rotation, there exist finite Borel partitions $P_k$ of $\mathbb{T} = \sqcup B_{ki}$ such that if $z_{ki}, w_{ki} \in B_{ki}$, then $\| f(z_{ki}\overline{w}) - f(w_{ki}\overline{w})\| < \epsilon_k$ whenever $|w| = 1$. Now

$$(f \star \mu)\hat{}(n) = \int_{\mathbb{T}} (f \star \mu)(z) z^{-n}\, dz = \int_{\mathbb{T}} \int_{\mathbb{T}} f(z\overline{w})\, d\mu(w) z^{-n}\, dz$$
$$= \int_{\mathbb{T}} \lim_{k\to\infty} \left( \sum_j \langle f(z\overline{w_{kj}}),\ \mu(B_{kj})\rangle \right) z^{-n}\, dz. \tag{14}$$

Since $f$ is continuous on $\mathbb{T}$, $f(\mathbb{T}) \subset B(0, r) = r B(0, 1)$ for some $r > 0$. We have

$$\left| \sum_j \langle f(z\overline{w_{kj}}),\ \mu(B_{kj})\rangle \right| \leq \sum_j |\langle f(z\overline{w_{kj}}),\ \mu(B_{kj})\rangle|$$
$$\leq \sum_j r V_\mu(B_{kj}) \leq r V_\mu(\mathbb{T}) \leq rC.$$

Applying dominated convergence theorem in (14) for the functions

$$z \mapsto \sum_j \langle f(z\overline{w_{kj}}), \mu(B_{kj})\rangle z^{-n}$$

we obtain

$$
\begin{aligned}
(f \star \mu)\hat{}(n) &= \lim_{k\to\infty} \int_{\mathbb{T}} \sum_j \big\langle f(z\overline{w_{kj}}),\ \mu(B_{kj})\big\rangle z^{-n}\, dz \\
&= \lim_{k\to\infty} \sum_j \int_{\mathbb{T}} \big\langle f(z\overline{w_{kj}}),\ \mu(B_{kj})\big\rangle z^{-n}\, dz \\
&= \lim_{k\to\infty} \sum_j \int_{\mathbb{T}} \big\langle z^{-n} f(z\overline{w_{kj}}),\ \mu(B_{kj})\big\rangle\, dz.
\end{aligned}
$$

Now apply change of variable formula for the function $z \mapsto \langle z^{-n} f(z\overline{w_{kj}}),\ \mu(B_{kj})\rangle$, to get

$$
\begin{aligned}
(f \star \mu)\hat{}(n) &= \lim_{k\to\infty} \sum_j \int_{\mathbb{T}} \left\langle \left(\frac{z}{\overline{\overline{w_{kj}}}}\right)^{-n} f(z),\ \mu(B_{kj})\right\rangle dz \\
&= \lim_{k\to\infty} \sum_j \left\langle \int_{\mathbb{T}} \left(\frac{z}{\overline{\overline{w_{kj}}}}\right)^{-n} f(z)dz,\ \mu(B_{kj})\right\rangle \\
&= \lim_{k\to\infty} \sum_j (\overline{w_{kj}})^{-n} \langle \hat{f}(n),\ \mu(B_{kj})\rangle \\
&= \left\langle \hat{f}(n),\ \lim_{k\to\infty} \sum_j (\overline{w_{kj}})^{-n} \mu(B_{kj})\right\rangle = \langle \hat{f}(n),\ \hat{\mu}(n)\rangle. \qquad \square
\end{aligned}
$$

COROLLARY 3.4. *For $f \in C(\mathbb{T}, X)$ and $\mu \in M(\mathbb{T}, X')$, $f \star \mu = 0$ if and only if $\langle \hat{f}(n),\ \hat{\mu}(n)\rangle = 0$ for all $n \in \mathbb{Z}$.*

PROOF. Follows from Lemma 3.3 and the uniqueness of Fourier-Stieltjes coefficients of scalar valued functions on $\mathbb{T}$. $\square$

PROPOSITION 3.5. *Let $f \in C(\mathbb{T}, X)$. Then $\sigma(f) = \{\alpha z^n \hat{f}(n) : \hat{f}(n) \neq 0$ and $0 \neq \alpha \in \mathbb{C}\}$.*

PROOF. First we show that $\{\alpha z^n \hat{f}(n) : \hat{f}(n) \neq 0\} \subseteq \sigma(f)$. Let $\mu \in M(\mathbb{T}, X')$ be such that $\mu(\tau(f)) = 0$. Then $f \star \mu = 0$, since $\tau(f)$ is translation invariant. Hence by Corollary 3.4, $\langle \hat{f}(n), \hat{\mu}(n)\rangle = 0$ for every $n$. Thus $\mu(\alpha z^n \hat{f}(n)) = \alpha\langle \hat{f}(n), \hat{\mu}(n)\rangle = 0$, and by Corollary 3.4, $\alpha z^n \hat{f}(n) \in \tau(f)$. Hence $\alpha z^n \hat{f}(n) \in \sigma(f)$.

On the other hand, let $z^m x \in \sigma(f)$. To show that $x = \alpha \hat{f}(m)$ for some scalar $\alpha$. Let $x' \in X'$ be such that $x'(\hat{f}(m)) = 0$. Let $d\nu(z) = z^m x' dz$. Then

$$
\hat{\mu}(n) = \begin{cases} x' & \text{if } n = m; \\ 0 & \text{if } n \neq m. \end{cases}
$$

Thus by Corollary 3.4, $f \star \nu = 0$. Therefore $z^m x \star \nu = 0$ and hence $\langle x, \hat{\nu}(m)\rangle = 0$, that is, $\langle x, x'\rangle = 0$. Thus for $x' \in X'$, $\langle \hat{f}(m), x'\rangle = 0$ implies $\langle x, x'\rangle = 0$. Therefore $x = \alpha \hat{f}(m)$ for some $\alpha \in \mathbb{C}$. This completes the proof. □

PROPOSITION 3.6. *Let $f \in C(\mathbb{T}, X)$. Then $\sigma(f) = \emptyset$ if and only if $f = 0$.*

PROOF. By Proposition 3.5, it suffices to show that $\hat{f}(n) = 0$, for every $n \in \mathbb{Z}$ if and only if $f = 0$. Using the uniqueness of Fourier coefficients for scalar valued functions we obtain, for every $n \in \mathbb{Z}$ and $x' \in X'$,

$$\begin{aligned} \hat{f}(n) = 0 &\Leftrightarrow \langle x', \hat{f}(n)\rangle = 0 \Leftrightarrow \left\langle x', \int_{\mathbb{T}} f(z) z^{-n}\, dz \right\rangle = 0 \Leftrightarrow \int_{\mathbb{T}} \langle x', f(z)\rangle z^{-n}\, dz = 0 \\ &\Leftrightarrow (x' \circ f)\hat{}(n) = 0 \Leftrightarrow x' \circ f = 0 \Leftrightarrow f = 0. \end{aligned}$$
□

THEOREM 3.7. *For a complex Banach space $X \neq \mathbb{C}$ the following hold:*

(i) $MP(\mathbb{T}, X) = C(\mathbb{T}, X)$.

(ii) *For every* $0 \neq \mu \in M(\mathbb{T}, X')$, $\{0\} \neq MP(\mu) \neq C(\mathbb{T}, X)$.

PROOF. (i) Let $f : \mathbb{T} \to X$ be a non zero continuous function. Then $\hat{f}(n_0) \neq 0$ for some $n_0$. Chose $x' \in X'$ such that $x' \neq 0$ and $\langle x', \hat{f}(n_0)\rangle = 0$. Define $\mu(E) := \left(\int_E z^{n_0} dz\right) x'$, for every $E \in \mathscr{B}_{\mathbb{T}}$. Then $\mu \in M(\mathbb{T}, X')$ and

$$\hat{\mu}(n) = \begin{cases} x' & \text{if } n = n_0; \\ 0 & \text{if } n \neq n_0. \end{cases}$$

Thus $(f \star \mu)\hat{}(n) = \langle \hat{f}(n), \hat{\mu}(n)\rangle = 0$, for every $n \in \mathbb{Z}$. Hence it follows from Corollary 3.4, $f \star \mu = 0$.

(ii) Let $0 \neq \mu \in M(\mathbb{T}, X')$. Then $\hat{\mu}(n_0) \neq 0$ for some $n_0$. Let $0 \neq x \in X$ be such that $\langle \hat{\mu}(n_0), x\rangle = 0$, and $y \in X$ be such that $\langle \hat{\mu}(n_0), y\rangle \neq 0$. Define $f, g : \mathbb{T} \to X$, by $f(z) = z^{n_0} x$ and $g(z) = z^{n_0} y$. Then

$$\hat{f}(n) = \begin{cases} x & \text{if } n = n_0; \\ 0 & \text{if } n \neq n_0 \end{cases} \quad \text{and} \quad \hat{g}(n) = \begin{cases} y & \text{if } n = n_0; \\ 0 & \text{if } n \neq n_0. \end{cases}$$

Therefore, $\langle \hat{f}(n), \hat{\mu}(n)\rangle = 0$ for all $n \in \mathbb{Z}$ and $(g \star \mu)\hat{}(n_0) = \langle \hat{g}(n_0), \hat{\mu}(n_0)\rangle \neq 0$. Thus $f$ is mean-periodic with respect to $\mu$ and $g$ is not mean-periodic with respect $\mu$. □

REMARK. (1) Theorem 3.7 (i) is not true when $X = \mathbb{C}$. For instance, the function $f : \mathbb{T} \to \mathbb{C}$ defined by $f(z) := \sum_{-\infty}^{\infty} a_n z^n$, $z \in \mathbb{T}$, where $a_n \in \mathbb{C}$, $a_n \neq 0$ for every $n$ and $\sum_{-\infty}^{\infty} |a_n| < \infty$ is not mean-periodic.

(2) Let $G$ be a locally compact abelian group and $X$ a complex Banach space. A function $f \in C(G, X)$ is said to be *almost periodic* if the set of all translates of $f$ is relatively compact in $C(G, X)$. Every $f \in C(\mathbb{T}, X)$ is almost periodic and if $X \neq \mathbb{C}$, then every $f \in C(\mathbb{T}, X)$ is mean-periodic. When $X = \mathbb{C}$, there are complex valued continuous functions on the circle group $\mathbb{T}$ which are not mean-periodic.

We have the following result for spectral analysis and spectral synthesis for $\mathbb{T}$.

THEOREM 3.8. *The following hold*:

(i) *Let $x \in X$, $x \neq 0$, and $n_0 \in \mathbb{Z}$. Then $\tau(z^{n_0}x)$, the closed translation invariant subspace generated by $z^{n_0}x$, does not contain any non-zero proper closed translation invariant subspace of $C(\mathbb{T}, X)$.*

(ii) *Every non-zero closed translation invariant subspace $V$ of $C(\mathbb{T}, X)$ contains an exponential, that is, spectral analysis holds in $C(\mathbb{T}, X)$.*

(iii) *The linear span of the exponentials in every closed translation invariant subspace $V$ of $C(\mathbb{T}, X)$ is dense in $V$, that is, spectral synthesis holds in $C(\mathbb{T}, X)$.*

PROOF. (i) Let $V_1$ be a non-zero closed translation invariant subspace of $C(\mathbb{T}, X)$ such that $V_1 \subseteq \tau(z^{n_0}x)$. Then for $f \in \tau(z^{n_0}x)$, $\hat{f}(n_0) = cx$ for some $0 \neq c \in \mathbb{C}$ and $\hat{f}(n) = 0$ if $n \neq n_0$. To show $V_1 = \tau(z^{n_0}x)$, let $\mu \in M(\mathbb{T}, X')$ be such that $\mu(V_1) = \{0\}$. Then $\langle \hat{\mu}(n), x\rangle$ for every $n$. In particular $\langle \hat{\mu}(n_0), x\rangle$ and hence $\mu(V) = \{0\}$. Hence $V_1 = \tau(z^{n_0}x)$.

(ii) Choose $n_0 \in \mathbb{Z}$ and $f \in V$ such that $\hat{f}(n_0) \neq 0$. We will show that $z^{n_0}\hat{f}(n_0) \in V$. For, let $\mu \in M(\mathbb{T}, X')$ be such that $\mu(V) = \{0\}$. Since $V$ is translation invariant and $\mu(V) = \{0\}$, $f \star \mu = 0$. This implies $\langle \hat{f}(n_0), \hat{\mu}(n_0)\rangle = 0$. Thus $z^{n_0}\hat{f}(n_0) \star \mu = 0$. Hence $z^{n_0}\hat{f}(n_0) \in V$.

(iii) Let $V$ be closed translation invariant subspace of $C(\mathbb{T}, X)$. Let $V_0$ be the closed linear span of $z^n\hat{f}(n)$, $f \in V$. Then by (ii), $V_0 \subseteq V$. Let $f \in V$. Let $\mu \in M(\mathbb{T}, X')$ such that $\mu(V_0) = 0$. Then $\langle \hat{f}(n), \hat{\mu}(n)\rangle = 0$, for every $n \in \mathbb{Z}$. Thus $f \star \mu = 0$. Therefore, $\mu(f) = 0$. □

COROLLARY 3.9. *For $f \in C(\mathbb{T}, X)$ and $\mu \in M(\mathbb{T}, X')$, the following are equivalent*:

(i) $f \star \mu = 0$.

(ii) *$f$ is a limit of finite linear combinations of functions $z^n x$ which satisfy the equation $z^n x \star \mu = 0$.*

PROOF. First observe that for a given $\mu$, $MP(\mu) = \{f \in C(\mathbb{T}, X) : f \star \mu = 0\}$ is a closed translation invariant subspace of $C(\mathbb{T}, X)$. The result follows from Theorem 3.8 (iii). □

## 4. Some results for general groups

As mentioned earlier, problem of analysing mean-periodic functions, the problem of spectral analysis and spectral synthesis seems difficult to answer for general groups. However, it is not difficult to show that if $G$ is compact and $X = \mathbb{C}$ then every nontrivial closed translation invariant subspace $V$ of $C(K, \mathbb{C})$ includes exponentials and the linear span of exponentials in $V$ is dense in it. Hence every mean-periodic (scalar valued) function on a compact group is a limit of finite linear combination of exponentials.

For $G$ arbitrary locally compact abelian, and $X = \mathbb{C}$ we have the following: recall, $\Omega = \{\omega : G \to \mathbb{C}^* : \omega \in C(G) \text{ and } \omega(g_1 + g_2) = \omega(g_1)\omega(g_2)\}$.

THEOREM 4.1. (i) *Every $\omega \in \Omega$ is mean-periodic.*

(ii) *Let $G$ be an infinite locally compact $T_1$ abelian group. Then every exponential polynomial on $G$ is mean-periodic.*

(iii) *Let $MP(G)$ be the set of all mean-periodic functions on $G$. Then $MP(G)$ is dense in $C(G)$ if and only if $G$ is not finite.*

PROOF. (i) Clearly, every translate $\omega_g$ of $\omega$ is a constant multiple of $\omega$, and hence every finite linear combination of translates of $\omega$ is a constant multiple of $\omega$. Therefore the closed translation invariant subspace $\tau(\omega)$ is a one dimensional subspace of $C(G)$. Thus $\tau(\omega) \neq C(G)$, if $G$ is non-trivial.

(ii) Let $f$ be an exponential polynomial on $G$,

$$f(g) := \left(\sum_{\alpha} c_\alpha a_1(g)^{\alpha_1} a_2(g)^{\alpha_2} \cdots a_m(g)^{\alpha_m}\right) \omega(g),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N}$, $c_\alpha$ are complex constants and $a_1, \ldots, a_m$ are additive functions. Let $V = LS\{a_1(g)^{\beta_1} a_2(g)^{\beta_2} \cdots a_m(g)^{\beta_m} \omega(g) : \beta_j \in \mathbb{Z}_+, \beta_j \le \alpha_j \text{ for } 1 \le j \le m\}$. It is easy to see that $f \in V$ and $V$ is a finite dimensional translation invariant subspace of $C(G)$. Since $V$ is finite dimensional, it is closed and it follows that $\tau(f) \subseteq V$. But $C(G)$ is infinite dimensional as $G$ is not finite. Hence $\tau(f) \neq C(G)$.

(iii) Suppose that $G$ is finite, $G = \{g_1, g_2, \ldots, g_n\}$. Let $f \in C(G)$ and $\mu \in M_c(G)$. Let $\mu(g_i) = c_i$. Then $f \star \mu = 0$ for a non-trivial $\mu$ if and only if

$$\begin{vmatrix} f(g_1 - g_1) & f(g_1 - g_2) & \cdots & f(g_1 - g_n) \\ f(g_2 - g_1) & f(g_2 - g_2) & \cdots & f(g_2 - g_n) \\ \vdots & \vdots & & \vdots \\ f(g_n - g_1) & f(g_n - g_2) & \cdots & f(g_n - g_n) \end{vmatrix} = 0.$$

The columns of the above matrix are permutations of $[f(g_1), f(g_2), \ldots, f(g_n)]$. Thus $f$ is mean-periodic if and only if $(f(g_1), f(g_2), \ldots, f(g_n))$ is a root of some fixed polynomial $P$ in the variables $z_1, z_2, \dots, z_n$. The roots of this polynomial $P$ form a closed set $Z(P)$ in $\mathbb{C}^n$ of $2n$-dimensional Lebesgue measure zero. Therefore $Z(P)$ is not dense in $\mathbb{C}^n$. But $MP(G) = Z(P)$. Hence $MP(G)$ is not dense in $C(G)$.

Conversely, suppose that $G$ is not finite. Let $EP(G)$ be the set of all exponential polynomials in $C(G)$. By (ii), $EP(G) \subseteq MP(G)$, that is, $\Gamma \subseteq \Omega \subseteq EP(G) \subseteq MP(G)$. Moreover $\Omega$ separates points of $G$. Since the pointwise product of finite number of exponentials is again an exponential, it is easy to see that product of two exponential polynomials $f$ and $g$ is a finite sum of exponential polynomials and hence $\tau(fg)$ is finite dimensional. Therefore the algebra $A(EP(G))$, generated by $EP(G)$, is contained in $MP(G)$, that is, $A(EP(G)) \subseteq MP(G)$. Hence by Stone Weierstrass theorem ([9]) $A(EP(G))$ is dense in $C(G)$. Since $A(EP(G)) \subseteq MP(G)$, $MP(G)$ is dense in $C(G)$. $\square$

COROLLARY 4.2. *If $G$ is a finite $T_1$ topological abelian group, then* $\{0\} \neq MP(G) \neq C(G)$.

LEMMA 4.3. *Let $G$ be a locally compact abelian group having no nontrivial compact subgroups. Let $\hat{G}$ be the dual group of $G$. Then for $\mu \in M_c(G)$,* $\lambda_\Gamma(\{\gamma \in \Gamma : \hat{\mu}(\gamma) = 0\}) = 0$.

PROOF. Refer [6]. $\square$

THEOREM 4.4. *If $G$ does not have compact elements, then* $\{0\} \neq MP(G) \neq C(G)$.

PROOF. Let $f \in C(G)$ be compactly supported. By Lemma 4.3, $f$ is not mean-periodic. Thus $MP(G) \neq C(G)$. $\square$

As we have pointed earlier, the problem of spectral synthesis does not hold for every closed translation invariant subspace $V$ of $C(\mathbb{R}^2, \mathbb{C})$. However, with some conditions on $V$ this is true. First we prove the following lemma.

LEMMA 4.5. *The following hold*:

(i) *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be distinct complex numbers and $m_1, m_2, \dots, m_n \in \mathbb{N}$. Then the set $\{e^{\iota\lambda_j t}, te^{\iota\lambda_j t}, \dots, t^{m_j}e^{\iota\lambda_j t} : 1 \leq j \leq n\} \subseteq C(\mathbb{R})$ is linearly independent over $\mathbb{C}$.*

(ii) *Let $\lambda_1, \lambda_2, \dots, \lambda_n; \eta_1, \eta_2, \dots, \eta_n$ be complex numbers and for $1 \leq j, k, l \leq n$, $\alpha_{lj}, \beta_{kr}$ be non-negative integers. Then $\{t_1^{\alpha_{lj}} t_2^{\beta_{lj}} e^{\iota(\lambda_l t_1 + \eta_l t_2)} : 1 \leq l, j \leq n\}$ is a linearly independent subset of $C(\mathbb{R}^2)$ over $\mathbb{C}$ if $(\lambda_j, \eta_j) \neq (\lambda_k, \eta_k)$ or $(\alpha_{lj}, \beta_{lj}) \neq (\alpha_{lk}, \beta_{lk})$.*

PROOF. (i) Without loss of generality, we may assume that

$$\operatorname{Im}(\lambda_n) = \max_{1 \leq j \leq n} \operatorname{Im}(\lambda_j),$$

where Im denotes the imaginary part of a complex number. Then $\mathrm{Im}(\lambda_n) - \mathrm{Im}(\lambda_j) > 0$ for $1 \le j \le n-1$. Now for $a_{ij} \in \mathbb{C}$,

$$\sum_{j=1}^{n} \left(a_{0j}e^{\iota\lambda_j t} + a_{1j}te^{\iota\lambda_j t} + \cdots + a_{m_j j}t^{m_j}e^{\iota\lambda_j t}\right) = 0 \quad \Longrightarrow$$

$$\sum_{j=1}^{n-1} \left(a_{0j}e^{\iota(\lambda_j-\lambda_n)t} + a_{1j}te^{\iota(\lambda_j-\lambda_n)t} + \cdots + a_{m_j j}t^{m_j}e^{\iota(\lambda_j-\lambda_n)t}\right) + p_n(t) = 0,$$

where $p_n(t) = a_{n0} + a_{n1}t + \cdots + a_{nm_n}t^{m_n} = a_{nm_n}(t-\beta_1)(t-\beta_2)\cdots(t-\beta_{n_m})$, for some $\beta_1, \beta_2, \ldots, \beta_{n_m} \in \mathbb{C}$. Now as $t \to -\infty$, $t^k e^{\iota(\lambda_j-\lambda_n)} \to 0$ for every $j \neq n$. This implies $a_{nm_n} = 0$, since as $t \to -\infty$, $p_n(t) \not\to 0$ if $a_{nm_n} \neq 0$. Similarly by repeating the same argument one can easily show that $a_{ij} = 0$ for all $i, j$.

(ii) Case (i): $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct. For $a_{lj} \in \mathbb{C}$,

$$\sum_{l=1}^{n}\sum_{j=1}^{n} \left(a_{lj}t_1^{\alpha_{lj}}t_2^{\beta_{lj}}e^{\iota(\lambda_l t_1+\eta_l t_2)}\right) = 0 \quad \Longrightarrow \quad \sum_{l=1}^{n}\sum_{j=1}^{n} \left((a_{lj}t_2^{\beta_{lj}}e^{\iota\eta_l t_2})t_1^{\alpha_{lj}}e^{\iota\lambda_l t_1}\right) = 0.$$

By (i), this implies $a_{ij} = 0$ for all $i, j$.

Case (ii): $\lambda_i = \lambda_j$ for some $i$ and $j$. In this case rearrange the terms of the above expression by collecting the distinct exponential monomials in $t_1$. By the hypothesis, the coefficients of the exponential monomial in $t_1$ are finite linear combination of exponential monomials in $t_2$. By applying (i) twice, namely, first $t_1$ variable and then $t_2$ variable we get $a_{ij} = 0$ for all $i, j$. □

THEOREM 4.6. *Let $V$ be a closed translation invariant subspace of $C(\mathbb{R}^2)$ satisfying any one of the following conditions*:

(i) *$V$ is finite dimensional.*
(ii) *$V$ is rotation invariant.*
(iii) *$V = \tau_\mu := \{f \in C(\mathbb{R}^2) : f \star \mu = 0\}$ for some $\mu \in M_c(\mathbb{R}^2)$.*

*Then $V$ contains an exponential.*

PROOF. Case (i): $V$ is finite dimensional. Let $f \in V$ and $f \neq 0$. By Theorem 1.9, $f$ is of the form $f = \sum_{j=1}^{m} p_j(t_1, t_2)e^{\iota(\lambda_j t_1+\eta_j t_2)}$, where $p_j$ is a non-zero polynomial in $t_1, t_2$ and $(\lambda_j, \eta_j) \neq (\lambda_k, \eta_k)$ for $j \neq k$. Let $\mu \in M_c(\mathbb{R}^2)$ be such that $\mu(V) = \{0\}$. We show that $\mu(e^{\iota(\lambda_j t_1+\eta_j t_2)}) = 0$. Since $V$ is translation invariant, $f \star \mu = 0$. Write $f$ as a linear combination of elements in $\{t_1^{\alpha_{lj}}t_2^{\beta_{lj}}e^{\iota(\lambda_l t_1+\eta_l t_2)} : 1 \le l, j \le n\}$. Let $c_{k_1}t_1^{\alpha_0}t_2^{\beta_{k_1}}e^{\iota(\lambda_j t_1+\eta_j t_2)}, c_{k_2}t_1^{\alpha_0}t_2^{\beta_{k_2}}e^{\iota(\lambda_j t_1+\eta_j t_2)}, \ldots, c_{k_m}t_1^{\alpha_0}t_2^{\beta_{km}}e^{\iota(\lambda_j t_1+\eta_j t_2)}$ be the terms containing $e^{\iota(\lambda_j t_1+\eta_j t_2)}$ and the largest degree term of $t_1$, namely $t_1^{\alpha_0}$, where $c_{k_1}, c_{k_2}, \ldots, c_{k_m}$ are non-zero scalars. Also, $f \star \mu$ has the same representation and the terms containing $t^{\alpha_0}e^{\iota(\lambda_j t_1+\eta_j t_2)}$ are $c_{k_1}\hat{\mu}(\lambda_j, \eta_j)t_1^{\alpha_0}t_2^{\beta_{k_1}}e^{\iota(\lambda_j t_1+\eta_j t_2)}$, $c_{k_2}\hat{\mu}(\lambda_j, \eta_j)t_1^{\alpha_0}t_2^{\beta_{k_2}}e^{\iota(\lambda_j t_1+\eta_j t_2)}, \ldots,$

$c_{k_m}\hat{\mu}(\lambda_j, \eta_j)t_1^{\alpha_0}t_2^{\beta_{km}}e^{\iota(\lambda_j t_1+\eta_j t_2)}$. Since $f \star \mu = 0$ and $c_{k_j} \neq 0$, $\hat{\mu}(\lambda_j, \eta_j) = 0$, by Lemma 4.5. Therefore $\mu(e^{\iota(\lambda_j t_1+\eta_j t_2)}) = 0$. Thus $e^{\iota(\lambda_j t_1+\eta_j t_2)} \in V$.

Cases (ii) and (iii): $V$ is rotation invariant, or $V = \tau_\mu$. By Theorem 1.7 and Theorem 1.8, $V$ contains an exponential polynomial. It follows easily from the proof of (i) that $V$ contains an exponential. □

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