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METANILPOTENT VARIETIES OF GROUPS

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Abstract

For each positive integer n let $\mathbf{N}_{2,n}$ denote the variety of all groups which are nilpotent of class at most 2 and which have exponent dividing n. For positive integers m and n, let $\mathbf{N}_{2,m}\mathbf{N}_{2,n}$ denote the variety of all groups which have a normal subgroup in $\mathbf{N}_{2,m}$ with factor group in $\mathbf{N}_{2,n}$. It is shown that if $G \in \mathbf{N}_{2,m}\mathbf{N}_{2,n}$, where m and n are coprime, then G has a finite basis for its identities.

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1. Introduction

The finite basis question for a group G asks whether the set of all identities of G is equivalent to some finite set of identities. (We refer to [13] for terminology and basic results concerned with varieties of groups, but we use the term 'identity' rather than 'law'.) Between 1970 and 1973 a number of examples were published of groups for which the answer is negative: see [9] for references covering this period and see [5] for an account of more recent results. In the majority of these examples, the groups are metanilpotent (that is, nilpotent-by-nilpotent) and have finite exponent. In the simplest cases the groups belong to the variety $\mathbf{N}_{2,4}\mathbf{N}_{2,4}$: here, for any positive integer n, $\mathbf{N}_{2,n}$ denotes the variety of all groups which are nilpotent of class at most 2 and have finite exponent dividing n, and, for varieties $\mathbf{U}$ and $\mathbf{V}$, $\mathbf{VU}$ denotes the product variety, consisting of all groups which have a normal subgroup in $\mathbf{V}$ with factor group in $\mathbf{U}$. However, there are also many positive results. In particular, Lyndon [11] showed that every nilpotent group has a finite basis for its identities and Krasil'nikov [10] showed, much more generally, that the same is true for every nilpotent-by-abelian group.

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In the negative examples mentioned above in which G is metanilpotent of finite exponent there is no bound on the class of the nilpotent subgroups of G. It seems still to be an open question whether a soluble group of finite exponent, in which the nilpotent subgroups have bounded class, has a finite basis for its identities. Our main result gives a positive answer in many simple cases.

THEOREM A. *Let $G \in \mathbf{N}_{2,m}\mathbf{N}_{2,n}$ where m and n are coprime positive integers. Then G has a finite basis for its identities.*

A special case of this result was proved by Brady, Bryce and Cossey [2]: they showed that G has a finite basis for its identities if G belongs to $\mathbf{A}_m\mathbf{N}_{2,n}$, where m and n are coprime positive integers and $\mathbf{A}_m$ denotes the variety of all abelian groups of exponent dividing m. Theorem A solves a problem posed by Kovács and Newman [9]. The method adopted in [2] depends upon an analysis of the irreducible linear groups in $\mathbf{N}_{2,n}$, in prime characteristic not dividing n, and develops ideas of Higman [8]. However, at about the same time, Cohen [4] introduced a quite different method for tackling the finite basis question, dependent on the combinatorics of ordered sets. Cohen used this method to prove that every metabelian group has a finite basis for its identities, and the method was developed by others in later work such as [3, 10] and [12]. We apply similar methods here, for which we need the idea of a well-quasi-ordered set, defined as follows.

A *quasi-order* on a set W is a binary relation $\preccurlyeq$ on W which is reflexive and transitive. (We do not assume that $x \preccurlyeq y$ and $y \preccurlyeq x$ imply $x = y$, as in a partial order. Furthermore, we give no meaning to $\prec$, only to $\preccurlyeq$.) As shown in [6], the following two properties of a quasi-ordered set $(W, \preccurlyeq)$ are equivalent:

(i) for every infinite sequence $w_1, w_2, \ldots$ of elements of W there exist i and j with $i < j$ such that $w_i \preccurlyeq w_j$;

(ii) for every subset X of W there exists a finite subset Y of X such that for every element x of X there exists $y \in Y$ such that $y \preccurlyeq x$.

If (either of) these conditions hold then $(W, \preccurlyeq)$ is said to be *well-quasi-ordered*. If the relation $\preccurlyeq$ is a total (or linear) order then we obtain the more familiar idea of a well-ordered set.

We need to apply this idea to bilinear forms. Let K be a non-zero, finite, commutative and associative ring, with identity element, and let S be a finitely generated K-module. By an *S-form* we mean a pair (V, θ) consisting of a finitely generated, non-zero, free K-module V and a K-bilinear mapping $\theta : V \times V \to S$. If (V, θ) and (V', θ') are S-forms we write $(V, \theta) \preccurlyeq (V', \theta')$ if there is a K-module monomorphism $\xi : V \to V'$ such that $\theta(v_1, v_2) = \theta'(v_1\xi, v_2\xi)$ for all $v_1, v_2 \in V$. The first step in the proof of Theorem A is the following result (or, to be precise, a more technical version of this result stated in Section 3).

THEOREM B. *The set of all S-forms is well-quasi-ordered under the relation $\preccurlyeq$.*

Strictly speaking, the class of all S-forms is not a set. However, Theorem B can be rephrased to say that every set of S-forms is well-quasi-ordered under $\preccurlyeq$.

A result like this for trilinear alternating forms over a finite field was obtained by Atkinson [1] in order to prove a different finite basis result.

The finite basis question for a group G is equivalent to the finite basis question for the variety $\mathbf{V}$ generated by G (see [13]). Furthermore, if F is a free group of countably infinite rank and $\mathbf{V}(F)$ denotes the verbal subgroup of F corresponding to $\mathbf{V}$ then every subvariety of $\mathbf{V}$ is finitely based if and only if $\mathbf{V}$ is finitely based and the maximal condition holds for fully invariant subgroups of the relatively free group $F/\mathbf{V}(F)$. Much of the proof of Theorem A is concerned with establishing that the maximal condition holds in some closely related situations, typically for certain ideals in group algebras.

Let n be a positive integer and let A be a free group of countably infinite rank in the variety $\mathbf{N}_{2,n}$. Let $\mathbb{F}$ be a field of characteristic which does not divide n. Let Ψ be the set of all endomorphisms of A and, for each positive integer r, let $A^{\times r}$ denote the r-th direct power of A. Each element ψ of Ψ acts 'diagonally' on $A^{\times r}$ by $(a_1, \ldots, a_r)\psi = (a_1\psi, \ldots, a_r\psi)$ for all $a_1, \ldots, a_r \in A$, and this action can be extended to the group algebra $\mathbb{F}(A^{\times r})$ in the obvious way. Using the version of Theorem B mentioned above we shall prove the following result.

THEOREM C. *For each positive integer r, the maximal condition holds for Ψ-closed left ideals of $\mathbb{F}(A^{\times r})$.*

If U is a left C-module, for some algebra C, and if there is also an action of Ψ on U, we call U a (C, Ψ)-module. The concepts of (C, Ψ)-submodule and homomorphism of (C, Ψ)-modules are defined in the obvious way.

The algebra $\mathbb{F}(A \times A)$ is isomorphic to $\mathbb{F}A \otimes \mathbb{F}A$ (where the tensor product is taken over $\mathbb{F}$) under the linear map which sends (a, a') to $a \otimes a'$ for all $a, a' \in A$. We shall identify these two algebras and write (a, a') or $a \otimes a'$ interchangeably. Let R be the subspace of $\mathbb{F}(A \times A)$ spanned by all elements of the form $a \otimes a$ and $a \otimes a' + a' \otimes a$ for $a, a' \in A$. It is easily verified that R is a subalgebra of $\mathbb{F}(A \times A)$. Thus we may regard $\mathbb{F}(A \times A)$ as a left R-module and, indeed, as an (R, Ψ)-module. Clearly R is an (R, Ψ)-submodule of $\mathbb{F}(A \times A)$. The last main step in the proof of Theorem A is the following result.

THEOREM D. *The maximal condition holds for (R, Ψ)-submodules of $\mathbb{F}(A \times A)$ which contain R.*

The vector space $\mathbb{F}(A \times A)/R$ is isomorphic to the exterior square $\mathbb{F}A \wedge \mathbb{F}A$, which can therefore be given the structure of an (R, Ψ)-module. Thus Theorem D gives the following result.

COROLLARY. *The maximal condition holds for* (R, Ψ)*-submodules of* $\mathbb{F}A \wedge \mathbb{F}A$.

Theorems B, C and D will be proved in Sections 3, 5 and 6, respectively. In Section 2 we show how Theorem A can be derived from Theorems C and D.

2. The derivation of Theorem A

In this section we assume Theorems C and D, and we obtain Theorem A from these results.

One step in the proof of Theorem A is the special case proved in [2]. We could, of course, assume this result, but in order to illustrate our method in a comparatively simple case we first prove this special case.

Let $\mathbf{U}$ and $\mathbf{V}$ be varieties of groups. Let A be a free group in $\mathbf{U}$ on a free generating set $\{x_i : i \in \mathbb{N}\}$ and let B be a free group in $\mathbf{V}$ on a free generating set $\{y_i^a : i \in \mathbb{N},\ a \in A\}$. For each i, the element y_i^1 is also written as y_i. Each element a' of A induces an automorphism of B in which $y_i^a \mapsto y_i^{aa'}$ for all $i \in \mathbb{N}$, $a \in A$. Accordingly we can form the semidirect product BA, a split extension of B by A in which the original action of A on B becomes conjugation. We denote this group BA by $F_{\text{split}}(\mathbf{V}, \mathbf{U})$. The group has the following universal property implicit in [14] and straightforward to prove directly.

LEMMA 2.1. *Let G be a split extension of a group B_1 in $\mathbf{V}$ by a group A_1 in $\mathbf{U}$. Then every pair of mappings $\{x_i : i \in \mathbb{N}\} \to A_1$, $\{y_i : i \in \mathbb{N}\} \to B_1$ extends (uniquely) to a homomorphism $F_{\text{split}}(\mathbf{V}, \mathbf{U}) \to G$.*

LEMMA 2.2. *Let $\mathbf{U}$ and $\mathbf{V}$ be locally finite varieties of groups of coprime exponents and write $W = F_{\text{split}}(\mathbf{V}, \mathbf{U})$. Let $\mathbf{S}$ be a subvariety of $\mathbf{VU}$. Then $\mathbf{S}$ is generated by the group $W/\mathbf{S}(W)$, where $\mathbf{S}(W)$ is the verbal subgroup of W corresponding to $\mathbf{S}$.*

PROOF. Since $\mathbf{S}$ is locally finite it is generated by the finite groups it contains. By the Schur-Zassenhaus Theorem, each such finite group G is a split extension of a group in $\mathbf{V}$ by a group in $\mathbf{U}$. It follows, by Lemma 2.1, that G is a homomorphic image of $W/\mathbf{S}(W)$. Therefore $W/\mathbf{S}(W)$ generates $\mathbf{S}$. □

LEMMA 2.3. *Let F be a relatively free group and let U be an abelian fully invariant subgroup of F of exponent dividing a positive integer m. Suppose that U contains an infinite strictly ascending chain of fully invariant subgroups of F. Then there exists a prime p dividing m such that U/U^p contains an infinite strictly ascending chain of fully invariant subgroups of F/U^p.*

PROOF. Let Ω be the set of all endomorphisms of F, with Ω regarded as a set of operators. If V is any fully invariant subgroup of F then, since the endomorphisms of F/V are precisely those induced by elements of Ω, F/V may be regarded as an Ω-group and the Ω-subgroups of F/V are precisely the fully invariant subgroups of F/V, each being of the form W/V for some Ω-subgroup W of F containing V. Observe that if N is an Ω-subgroup of U then, since U contains an infinite strictly ascending chain of Ω-subgroups, either N or U/N contains such a chain.

Since U is abelian of exponent dividing m, we may write U as a finite direct product $U = U_1 \times \cdots \times U_k$ where each U_i is a non-trivial Ω-subgroup of prime-power exponent dividing m. By repeated use of the previous observation and isomorphisms of Ω-groups, we find that there exists $i \in \{1, \dots, k\}$ such that $U/\prod_{j \neq i} U_j$ contains an infinite strictly ascending chain of Ω-subgroups. Thus it suffices to prove the lemma in the case where U has exponent p^s for some prime p and positive integer s. By the same observation applied to the chain $U \geq U^p \geq \cdots \geq U^{p^s} = \{1\}$, there exists $r \in \{0, 1, \dots, s-1\}$ such that $U^{p^r}/U^{p^{r+1}}$ contains an infinite strictly ascending chain of Ω-subgroups. Thus there are Ω-subgroups $W_1, W_2, \dots$ of U satisfying

$$U^{p^{r+1}} \leq W_1 < W_2 < \cdots < U^{p^r}.$$

Let $\chi : U \to U^{p^r}$ be the homomorphism defined by $u\chi = u^{p^r}$ for all $u \in U$. Note that χ is surjective. Thus $U^p \leq W_1\chi^{-1} < W_2\chi^{-1} < \cdots < U$. It is easily verified that χ is a homomorphism of Ω-groups. Thus each $W_i\chi^{-1}$ is an Ω-group and $W_1\chi^{-1}/U^p < W_2\chi^{-1}/U^p < \cdots$ is an infinite strictly ascending chain of fully invariant subgroups of F/U^p contained in U/U^p. □

We shall now obtain the finite basis result of [2]. For any variety $\mathbf{V}$, $F(\mathbf{V})$ denotes the free group of $\mathbf{V}$ of countably infinite rank.

THEOREM 2.4 ([2]). *Let m and n be coprime positive integers. Then the subvarieties of $\mathbf{A}_m\mathbf{N}_{2,n}$ are finitely based.*

PROOF. Since $\mathbf{A}_m\mathbf{N}_{2,n}$ is finitely based by [7], it suffices to show that $F(\mathbf{A}_m\mathbf{N}_{2,n})$ satisfies the maximal condition on fully invariant subgroups. Write $H = F(\mathbf{A}_m\mathbf{N}_{2,n})$ and $U = \mathbf{N}_{2,n}(H)$. Thus $H/U \cong F(\mathbf{N}_{2,n})$. By [11], H/U satisfies the maximal condition on fully invariant subgroups. Thus it suffices to show that the maximal condition holds for fully invariant subgroups of H contained in U. By Lemma 2.3, it suffices to show that for each prime p dividing m the maximal condition holds for fully invariant subgroups of H/U^p contained in U/U^p. But $H/U^p \cong F(\mathbf{A}_p\mathbf{N}_{2,n})$, so it suffices to show that the minimal condition holds for subvarieties of $\mathbf{A}_p\mathbf{N}_{2,n}$ which contain $\mathbf{N}_{2,n}$.

Let $W = F_{\text{split}}(\mathbf{A}_p, \mathbf{N}_{2,n})$ and write $W = BA$ where $A = \langle x_i : i \in \mathbb{N}\rangle \cong F(\mathbf{N}_{2,n})$ and $B = \langle y_i^a : i \in \mathbb{N},\ a \in A\rangle$. Thus B is free in $\mathbf{A}_p$. By Lemma 2.2, the subvarieties of $\mathbf{A}_p\mathbf{N}_{2,n}$ which contain $\mathbf{N}_{2,n}$ are in one-one correspondence with the corresponding verbal subgroups of W, and these verbal subgroups are contained in B. Thus it suffices to prove that the maximal condition holds for fully invariant subgroups of W contained in B.

We can write B additively as a vector space over $\mathbb{F}_p$, the field with p elements, and B has basis $\{y_i^a : i \in \mathbb{N},\ a \in A\}$. Let T be the subspace with basis $\{y_1^a : a \in A\}$. There is an $\mathbb{F}_p$-space isomorphism $\mu : \mathbb{F}_p A \to T$ satisfying $a\mu = y_1^a$ for all $a \in A$. Hence we can give T the structure of a left $\mathbb{F}_p A$-module in such a way that μ is a module isomorphism. Let Ψ be the set of all endomorphisms of A. By Lemma 2.1, each element ψ of Ψ can be extended to an endomorphism of W by taking $y_i\psi = y_i$ for each i. Thus Ψ acts on W. Clearly T is Ψ-closed and the map $\mu : \mathbb{F}_p A \to T$ is an isomorphism of $(\mathbb{F}_p A, \Psi)$-modules.

For each $a \in A$, let ξ_a be the endomorphism of W satisfying $x_i\xi_a = x_i$ for all i, $y_1\xi_a = y_1^a$ and $y_i\xi_a = y_i$ for all $i > 1$. Clearly T is invariant under each ξ_a, and ξ_a acts on T in the same way as a acts (when T is regarded as a left $\mathbb{F}_p A$-module). It follows that if V is a fully invariant subgroup of W then $V \cap T$ is an $(\mathbb{F}_p A, \Psi)$-submodule of T.

For each $i, j \in \mathbb{N}$, let δ_{ij} be the endomorphism of W determined by $x_k\delta_{ij} = x_k$ for all k, $y_i\delta_{ij} = y_j$ and $y_k\delta_{ij} = 1$ for all $k \in \mathbb{N} \setminus \{i\}$. Let V be a fully invariant subgroup of W contained in B and let $v \in V$. Then there exists $r \in \mathbb{N}$ such that v belongs to the span of $\{y_i^a : 1 \le i \le r, a \in A\}$. We have $v = v\delta_{11}\delta_{11} + v\delta_{21}\delta_{12} + \cdots + v\delta_{r1}\delta_{1r}$, where $v\delta_{11}, v\delta_{21}, \ldots, v\delta_{r1} \in V \cap T$. Thus V is generated as a fully invariant subgroup by $V \cap T$.

Suppose that $V_1 \le V_2 \le \ldots$ is an ascending chain of fully invariant subgroups of W contained in B. Then $V_1 \cap T \le V_2 \cap T \le \cdots$ is an ascending chain of $(\mathbb{F}_p A, \Psi)$-modules. Hence $(V_1 \cap T)\mu^{-1} \le (V_2 \cap T)\mu^{-1} \le \cdots$ is an ascending chain of Ψ-closed left ideals of $\mathbb{F}_p A$. By Theorem C, this chain becomes stationary. Therefore, so does $V_1 \cap T \le V_2 \cap T \le \cdots$, and so does $V_1 \le V_2 \le \cdots$, which completes the proof of Theorem 2.4. □

Proof of Theorem A. Let m and n be coprime positive integers, and write $F = F(\mathbf{N}_{2,m}\mathbf{N}_{2,n})$. By [7], $\mathbf{N}_{2,m}\mathbf{N}_{2,n}$ is finitely based. Thus it suffices to show that F satisfies the maximal condition on fully invariant subgroups. Let U be the verbal subgroup of F corresponding to $\mathbf{A}_m\mathbf{N}_{2,n}$. Thus $F/U \cong F(\mathbf{A}_m\mathbf{N}_{2,n})$ and, by Theorem 2.4, it suffices to show that the maximal condition holds for fully invariant subgroups of F contained in U. By Lemma 2.3 it suffices to show that, for each prime p dividing m, the maximal condition holds for fully invariant subgroups of F/U^p contained in U/U^p. Let $\mathbf{V}$ be the variety of all groups G such that G is nilpotent of class at most two, G has exponent dividing m and G' has exponent dividing p. Thus $F/U^p \cong F(\mathbf{V}\mathbf{N}_{2,n})$. It suffices to show that the minimal condition holds for subvarieties of $\mathbf{V}\mathbf{N}_{2,n}$ which contain $\mathbf{A}_m\mathbf{N}_{2,n}$.

Let $W = F_{\text{split}}(\mathbf{V}, \mathbf{N}_{2,n})$ and write $W = BA$ where $A = \langle x_i : i \in \mathbb{N}\rangle \cong F(\mathbf{N}_{2,n})$ and $B = \langle y_i^a : i \in \mathbb{N},\ a \in A\rangle$. Thus B is free in $\mathbf{V}$. By Lemma 2.2, the subvarieties of $\mathbf{VN}_{2,n}$ which contain $\mathbf{A}_m\mathbf{N}_{2,n}$ are in one-one correspondence with the corresponding verbal subgroups of W, and these verbal subgroups are contained in B'. Thus it suffices to prove that the maximal condition holds for fully invariant subgroups of W contained in B'. If $B' = \{1\}$ (as occurs when $p = 2$ and m is not divisible by 4) then the result is trivial. Thus we may assume that $B' \neq \{1\}$.

We can write B' additively as a vector space over $\mathbb{F}_p$ spanned by $\{[y_i^a, y_j^{a'}] : i, j \in \mathbb{N},\ a, a' \in A\}$. Let T_1 be the subspace spanned by $\{[y_1^a, y_1^{a'}] : a, a' \in A\}$ and let T_2 be the subspace spanned by $\{[y_1^a, y_2^{a'}] : a, a' \in A\}$. Thus T_1 has basis $\{[y_1^a, y_1^{a'}] : a, a' \in A,\ a > a'\}$, where $>$ is an arbitrary total order on A, and T_2 has basis $\{[y_1^a, y_2^{a'}] : a, a' \in A\}$. Thus there are $\mathbb{F}_p$-space isomorphisms $\mu_1 : \mathbb{F}_pA \wedge \mathbb{F}_pA \to T_1$ and $\mu_2 : \mathbb{F}_p(A \times A) \to T_2$ satisfying $(a \wedge a')\mu_1 = [y_1^a, y_1^{a'}]$ and $(a \otimes a')\mu_2 = [y_1^a, y_2^{a'}]$ for all $a, a' \in A$. Hence, with R defined as in Section 1, we can give T_1 the structure of a left R-module and T_2 the structure of a left $\mathbb{F}_p(A \times A)$-module in such a way that μ_1 and μ_2 are module isomorphisms. Let Ψ be the set of all endomorphisms of A. As in the proof of Theorem 2.4, Ψ acts on W. Clearly T_1 and T_2 are Ψ-closed, μ_1 is an isomorphism of (R, Ψ)-modules, and μ_2 is an isomorphism of $(\mathbb{F}_p(A \times A), \Psi)$-modules.

For $a \in A$, let ξ_a be the endomorphism of W satisfying $x_i\xi_a = x_i$ for all i, $y_1\xi_a = y_1^a$ and $y_i\xi_a = y_i$ for all $i > 1$. For $a, a' \in A$, let $\xi_{a+a'}$ be the endomorphism of W satisfying $x_i\xi_{a+a'} = x_i$ for all i, $y_1\xi_{a+a'} = y_1^a y_1^{a'}$ and $y_i\xi_{a+a'} = y_i$ for all $i > 1$. Thus T_1 is invariant under each ξ_a and under each $\xi_{a+a'}$. Furthermore, ξ_a acts on T_1 in the same way as $a \otimes a$ acts, while $\xi_{a+a'}$ acts on T_1 in the same way as $(a + a') \otimes (a + a')$ acts. It is easily verified that R is spanned by the elements $a \otimes a$ and $(a+a') \otimes (a+a')$ for $a, a' \in A$. It follows that if V is a fully invariant subgroup of W then $V \cap T_1$ is an (R, Ψ)-submodule of T_1.

For $a, a' \in A$, let $\xi_{a,a'}$ be the endomorphism of W determined by $x_i\xi_{a,a'} = x_i$ for all i, $y_1\xi_{a,a'} = y_1^a$, $y_2\xi_{a,a'} = y_2^{a'}$ and $y_i\xi_{a,a'} = y_i$ for all $i > 2$. Clearly T_2 is invariant under each $\xi_{a,a'}$. Furthermore, $\xi_{a,a'}$ acts on T_2 in the same way as $a \otimes a'$ acts. It follows that if V is a fully invariant subgroup of W then $V \cap T_2$ is an $(\mathbb{F}_p(A \times A), \Psi)$-submodule of T_2.

For each $i, j \in \mathbb{N}$, let δ_{ij} be the endomorphism of W determined by $x_k\delta_{ij} = x_k$ for all k, $y_i\delta_{ij} = y_j$ and $y_k\delta_{ij} = 1$ for all $k \in \mathbb{N} \setminus \{i\}$. For each $i, j, i', j' \in \mathbb{N}$ with $i \neq j$, let $\varepsilon_{ii',jj'}$ be the endomorphism of W determined by $x_k\varepsilon_{ii',jj'} = x_k$ for all k, $y_i\varepsilon_{ii',jj'} = y_{i'}$, $y_j\varepsilon_{ii',jj'} = y_{j'}$ and $y_k\varepsilon_{ii',jj'} = 1$ for all $k \in \mathbb{N} \setminus \{i, j\}$.

Let V be a fully invariant subgroup of W contained in B' and let $v \in V$. Then, for some $r \in \mathbb{N}$, we can write $v = v_1 + v_2$ where v_1 is in the span of $\{[y_i^a, y_i^{a'}] : 1 \leq i \leq r,\ a, a' \in A\}$ and v_2 is in the span of $\{[y_i^a, y_j^{a'}] : 1 \leq i < j \leq r,\ a, a' \in A\}$. Then it is easily verified that $v_1 = \sum_i v\delta_{i1}\delta_{1i}$ and

$$v - v_1 = v_2 = \sum_{\substack{i,j \\ 1 \le i < j \le r}} v_2 \varepsilon_{i1,j2} \varepsilon_{1i,2j}.$$

Here $v\delta_{i1} \in V \cap T_1$ for all i and $v_2\varepsilon_{i1,j2} \in V \cap T_2$ for all i, j. It follows that V is generated as a fully invariant subgroup by $(V \cap T_1) \cup (V \cap T_2)$.

Suppose that $V_1 \le V_2 \le \cdots$ is an ascending chain of fully invariant subgroups of W contained in B'. Then $V_1 \cap T_1 \le V_2 \cap T_1 \le \cdots$ is an ascending chain of (R, Ψ)-submodules of T_1 while $V_1 \cap T_2 \le V_2 \cap T_2 \le \cdots$ is an ascending chain of $(\mathbb{F}_p(A \times A), \Psi)$-submodules of T_2. Hence $(V_1 \cap T_1)\mu_1^{-1} \le (V_2 \cap T_1)\mu_1^{-1} \le \cdots$ is an ascending chain of (R, Ψ)-submodules of $\mathbb{F}_p A \wedge \mathbb{F}_p A$ and

$$(V_1 \cap T_2)\mu_2^{-1} \le (V_2 \cap T_2)\mu_2^{-1} \le \cdots$$

is an ascending chain of Ψ-closed left ideals of $\mathbb{F}_p(A \times A)$. By Theorem C and the Corollary to Theorem D, both of the last two chains become stationary. Hence $(V_1 \cap T_1) \cup (V_1 \cap T_2) \le (V_2 \cap T_1) \cup (V_2 \cap T_2) \le \cdots$ becomes stationary. Therefore $V_1 \le V_2 \le \cdots$ becomes stationary, which proves Theorem A. □

3. Bilinear forms

Let K be a non-zero, finite, commutative and associative ring, with identity element 1. Unless otherwise stated all K-modules are finitely generated (therefore finite). Let S be a K-module. An *S-form* is a pair (V, θ) consisting of a non-zero free K-module V and a K-bilinear map $\theta : V \times V \to S$. A K-linear map $\xi : V \to V'$, where (V, θ) and (V', θ') are S-forms, is said to be a *homomorphism of S-forms* if $\theta(v_1, v_2) = \theta'(v_1\xi, v_2\xi)$ for all $v_1, v_2 \in V$. We write $\xi : (V, \theta) \to (V', \theta')$. The terms *isomorphism* and *monomorphism* are defined in the obvious way. We define a quasi-order $\preccurlyeq$ on the set of all S-forms by defining $(V, \theta) \preccurlyeq (V', \theta')$ if there is a monomorphism $\xi : (V, \theta) \to (V', \theta')$. The main result of this section is the following.

THEOREM B. *The set of all S-forms is well-quasi-ordered under the relation $\preccurlyeq$.*

Let (V, θ) be an S-form. For any subset U of V we define $P(U)$ to be the subset of $S \oplus S$ given by $P(U) = \{(\theta(v_1, v_2), \theta(v_2, v_1)) : v_1, v_2 \in U\}$, and we define $Q(U) \subseteq S$ by $Q(U) = \{\theta(v, v) : v \in U\}$. Also, for $U, U' \subseteq V$ we define $\theta(U, U') \subseteq S$ by $\theta(U, U') = \{\theta(u, u') : u \in U,\ u' \in U'\}$. Subsets U and U' are said to be *orthogonal* if $\theta(U, U') = \theta(U', U) = \{0\}$.

LEMMA 3.1. *Let V be a free K-module and let $v_1, \ldots, v_l \in V$. Then there are free K-submodules U_1, U_2 of V such that $V = U_1 \oplus U_2$, $\operatorname{rank}(U_1) \leq |K|l$, and $v_1, \ldots, v_l \in U_1$.*

PROOF. Take elements $x_1, \ldots, x_m$ of V where m is minimal such that $\{x_1, \ldots, x_m\}$ is contained in a K-basis of V and v_1 belongs to the submodule $\langle x_1, \ldots, x_m \rangle$. Write $v_1 = \sum_{i=1}^{m} \alpha_i x_i$ where each α_i is an element of K. If $m > |K|$ then there exist distinct $j, k \in \{1, \ldots, m\}$ such that $\alpha_j = \alpha_k$ and we may replace x_j and x_k by $x_j + x_k$, contrary to the minimality of m. Thus $m \leq |K|$. Let W be a free K-submodule of V such that $V = \langle x_1, \ldots, x_m \rangle \oplus W$ and, for $i = 2, \ldots, l$, write $v_i = v_i' + w_i$ where $v_i' \in \langle x_1, \ldots, x_m \rangle$ and $w_i \in W$. The result follows by applying an inductive argument to $w_2, \ldots, w_l$ in W. □

LEMMA 3.2. *Let (V, θ) be an S-form. Suppose that W is a free K-submodule of V and let $v_1, \ldots, v_l \in V$. Then there are free K-submodules W_1, W_2 of W such that $W = W_1 \oplus W_2$, $\operatorname{rank}(W_1) \leq 2|S|l$ and W_2 is orthogonal to $\{v_1, \ldots, v_l\}$.*

PROOF. We assume that $l = 1$ since the general case follows easily. We shall find free submodules U_1, U_2 of W such that $W = U_1 \oplus U_2$, $\operatorname{rank}(U_1) \leq |S|$ and $\theta(\{v_1\}, U_2) = \{0\}$. A similar argument gives $U_2 = U' \oplus U''$ with $\operatorname{rank}(U') \leq |S|$ and $\theta(U'', \{v_1\}) = \{0\}$. The result follows with $W_1 = U_1 \oplus U'$ and $W_2 = U''$.

Take basis elements $x_1, \ldots, x_m$ of W where m is maximal subject to $\theta(v_1, x_i) = 0$ for $i = 1, \ldots, m$. Let $\{x_1, \ldots, x_d\}$ be a basis of W containing $\{x_1, \ldots, x_m\}$. If $d - m > |S|$ then there exist distinct $j, k \in \{m+1, \ldots, d\}$ such that $\theta(v_1, x_j) = \theta(v_1, x_k)$ and we may extend $\{x_1, \ldots, x_m\}$ to $\{x_1, \ldots, x_m, x_j - x_k\}$, contrary to the maximality of m. Thus $d - m \leq |S|$ and we may take $U_1 = \langle x_{m+1}, \ldots, x_d \rangle$, $U_2 = \langle x_1, \ldots, x_m \rangle$. □

Let N be a positive integer and define $N^{[i]}$, for each non-negative integer i, by $N^{[0]} = 0$ and $N^{[i]} = N + N^2 + \cdots + N^i$ for $i > 0$. Let (V, θ) be an S-form and let $\{x_1, \ldots, x_d\}$ be a K-basis of V. We shall assume, in such notation, that the elements x_i are distinct (that is, $d = \operatorname{rank}(V)$) and that the basis is ordered as shown, corresponding to the ordered d-tuple $(x_1, \ldots, x_d)$. Let m be the non-negative integer which satisfies $N^{[m]} < d \leq N^{[m+1]}$ and write $V_1 = \langle x_1, \ldots, x_{N^{[1]}} \rangle, \ldots, V_m = \langle x_{N^{[m-1]}+1}, \ldots, x_{N^{[m]}} \rangle$, $V_{m+1} = \langle x_{N^{[m]}+1}, \ldots, x_d \rangle$. Thus $\operatorname{rank}(V_i) = N^i$ for $i = 1, \ldots, m$ and $0 < \operatorname{rank}(V_{m+1}) \leq N^{m+1}$. For $i = 1, \ldots, m+1$, write $V_i^+ = V_i \oplus \cdots \oplus V_{m+1}$. We say that (V, θ) is *N-regular* with respect to the ordered basis $\{x_1, \ldots, x_d\}$ if $P(V_i) = P(V_i^+)$ for $i = 1, \ldots, m+1$, $Q(V_i) = Q(V_i^+)$ for $i = 1, \ldots, m+1$, and V_{i-1} and V_{i+1}^+ are orthogonal for $i = 2, \ldots, m$. A decomposition $V = V_1 \oplus \cdots \oplus V_{m+1}$ with these properties, which is obtained from some ordered basis in the way described, is called an *N-regular decomposition* of V. Note that V_i

and V_j are orthogonal whenever $|i-j| \geq 2$. Also $P(V_1) \supseteq P(V_2) \supseteq \cdots \supseteq P(V_{m+1})$ and $Q(V_1) \supseteq Q(V_2) \supseteq \cdots \supseteq Q(V_{m+1})$.

LEMMA 3.3. *Let $N \geq |K|(2|S|^2 + |S|)$. Then every S-form is N-regular with respect to some basis.*

PROOF. Write $s = |S|$. Let (V, θ) be an S-form. Let $d = \mathrm{rank}(V)$ and define m by $N^{[m]} < d \leq N^{[m+1]}$. Suppose we can find free modules $V_1^+, V_1, V_2^+, V_2, \ldots, V_m^+, V_m, V_{m+1}^+$ with the following properties: $V_1^+ = V$; for $i = 1, \ldots, m$, $V_i^+ = V_i \oplus V_{i+1}^+$, $\mathrm{rank}(V_i) = N^i$, $P(V_i) = P(V_i^+)$ and $Q(V_i) = Q(V_i^+)$; and, for $i = 2, \ldots, m$, V_{i-1} and V_{i+1}^+ are orthogonal. Then, taking $V_{m+1} = V_{m+1}^+$, we see that $V = V_1 \oplus \cdots \oplus V_{m+1}$ and (V, θ) is N-regular with respect to a basis of V composed of bases of $V_1, \ldots, V_{m+1}$. We construct the required free modules inductively.

First define $V_1^+ = V$. If $\mathrm{rank}(V_1^+) \leq N$ then $m = 0$ and we have finished. So suppose that $\mathrm{rank}(V_1^+) > N$. Since $|P(V_1^+)| \leq s^2$ and $|Q(V_1^+)| \leq s$ we can choose elements $v_1, \ldots, v_{2s^2+s}$ of V_1^+ (not necessarily distinct) such that

$$\{(\theta(v_{2i-1}, v_{2i}), \theta(v_{2i}, v_{2i-1})) : i = 1, \ldots, s^2\} = P(V_1^+),$$
$$\{\theta(v_i, v_i) : i = 2s^2+1, \ldots, 2s^2+s\} = Q(V_1^+).$$

By Lemma 3.1, we can find free submodules U_1 and U_2 of V_1^+ such that $V_1^+ = U_1 \oplus U_2$, $v_1, \ldots, v_{2s^2+s} \in U_1$ and $\mathrm{rank}(U_1) \leq |K|(2s^2+s) \leq N$. Choose free modules V_1 and V_2^+ such that $V_1^+ = V_1 \oplus V_2^+$, $\mathrm{rank}(V_1) = N$ and $V_1 \supseteq U_1$. By the choice of $v_1, \ldots, v_{2s^2+s}$, we have $P(V_1) = P(V_1^+)$ and $Q(V_1) = Q(V_1^+)$.

Suppose that for some k with $1 \leq k \leq m$ we have found free modules $V_1^+, V_1, V_2^+, \ldots, V_k, V_{k+1}^+$ with the required properties for these modules. If $\mathrm{rank}(V_{k+1}^+) \leq N^{k+1}$ then $m = k$ and we have finished. So suppose that $\mathrm{rank}(V_{k+1}^+) > N^{k+1}$. By the method used in the first part of the proof we may find free submodules U and W of V_{k+1}^+ such that $V_{k+1}^+ = U \oplus W$, $P(U) = P(V_{k+1}^+)$, $Q(U) = Q(V_{k+1}^+)$ and $\mathrm{rank}(U) = N$. By Lemma 3.2, there are free submodules W_1 and W_2 of W such that $W = W_1 \oplus W_2$, W_2 and V_k are orthogonal and $\mathrm{rank}(W_1) \leq 2sN^k$. Then

$$\mathrm{rank}(U \oplus W_1) \leq N + 2sN^k \leq (1+2s)N^k \leq N^{k+1}.$$

Choose free modules V_{k+1} and V_{k+2}^+ such that $V_{k+1}^+ = V_{k+1} \oplus V_{k+2}^+$, $\mathrm{rank}(V_{k+1}) = N^{k+1}$, $V_{k+1} \supseteq U \oplus W_1$ and $V_{k+2}^+ \subseteq W_2$. Then V_{k+1} and V_{k+2}^+ have the required properties. □

LEMMA 3.4. *Let (V, θ) be an S-form which has an N-regular decomposition $V = V_1 \oplus \cdots \oplus V_{m+1}$.*

(i) *Let $k \in \{1, \ldots, m-1\}$. Suppose that $P(V_k) = P(V_{k+2})$ and $Q(V_k) = Q(V_{k+2})$. Then $P(V_k)$ is an additive subgroup of $S \oplus S$ and $Q(V_k)$ is an additive subgroup of S.*

(ii) *Let c be a positive integer and let $r(1)$ and $r(2)$ be integers such that $1 \leq r(1) < r(2) \leq m+1$. Suppose that*

$$P(V_{r(1)}) = P(V_{r(1)+1}) = \cdots = P(V_{r(2)}) = P \subseteq S \oplus S,$$
$$Q(V_{r(1)}) = Q(V_{r(1)+1}) = \cdots = Q(V_{r(2)}) = Q \subseteq S,$$

and $r(2) - r(1) \geq c(c+1) + 2$. Write $W = V_{r(1)+2} \oplus \cdots \oplus V_{r(2)-2}$. For all $i, j \in \{1, \ldots, c\}$ with $i < j$ let $p_{ij} \in P$ and for all $i \in \{1, \ldots, c\}$ let $q_i \in Q$. Then there exist $w_1, \ldots, w_c \in W$ such that $(\theta(w_i, w_j), \theta(w_j, w_i)) = p_{ij}$, for all $i, j \in \{1, \ldots, c\}$ with $i < j$, and $\theta(w_i, w_i) = q_i$, for all $i \in \{1, \ldots, c\}$.

PROOF. (i) Let $p, p' \in P(V_k)$. Then there exist $v, w \in V_k$ and $v', w' \in V_{k+2}$ such that $(\theta(v, w), \theta(w, v)) = p$ and $(\theta(v', w'), \theta(w', v')) = p'$. Write $V_k^+ = V_k \oplus \cdots \oplus V_{m+1}$. Since V_k and V_{k+2} are orthogonal,

$$p + p' = (\theta(v + v', w + w'), \theta(w + w', v + v')) \in P(V_k^+) = P(V_k).$$

Hence, since $P(V_k)$ is finite, it is a group. Similarly $Q(V_k)$ is a group.

(ii) By (i), P and Q are additive groups. There are $c(c+1)/2$ modules in the set $\{V_{r(1)+2}, V_{r(1)+4}, \ldots, V_{r(1)+c(c+1)}\}$ and so these modules can be relabelled as U_i for $1 \leq i \leq c$ and U_{ij} for $1 \leq i < j \leq c$. These modules are pairwise orthogonal submodules of W such that $P(U_i) = P(U_{ij}) = P$ and $Q(U_i) = Q(U_{ij}) = Q$ for all i, j. For $i, j \in \{1, \ldots, c\}$ with $i < j$ choose $u_{ij}, v_{ij} \in U_{ij}$ such that

$$(\theta(u_{ij}, v_{ij}), \theta(v_{ij}, u_{ij})) = p_{ij}.$$

Then for each $i \in \{1, \ldots, c\}$ choose $u_i \in U_i$ such that

$$\theta(u_i, u_i) = q_i - \sum_{j:j>i} \theta(u_{ij}, u_{ij}) - \sum_{j:j<i} \theta(v_{ji}, v_{ji}).$$

Finally, for $i = 1, \ldots, c$, define $w_i = u_i + \sum_{j:j>i} u_{ij} + \sum_{j:j<i} v_{ji}$. It is easy to check that these elements have the required properties. □

For each S-form (V, θ) we need to fix an ordered basis of V. Thus we define an *S-triple* to be a triple (V, θ, X) where (V, θ) is an S-form and X is an ordered basis of V.

Let (V, θ, X) and (V', θ', X') be S-triples, where $\text{rank}(V) = d$, $\text{rank}(V') = d'$, $X = \{x_1, \ldots, x_d\}$ and $X' = \{x'_1, \ldots, x'_{d'}\}$. We say that (V, θ, X) and (V', θ', X') are *isomorphic* if $d = d'$ and there is an S-form isomorphism $\xi : (V, \theta) \to (V', \theta')$ such that $x_i \xi = x'_i$ for $i = 1, \ldots, d$. We write $(V, \theta, X) \preccurlyeq (V', \theta', X')$ if there is a one-one order-preserving map $\phi : \{1, \ldots, d\} \to \{1, \ldots, d'\}$ together with an S-form homomorphism $\xi : (V, \theta) \to (V', \theta')$ such that, for $i = 1, \ldots, d$,

$$x_i\xi = x'_{i\phi} + z_i, \quad \text{for some } z_i \in \langle x'_1, x'_2, \ldots, x'_{i\phi-1}\rangle. \tag{3.1}$$

Clearly $\preccurlyeq$ is a quasi-order on the set of all S-triples. Also, if ξ satisfies (3.1) then ξ is a monomorphism. Hence $(V, \theta, X) \preccurlyeq (V', \theta', X')$ implies $(V, \theta) \preccurlyeq (V', \theta')$. An S-triple (V, θ, X) is said to be N-*regular* if (V, θ) is N-regular with respect to X.

PROPOSITION 3.5. *The set of all N-regular S-triples is well-quasi-ordered under the relation $\preccurlyeq$.*

PROOF. Let $Y^{(1)}, Y^{(2)}, Y^{(3)}, \ldots$ be an infinite sequence of N-regular S-triples. It suffices to show that there exist integers i and j with $i < j$ such that $Y^{(i)} \preccurlyeq Y^{(j)}$. For each i, let $Y^{(i)} = (V^{(i)}, \theta^{(i)}, X^{(i)})$ where $V^{(i)}$ has N-regular decomposition $V_1^{(i)} \oplus \cdots \oplus V^{(i)}_{m(i)+1}$, $d(i) = \operatorname{rank}(V^{(i)})$ and $X^{(i)} = \{x_1^{(i)}, \ldots, x^{(i)}_{d(i)}\}$. If $\{m(1), m(2), \ldots\}$ is bounded then there are only finitely many isomorphism types in the sequence $Y^{(1)}, Y^{(2)}, Y^{(3)}, \ldots$ and the result is clear. Thus we assume that $\{m(1), m(2), \ldots\}$ is unbounded. By passing to an infinite subsequence we may assume that $m(i) \geq 1$ for all $i \geq 1$. There are only finitely many possibilities for the values $\theta^{(i)}(x_j^{(i)}, x_k^{(i)})$ for $j, k \in \{1, \ldots, N^{[1]}\}$. Thus, by passing to an infinite subsequence, we may assume that, for all $j, k \in \{1, \ldots, N^{[1]}\}$, the value $\theta^{(i)}(x_j^{(i)}, x_k^{(i)})$ is independent of i. Then, by passing to an infinite subsequence, we may assume that $m(i) \geq 2$ for all $i \geq 2$ and that, for all $j, k \in \{1, \ldots, N^{[2]}\}$, the value $\theta^{(i)}(x_j^{(i)}, x_k^{(i)})$ is independent of i for all $i \geq 2$. Continuing in this way we may pass to an infinite subsequence with the following property for all $n \in \mathbb{N}$:

$$\begin{gathered} m(i) \geq n \text{ for all } i \geq n \text{ and,} \\ \text{for all } j, k \in \{1, \ldots, N^{[n]}\}, \theta^{(i)}(x_j^{(i)}, x_k^{(i)}) \text{ is independent of } i \text{ for all } i \geq n. \end{gathered} \tag{3.2}$$

Let $\overline{V}$ be a free K-module with countably infinite basis $\overline{X} = \{\overline{x}_1, \overline{x}_2, \ldots\}$. Define a K-bilinear map $\overline{\theta} : \overline{V} \times \overline{V} \to S$ by taking $\overline{\theta}(\overline{x}_j, \overline{x}_k)$ to be the limiting value of $\theta^{(i)}(x_j^{(i)}, x_k^{(i)})$. Furthermore, for each positive integer n, let P_n and Q_n be the limiting values of $P(V_n^{(i)})$ and $Q(V_n^{(i)})$, respectively. Since $P_1 \supseteq P_2 \supseteq \cdots$ and $Q_1 \supseteq Q_2 \supseteq \cdots$, there exist $P \subseteq S \oplus S$, $Q \subseteq S$, and a positive integer r, such that $P_r = P_{r+1} = \cdots = P$ and $Q_r = Q_{r+1} = \cdots = Q$. By Lemma 3.4, P and Q are additive groups.

For each i, let $r(i)$ be the largest integer belonging to $\{1, \ldots, m(i)\}$ such that $\theta^{(i)}(x_j^{(i)}, x_k^{(i)}) = \overline{\theta}(\overline{x}_j, \overline{x}_k)$ for all $j, k \in \{1, \ldots, N^{[r(i)]}\}$. By construction, the set $\{r(1), r(2), \ldots\}$ is unbounded. Hence, by passing to an infinite subsequence, we may assume that $r \le r(1) < r(2) < \cdots$. Let

$$a(i) = N^{[r(i)-1]} = \operatorname{rank}\bigl(V_1^{(i)} \oplus \cdots \oplus V_{r(i)-1}^{(i)}\bigr),$$
$$b(i) = N^{[r(i)]} = \operatorname{rank}\bigl(V_1^{(i)} \oplus \cdots \oplus V_{r(i)}^{(i)}\bigr) = a(i) + N^{r(i)}.$$

We may pass to an infinite subsequence so that, for each i, we have

$$\begin{aligned} d(i) - a(i) &\le d(i+1) - a(i+1) \quad \text{and} \\ r(i+1) - r(i) &\ge (d(i) - a(i))(d(i) - a(i) + 1) + 2. \end{aligned} \tag{3.3}$$

We now focus on $Y^{(1)}$ and $Y^{(2)}$ and show that $Y^{(1)} \preccurlyeq Y^{(2)}$. By the choice of $r(1)$ and $r(2)$, we have

$$P\bigl(V_{r(1)}^{(1)}\bigr) = P\bigl(V_{r(1)}^{(2)}\bigr) = P\bigl(V_{r(1)+1}^{(2)}\bigr) = \cdots = P\bigl(V_{r(2)}^{(2)}\bigr) = P,$$
$$Q\bigl(V_{r(1)}^{(1)}\bigr) = Q\bigl(V_{r(1)}^{(2)}\bigr) = Q\bigl(V_{r(1)+1}^{(2)}\bigr) = \cdots = Q\bigl(V_{r(2)}^{(2)}\bigr) = Q,$$

and

$$\theta^{(1)}\bigl(x_i^{(1)}, x_j^{(1)}\bigr) = \theta^{(2)}\bigl(x_i^{(2)}, x_j^{(2)}\bigr) \quad \text{for all } i, j \in \{1, \ldots, b(1)\}.$$

Since $a(1) < a(2)$ and $d(1) - a(1) \le d(2) - a(2)$ there exists a one-one order-preserving map $\phi : \{1, \ldots, d(1)\} \to \{1, \ldots, d(2)\}$ such that $i\phi = i$ for $i = 1, \ldots, a(1)$ and $\{a(1) + 1, \ldots, d(1)\}\phi \subseteq \{a(2) + 1, \ldots, d(2)\}$.

Write $W = V_{r(1)+2}^{(2)} \oplus \cdots \oplus V_{r(2)-2}^{(2)}$ as in Lemma 3.4. Note that, for $i \in \{a(1) + 1, \ldots, d(1)\}$,

$$\theta^{(1)}\bigl(x_i^{(1)}, x_i^{(1)}\bigr) \in Q\bigl(V_{r(1)}^{(1)} \oplus \cdots \oplus V_{m(1)+1}^{(1)}\bigr) = Q\bigl(V_{r(1)}^{(1)}\bigr) = Q,$$

and

$$\theta^{(2)}\bigl(x_{i\phi}^{(2)}, x_{i\phi}^{(2)}\bigr) \in Q\bigl(V_{r(2)}^{(2)} \oplus \cdots \oplus V_{m(2)+1}^{(2)}\bigr) = Q\bigl(V_{r(2)}^{(2)}\bigr) = Q.$$

Similarly,

$$\bigl(\theta^{(1)}\bigl(x_i^{(1)}, x_j^{(1)}\bigr), \theta^{(1)}\bigl(x_j^{(1)}, x_i^{(1)}\bigr)\bigr) \in P, \quad \bigl(\theta^{(2)}\bigl(x_{i\phi}^{(2)}, x_{j\phi}^{(2)}\bigr), \theta^{(2)}\bigl(x_{j\phi}^{(2)}, x_{i\phi}^{(2)}\bigr)\bigr) \in P,$$

for all $i, j \in \{a(1) + 1, \ldots, d(1)\}$ with $i < j$. Hence, by Lemma 3.4, we can choose elements $w_{a(1)+1}, \ldots, w_{d(1)}$ of W satisfying

$$\theta^{(2)}(w_i, w_i) = \begin{cases} -\theta^{(2)}\bigl(x_{i\phi}^{(2)}, x_{i\phi}^{(2)}\bigr) & \text{for } i \in \{a(1) + 1, \ldots, b(1)\}; \\ \theta^{(1)}\bigl(x_i^{(1)}, x_i^{(1)}\bigr) - \theta^{(2)}\bigl(x_{i\phi}^{(2)}, x_{i\phi}^{(2)}\bigr) & \text{for } i \in \{b(1) + 1, \ldots, d(1)\}; \end{cases}$$

and

$$
\begin{aligned}
&\big(\theta^{(2)}(w_i, w_j), \theta^{(2)}(w_j, w_i)\big)\\
&\quad = \begin{cases}
\big(\theta^{(1)}(x_i^{(1)}, x_j^{(1)}), \theta^{(1)}(x_j^{(1)}, x_i^{(1)})\big) - \big(\theta^{(2)}(x_{i\phi}^{(2)}, x_{j\phi}^{(2)}), \theta^{(2)}(x_{j\phi}^{(2)}, x_{i\phi}^{(2)})\big), \\
\qquad \text{for } i < j \text{ with } i \in \{a(1)+1, \dots, b(1)\},\ j \in \{b(1)+1, \dots, d(1)\}; \\
-\big(\theta^{(2)}(x_{i\phi}^{(2)}, x_{j\phi}^{(2)}), \theta^{(2)}(x_{j\phi}^{(2)}, x_{i\phi}^{(2)})\big), \\
\qquad \text{for } i < j \text{ with } i, j \in \{a(1)+1, \dots, b(1)\}; \\
\big(\theta^{(1)}(x_i^{(1)}, x_j^{(1)}), \theta^{(1)}(x_j^{(1)}, x_i^{(1)})\big) - \big(\theta^{(2)}(x_{i\phi}^{(2)}, x_{j\phi}^{(2)}), \theta^{(2)}(x_{j\phi}^{(2)}, x_{i\phi}^{(2)})\big) \\
\qquad \text{for } i < j \text{ with } i, j \in \{b(1)+1, \dots, d(1)\}.
\end{cases}
\end{aligned}
$$

Then we define a K-linear map $\xi : V^{(1)} \to V^{(2)}$ by

$$
x_i^{(1)}\xi = \begin{cases}
x_i^{(2)} & \text{for } i \in \{1, \dots, a(1)\}; \\
x_i^{(2)} + w_i + x_{i\phi}^{(2)} & \text{for } i \in \{a(1)+1, \dots, b(1)\}; \\
w_i + x_{i\phi}^{(2)} & \text{for } i \in \{b(1)+1, \dots, d(1)\}.
\end{cases}
$$

Note that, in these equations, $x_i^{(2)} \in V_1^{(2)} \oplus \cdots \oplus V_{r(1)}^{(2)}$, while $w_i \in W$ and $x_{i\phi}^{(2)} \in V_{r(2)}^{(2)} \oplus \cdots \oplus V_{m(2)+1}^{(2)}$, where $V_1^{(2)} \oplus \cdots \oplus V_{r(1)}^{(2)}$, W and $V_{r(2)}^{(2)} \oplus \cdots \oplus V_{m(2)+1}^{(2)}$ are pairwise orthogonal. It is straightforward to check that $\theta^{(2)}(x_i^{(1)}\xi, x_j^{(1)}\xi) = \theta^{(1)}(x_i^{(1)}, x_j^{(1)})$ in all the various cases for i and j. Hence ξ is a homomorphism of S-forms. Clearly ξ has the form required in (3.1). Thus we have $Y^{(1)} \preccurlyeq Y^{(2)}$, as required. □

PROOF OF THEOREM B. Take any positive integer N such that $N \geq |K|(2|S|^2+|S|)$. Then, by Lemma 3.3, for each S-form (V, θ) there exists an ordered basis $X_{(V,\theta)}$ of V such that $(V, \theta, X_{(V,\theta)})$ is an N-regular S-triple. If (V, θ) and (V', θ') are S-forms such that $(V, \theta, X_{(V,\theta)}) \preccurlyeq (V', \theta', X_{(V',\theta')})$ then $(V, \theta) \preccurlyeq (V', \theta')$. Hence the result follows by Proposition 3.5. □

To prove our result about varieties of groups we need, in fact, not Theorem B itself but the assertion stated below as Proposition 3.7.

Let T be any non-empty finite set. We consider finite sequences $(t_1, \dots, t_n)$ of elements of T and write $(t_1, \dots, t_n) \preccurlyeq (t'_1, \dots, t'_{n'})$ if $(t_1, \dots, t_n)$ is a subsequence of $(t'_1, \dots, t'_{n'})$, that is, if there is a one-one order-preserving map $\phi : \{1, \dots, n\} \to \{1, \dots, n'\}$ such that $t_i = t'_{i\phi}$ for $i = 1, \dots, n$. Clearly $\preccurlyeq$ is a quasi-order (in fact a partial-order). The following result is a special case of [6, Theorem 4.3].

LEMMA 3.6. *The set of all finite sequences of elements of T is well-quasi-ordered under the relation $\preccurlyeq$.*

We define an (S, T)-*form* to be a quadruple $(V, \theta, X, \mathbf{t})$ where (V, θ, X) is an S-triple and $\mathbf{t}$ is an ordered d-tuple $(t_1, \ldots, t_d)$ of elements of T, with $d = \operatorname{rank}(V)$. We say that (S, T)-forms $(V, \theta, X, \mathbf{t})$ and $(V', \theta', X', \mathbf{t}')$ are *isomorphic* if the S-triples (V, θ, X) and (V', θ', X') are isomorphic and $\mathbf{t} = \mathbf{t}'$. Let $\operatorname{rank}(V) = d$, $\operatorname{rank}(V') = d'$, $X = \{x_1, \ldots, x_d\}$ and $X' = \{x'_1, \ldots, x'_{d'}\}$. Write $(V, \theta, X, \mathbf{t}) \preccurlyeq (V', \theta', X', \mathbf{t}')$ if there is a one-one order-preserving map $\phi : \{1, \ldots, d\} \to \{1, \ldots, d'\}$ together with an S-form homomorphism $\xi : (V, \theta) \to (V', \theta')$ such that, for $i = 1, \ldots, d$, $t_i = t'_{i\phi}$ and

$$x_i \xi = x'_{i\phi} + z_i, \quad \text{for some } z_i \in \langle x'_1, x'_2, \ldots, x'_{i\phi-1} \rangle. \tag{3.4}$$

Clearly $\preccurlyeq$ is a quasi-order on the set of all (S, T)-forms, and we observe that $(V, \theta, X, \mathbf{t}) \preccurlyeq (V', \theta', X', \mathbf{t}')$ implies $(V, \theta, X) \preccurlyeq (V', \theta', X')$.

An (S, T)-form $(V, \theta, X, \mathbf{t})$ is said to be N-*regular* if the S-triple (V, θ, X) is N-regular. For given S, T and N we write $\mathscr{Z}$ for the set of all N-regular (S, T)-forms.

Proposition 3.7. *The set $(\mathscr{Z}, \preccurlyeq)$ is well-quasi-ordered.*

Proof. Let $Z^{(1)}, Z^{(2)}, Z^{(3)}, \ldots$ be an infinite sequence of N-regular (S, T)-forms. It suffices to show that there exist integers i and j with $i < j$ such that $Z^{(i)} \preccurlyeq Z^{(j)}$. For each i, let $Z^{(i)} = (V^{(i)}, \theta^{(i)}, X^{(i)}, \mathbf{t}^{(i)})$ and use further notation for $(V^{(i)}, \theta^{(i)}, X^{(i)})$ exactly as in the proof of Proposition 3.5. Also, write $\mathbf{t}^{(i)} = (t_1^{(i)}, \ldots, t_{d(i)}^{(i)})$.

As in the proof of Proposition 3.5, we may assume that $\{m(1), m(2), \ldots\}$ is unbounded and we may pass to a subsequence with the property (3.2) for all $n \in \mathbb{N}$. But, for each n and each $k \in \{1, \ldots, N^{[n]}\}$, there are only finitely many possibilities for $t_k^{(i)}$; thus we may also assume that, for all $k \in \{1, \ldots, N^{[n]}\}$, $t_k^{(i)}$ is independent of i for all $i \geq n$.

Define $\overline{V}$, $\overline{X}$, $\overline{\theta}$, P, Q and r as before. Also, for each $k \in \mathbb{N}$, define $\overline{t}_k$ to be the limiting value of $t_k^{(i)}$. Then define $r(i)$ as before, but with the additional requirement that $t_k^{(i)} = \overline{t}_k$ for all $k \in \{1, \ldots, N^{[r(i)]}\}$.

Define $a(i)$ and $b(i)$ as before and pass to an infinite subsequence with property (3.3) for each i. Also, define $\mathbf{t}_i = \left(t^{(i)}_{a(i)+1}, t^{(i)}_{a(i)+2}, \ldots, t^{(i)}_{d(i)}\right)$ for each i. By Lemma 3.6, there exist i and j with $i < j$ such that $\mathbf{t}_i$ is a subsequence of $\mathbf{t}_j$. Hence, by passing to an infinite subsequence of $Z^{(1)}, Z^{(2)}, \ldots$, we may assume that $\mathbf{t}_1$ is a subsequence of $\mathbf{t}_2$. Thus there is a one-one order-preserving map $\phi : \{a(1) + 1, \ldots, d(1)\} \to \{a(2) + 1, \ldots, d(2)\}$ such that $t_i^{(1)} = t_{i\phi}^{(2)}$ for $i = a(1) + 1, \ldots, d(1)$. We may extend ϕ to a one-one order-preserving map $\phi : \{1, \ldots, d(1)\} \to \{1, \ldots, d(2)\}$ by defining $i\phi = i$ for $i = 1, \ldots, a(1)$.

As in the proof of Proposition 3.5, there is a homomorphism of S-forms $\xi : (V^{(1)}, \theta^{(1)}) \to (V^{(2)}, \theta^{(2)})$ such that ξ has the form required in (3.4). For $i = 1, \ldots, a(1)$, we have $t_i^{(1)} = t_i^{(2)} = \overline{t}_i$, since $a(1) \leq N^{[r(1)]} \leq N^{[r(2)]}$, and so $t_i^{(1)} = t_{i\phi}^{(2)}$, since $i = i\phi$. Also, for $i = a(1) + 1, \dots, d(1)$, we have $t_i^{(1)} = t_{i\phi}^{(2)}$ by the choice of ϕ. Thus $Z^{(1)} \preccurlyeq Z^{(2)}$. $\square$

An *alternating* S-form is an S-form (V, θ) such that $\theta(v, v) = 0$ for all $v \in V$. Consider now the case where $S = K$. An alternating K-form (V, θ) is called *standard* with respect to the ordered basis $\{x_1, \dots, x_d\}$ of V if $\theta(x_i, x_j) = 0$ for all i, j such that $1 \leq i < j \leq d$ and $(i, j) \notin \{(1, 2), (3, 4), \dots, (2[d/2] - 1, 2[d/2])\}$.

LEMMA 3.8 (compare [2]). *Let n_0 be an integer, with $n_0 \geq 2$, and let $K = \mathbb{Z}/n_0\mathbb{Z}$. Let (V, θ) be an alternating K-form. Then there is a K-basis $\{x_1, \dots, x_d\}$ of V such that (V, θ) is standard with respect to $\{x_1, \dots, x_d\}$.*

PROOF. Choose $u_1, u_2 \in V$ such that the additive cyclic subgroup $\langle\theta(u_1, u_2)\rangle$ of K has largest possible order. Let x_1 be an element of V of order n_0 such that $u_1 \in \langle x_1\rangle$. Note that x_1 belongs to some basis of V. By maximality, $\langle\theta(u_1, u_2)\rangle = \langle\theta(x_1, u_2)\rangle$. Hence we may replace u_1 by x_1. Let U be a submodule of V such that $V = \langle x_1\rangle \oplus U$. If $U = \{0\}$ then $\{x_1\}$ is the required basis, so suppose $U \neq \{0\}$. Write $u_2 = u_2' + u$ where $u_2' \in \langle x_1\rangle$ and $u \in U$. Clearly we may replace u_2 by u. Then, as before, we may replace u by an element x_2 which belongs to a basis of U. Thus $\{x_1, x_2\}$ is contained in a basis of V. Set $W = \{w \in V : \theta(x_1, w) = \theta(x_2, w) = 0\}$. Let $v \in V$. The choice of x_1 and x_2 shows that $\theta(x_1, x_2)$ is a generator of the cyclic group $\{\theta(x_1, u) : u \in V\}$. Hence there exists $\lambda \in K$ such that $\theta(x_1, v) = \lambda\theta(x_1, x_2)$. Similarly there exists $\mu \in K$ such that $\theta(v, x_2) = \mu\theta(x_1, x_2)$. It follows that $v - \mu x_1 - \lambda x_2 \in W$ and so $v \in \langle x_1, x_2\rangle + W$. Therefore $V = \langle x_1, x_2\rangle + W$. Thus we may find a basis $\{x_1, x_2, w_1, \dots, w_{d-2}\}$ of V with $w_1, \dots, w_{d-2} \in W$. The lemma follows by an inductive argument applied to $\langle w_1, \dots, w_{d-2}\rangle$. $\square$

4. Direct powers of finite groups

In this section we shall obtain some results which will be useful for both Theorem C and Theorem D.

Let G be a finite group and let D be the (restricted) direct product $D = \prod_{i \in \mathbb{N}} G_i$ where $G_i = G$ for all i. Thus the elements of D may be regarded as sequences of the form $(g_1, g_2, \dots)$ where $g_i \in G$ for all i and where $\{i : g_i \neq 1\}$ is finite.

Let $\phi : \mathbb{N} \to \mathbb{N}$ be a one-one order-preserving function. Let X be a finite subset of $\mathbb{N} \setminus \mathbb{N}\phi$ and let $\sigma : X \to \mathbb{N}\phi$ be a function such that $j < j\sigma$ for all $j \in X$. Given such ϕ, X and σ, let ξ be the endomorphism of D defined by

$$(g_1, g_2, \dots)\xi = (g_1', g_2', \dots),$$

where $g'_j = g_i$ if $j = i\phi$, $g'_j = 1$ if $j \notin \mathbb{N}\phi \cup X$, and $g'_j = g'_{j\sigma}$ if $j \in X$. Let Ξ be the set of all such endomorphisms of D (for all possible choices of ϕ, X and σ).

Let $\leq$ be a total order on G which is arbitrary except that $1 \leq g$ for all $g \in G$. Then the set D may be ordered lexicographically from the right: if $d, d' \in D$ where $d = (g_1, g_2, \ldots)$ and $d' = (g'_1, g'_2, \ldots)$, we set $d < d'$ if there exists $l \in \mathbb{N}$ such that $g_l < g'_l$ but $g_i = g'_i$ for all $i > l$. Clearly $(D, \leq)$ is well-ordered, and it is easy to prove the following result.

LEMMA 4.1. *Let $d, d' \in D$ and let $\xi \in \Xi$. If $d < d'$ then $d\xi < d'\xi$.*

For $d \in D$, where $d = (g_1, g_2, \ldots)$, write

$$\operatorname{span}(d) = \{g \in G \setminus \{1\} : g = g_i \text{ for some } i\},$$

and, for $g \in \operatorname{span}(d)$, let $i_g(d)$ denote the largest i such that $g_i = g$.

Let d and d' be elements of D, where $d = (g_1, g_2, \ldots)$ and $d' = (g'_1, g'_2, \ldots)$. Write $d \preccurlyeq d'$ if $\operatorname{span}(d) = \operatorname{span}(d')$ and there is a one-one order-preserving function $\phi : \mathbb{N} \to \mathbb{N}$ such that $g_i = g'_{i\phi}$ for all i and $i_g(d)\phi = i_g(d')$ for all $g \in \operatorname{span}(d)$. Clearly $(D, \preccurlyeq)$ is quasi-ordered (in fact, partially-ordered).

LEMMA 4.2. *The set $(D, \preccurlyeq)$ is well-quasi-ordered.*

PROOF. Let $m = |G \setminus \{1\}|$ and assume $m \geq 1$ (the result is trivial for $m = 0$). Write $G \setminus \{1\} = \{a_1, \ldots, a_m\}$. For $d \in D$ and $k = 1, \ldots, m$, define $p_k(d) = i_{a_k}(d)$ if $a_k \in \operatorname{span}(d)$ and $p_k(d) = 1$ otherwise, so that we obtain an $m+1$-tuple $s(d) = (p_1(d), \ldots, p_m(d), d)$. Let $d, d' \in D$, where $d = (g_1, g_2, \ldots)$ and $d' = (g'_1, g'_2, \ldots)$. Following the notation of [3], we write $s(d) \preccurlyeq_\Phi s(d')$ if there exists a one-one order-preserving map $\phi : \mathbb{N} \to \mathbb{N}$ such that $g_i = g'_{i\phi}$ for all i and $p_i(d)\phi = p_i(d')$ for $i = 1, \ldots, m$. By [3, Lemma 3.2], the set of $m+1$-tuples $s(d)$ is well-quasi-ordered under $\preccurlyeq_\Phi$. But $s(d) \preccurlyeq_\Phi s(d')$ implies $d \preccurlyeq d'$. The result follows. ☐

Let $\mathbb{F}$ be any field. Then each non-zero element u of the group algebra $\mathbb{F}D$ can be written (uniquely) in the form $u = \lambda_1 d_1 + \cdots + \lambda_r d_r$ where $d_1, \ldots, d_r \in D$, $d_1 > \cdots > d_r$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{F} \setminus \{0\}$. The largest group element d_1 is called the *leading group element* of u and we write $d_1 = \operatorname{lead}(u)$. Since every endomorphism of D extends to $\mathbb{F}D$, each element of Ξ acts on $\mathbb{F}D$. For $S \subseteq \mathbb{F}D$ we write $\langle S \rangle_\Xi$ for the Ξ-closed subspace of $\mathbb{F}D$ generated by S.

LEMMA 4.3. *Let u and v be non-zero elements of $\mathbb{F}D$ with* $\operatorname{lead}(u) \preccurlyeq \operatorname{lead}(v)$. *Then there exists $v^* \in \mathbb{F}D$ such that $\langle u, v \rangle_\Xi = \langle u, v^* \rangle_\Xi$ and either $v^* = 0$ or* $\operatorname{lead}(v^*) < \operatorname{lead}(v)$.

PROOF. Write $u = \lambda_1 d_1 + \cdots + \lambda_r d_r$ and $v = \lambda'_1 d'_1 + \cdots + \lambda'_s d'_s$ where the d_i and d'_i are elements of D, $d_1 > \cdots > d_r$, $d'_1 > \cdots > d'_s$, and the λ_i and λ'_i are elements of $\mathbb{F} \setminus \{0\}$. Write $d = d_1 = \mathrm{lead}(u)$ and $d' = d'_1 = \mathrm{lead}(v)$. Thus $d \preccurlyeq d'$. Let $d = (g_1, g_2, \dots)$ and $d' = (g'_1, g'_2, \dots)$, and let $\phi : \mathbb{N} \to \mathbb{N}$ be as in the definition of $d \preccurlyeq d'$. Let $X = \{j : j \notin \mathbb{N}\phi \text{ and } g'_j \neq 1\}$. By the definition of $d \preccurlyeq d'$ we have $i_g(d') \in \mathbb{N}\phi$ for all $g \in \mathrm{span}(d')$. For each $j \in X$ let $j\sigma = i_g(d')$ where $g = g'_j$. Let ξ be the element of Ξ corresponding to ϕ, X and σ. Then it is easy to check that $d\xi = d'$. Hence, by Lemma 4.1, $\mathrm{lead}(u\xi) = d' = \mathrm{lead}(v)$. Let $v^* = v - \lambda'_1 \lambda_1^{-1}(u\xi)$. Then the result follows. □

PROPOSITION 4.4. *The maximal condition holds for Ξ-closed subspaces of $\mathbb{F}D$.*

PROOF. Let U be a Ξ-closed subspace of $\mathbb{F}D$. It suffices to prove that U is finitely generated as a Ξ-closed subspace. By Lemma 4.2, there exists a finite subset S of $U \setminus \{0\}$ such that for all $v \in U \setminus \{0\}$ there exists $u \in S$ such that $\mathrm{lead}(u) \preccurlyeq \mathrm{lead}(v)$. We claim that $U = \langle S \rangle_\Xi$. Suppose, in order to get a contradiction, that there exists $v \in U$ such that $v \notin \langle S \rangle_\Xi$, and choose such v so that $\mathrm{lead}(v)$ is as small as possible in the well-ordered set $(D, \leq)$. There exists $u \in S$ such that $\mathrm{lead}(u) \preccurlyeq \mathrm{lead}(v)$. By Lemma 4.3, there exists $v^* \in \mathbb{F}D$ such that $\langle u, v \rangle_\Xi = \langle u, v^* \rangle_\Xi$ and either $v^* = 0$ or $\mathrm{lead}(v^*) < \mathrm{lead}(v)$. Since $v \notin \langle u \rangle_\Xi$, we have $v^* \neq 0$. Since $v^* \in \langle u, v \rangle_\Xi \subseteq U$, the choice of v gives $v^* \in \langle S \rangle_\Xi$. Hence $v \in \langle u, v^* \rangle_\Xi \subseteq \langle S \rangle_\Xi$, and we have the required contradiction. □

Let n be a positive integer and let E be a free group of countably infinite rank in the variety $\mathbf{A}_n$. Let Γ be the set of all endomorphisms of E.

PROPOSITION 4.5. *For each positive integer r, the maximal condition holds for Γ-closed subspaces of $\mathbb{F}(E^{\times r})$.*

PROOF. Clearly we may assume $n > 1$. Let $\{x_1, x_2, \dots\}$ be a free generating set for E. For each $i \in \mathbb{N}$, let G_i be the subgroup of $E^{\times r}$ generated by the elements $(x_i, 1, \dots, 1), (1, x_i, 1, \dots, 1), \dots, (1, \dots, 1, x_i)$. Write $G = G_1$. Thus G is a finite group. Clearly $E^{\times r}$ is the direct product of the groups G_i and, for each i, there is an obvious isomorphism from G to G_i. Thus we may identify $E^{\times r}$ with the direct power D of G considered above. The result will follow from Proposition 4.4 if we can show that every element of Ξ is induced by some element of Γ. Let $\xi \in \Xi$ and suppose that ξ is associated with ϕ, X and σ, in the notation used before. Define a homomorphism $\gamma : E \to E$ by $x_i\gamma = x_{i\phi} \prod_{j \in X,\ j\sigma = i\phi} x_j$, for each i, where the product is taken over all those values of j, if any, which lie in X and satisfy $j\sigma = i\phi$. It is straightforward to verify that γ induces ξ. □

5. Proof of Theorem C

We use the notation of Section 1. In particular, n is a positive integer, A is a free group of $\mathbf{N}_{2,n}$ of countably infinite rank, Ψ is the set of endomorphisms of A and $\mathbb{F}$ is a field of characteristic not dividing n. We shall describe the proof of Theorem C only in the case $r = 2$. The proof for general r is essentially the same, but greater notational complexity is required for $r > 2$.

Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. If I is a Ψ-closed left ideal of $\mathbb{F}(A \times A)$ then $\overline{\mathbb{F}} \otimes_{\mathbb{F}} I$ is a Ψ-closed left ideal of $\overline{\mathbb{F}}(A \times A)$, and $I = \mathbb{F}(A \times A) \cap \overline{\mathbb{F}} \otimes_{\mathbb{F}} I$. Therefore we may assume that $\mathbb{F} = \overline{\mathbb{F}}$. We write $\mathbb{F}^{\times}$ for the multiplicative group $\mathbb{F} \setminus \{0\}$.

Let $\{x_i : i \in \mathbb{N}\}$ be a free generating set of A and, for each positive integer k, let A_k be the subgroup $\langle x_1, \ldots, x_k \rangle$. Define n_0 by $n_0 = n$ if n is odd and $n_0 = n/2$ if n is even. For all $a, b \in A$ we have $(ab)^n = 1$ and hence $[a, b]^{n_0} = [a^{n_0}, b] = 1$. Thus $(A')^{n_0} = \{1\}$ and A^{n_0} is central in A. It is easily verified that the relations $x_i^n = 1$ and $[x_i, x_j]^{n_0} = 1$, for all $i, j \in \{1, \ldots, k\}$, imposed on the free nilpotent group of class 2 on free generators $x_1, \ldots, x_k$, give a group of exponent n, which is therefore isomorphic to A_k. It follows that A_k' is a free abelian group of exponent n_0 with basis $\{[x_i, x_j] : 1 \le i < j \le k\}$. If $n \le 2$, then A is the free group of countably infinite rank in the variety $\mathbf{A}_n$, and, in this case, Theorem C follows from Proposition 4.5. Thus we assume that $n > 2$, so that $n_0 > 1$.

Let $K = \mathbb{Z}/n_0\mathbb{Z}$ and let ω be a primitive n_0-th root of unity in $\mathbb{F}$. Thus ω^{λ} is well-defined for all $\lambda \in K$, and $\{\omega^{\lambda} : \lambda \in K\}$ is the cyclic subgroup of $\mathbb{F}^{\times}$ consisting of all n_0-th roots of unity in $\mathbb{F}$.

Let Q_k be the set of all ordered pairs (i, j) with $1 \le i < j \le k$, and let Δ_k be the set of all functions $\delta : Q_k \to K$. For each $\delta \in \Delta_k$ there is a group homomorphism $\chi_\delta : A_k' \to \mathbb{F}^{\times}$ determined by $\chi_\delta([x_i, x_j]) = \omega^{\delta(i,j)}$ for all $(i, j) \in Q_k$. Since the elements $[x_i, x_j]$ form a basis for A_k', every homomorphism $A_k' \to \mathbb{F}^{\times}$ arises in this way from some δ. We extend χ_δ by linearity to a function $\chi_\delta : \mathbb{F}A_k' \to \mathbb{F}$. In the language of representation theory, the functions χ_δ are the characters afforded by the irreducible representations of the abelian group A_k' over $\mathbb{F}$, all of which are one-dimensional.

For each $\delta \in \Delta_k$, let e_δ be the element of $\mathbb{F}A_k'$ defined by

$$e_\delta = \frac{1}{|A_k'|} \sum_{a \in A_k'} \chi_\delta(a^{-1})a. \tag{5.1}$$

The elements e_δ have the following properties, which may be verified by elementary representation theory or direct calculation.

$$we_\delta = \chi_\delta(w)e_\delta \quad \text{for all} \quad \delta \in \Delta_k \quad \text{and all} \quad w \in \mathbb{F}A_k'. \tag{5.2}$$

$$\chi_\delta(e_\delta) = 1 \quad \text{and} \quad e_\delta^2 = e_\delta \quad \text{for all} \quad \delta \in \Delta_k. \tag{5.3}$$

(5.4) $$\chi_{\delta'}(e_\delta) = 0 \text{ and } e_\delta e_{\delta'} = 0 \text{ for all } \delta, \delta' \in \Delta_k \text{ with } \delta \neq \delta'.$$

(5.5) $$\sum_{\delta \in \Delta_k} e_\delta = 1.$$

Thus the elements e_δ are pairwise orthogonal idempotents. They form a basis of $\mathbb{F}A'_k$ and each e_δ spans a one-dimensional ideal of $\mathbb{F}A'_k$. Within the larger group algebras $\mathbb{F}A_k$ and $\mathbb{F}A$, the e_δ are central idempotents. For each δ, let $I_\delta = (\mathbb{F}A_k)e_\delta$. Thus I_δ is the (two-sided) ideal of $\mathbb{F}A_k$ generated by e_δ. By (5.3), (5.4) and (5.5),

(5.6) $$\mathbb{F}A_k = \bigoplus_{\delta \in \Delta_k} I_\delta.$$

It follows from (5.6) and (5.2) that $\mathbb{F}A_k$ is spanned by all elements of the form $x_1^{\alpha_1} \cdots x_k^{\alpha_k} e_\delta$ with $\delta \in \Delta_k$ and $\alpha_i \in \mathbb{Z}/n\mathbb{Z}$ for $i = 1, \ldots, k$. It is easily checked that there are exactly $|A_k|$ such elements. Hence they form a basis for $\mathbb{F}A_k$ and, for fixed δ, the elements $x_1^{\alpha_1} \cdots x_k^{\alpha_k} e_\delta$ form a basis for I_δ.

If $\psi : A_k \to A_l$ is a homomorphism, where $k, l \in \mathbb{N}$, then ψ extends to a homomorphism $\mathbb{F}A_k \to \mathbb{F}A_l$, which we also denote by ψ. In particular, $\psi : A_k \to A_k$ extends to $\psi : \mathbb{F}A_k \to \mathbb{F}A_k$.

For each k, write $\tilde{A}_k = A_k / A'_k (A_k)^{n_0}$ and, for $a \in A_k$, write $\tilde{a} = a A'_k (A_k)^{n_0} \in \tilde{A}_k$. Thus $\tilde{A}_k$ is a free abelian group of exponent n_0 with basis $\{\tilde{x}_1, \ldots, \tilde{x}_k\}$. We shall usually think of $\tilde{A}_k$ in additive notation: thus we may regard it as a free K-module.

If $\psi : A_k \to A_l$ is a homomorphism, we write $\tilde{\psi}$ for the induced homomorphism from $\tilde{A}_k$ to $\tilde{A}_l$. In particular, if $\eta \in \mathrm{Aut}(A_k)$ then $\tilde{\eta} \in \mathrm{Aut}(\tilde{A}_k)$.

For each $\delta \in \Delta_k$, let θ_δ be the alternating K-form on $\tilde{A}_k$ satisfying $\theta_\delta(\tilde{x}_i, \tilde{x}_j) = \delta(i, j)$ for all $(i, j) \in Q_k$. Clearly every alternating K-form on $\tilde{A}_k$ arises in this way from some δ. Since $\chi_\delta([x_i, x_j]) = \omega^{\delta(i,j)}$ it is straightforward to verify that

(5.7) $$\chi_\delta([a_1, a_2]) = \omega^{\theta_\delta(\tilde{a}_1, \tilde{a}_2)} \text{ for all } a_1, a_2 \in A_k.$$

LEMMA 5.1. *Let $\delta \in \Delta_k$ and $\eta \in \mathrm{Aut}(A_k)$. Then $e_\delta \eta = e_\varepsilon$ where $\varepsilon \in \Delta_k$ and $\theta_\varepsilon(\tilde{a}_1, \tilde{a}_2) = \theta_\delta(\tilde{a}_1 \tilde{\eta}^{-1}, \tilde{a}_2 \tilde{\eta}^{-1})$ for all $a_1, a_2 \in A_k$.*

PROOF. The map $a \mapsto \chi_\delta(a\eta^{-1})$ is a homomorphism from A'_k to $\mathbb{F}^\times$. Hence there exists $\varepsilon \in \Delta_k$ such that $\chi_\varepsilon(a) = \chi_\delta(a\eta^{-1})$ for all $a \in A'_k$. By direct calculation we obtain $e_\delta \eta = e_\varepsilon$. Also, for all $a_1, a_2 \in A_k$, (5.7) gives $\omega^{\theta_\varepsilon(\tilde{a}_1, \tilde{a}_2)} = \chi_\varepsilon([a_1, a_2]) = \chi_\delta([a_1, a_2]\eta^{-1}) = \chi_\delta([a_1\eta^{-1}, a_2\eta^{-1}]) = \omega^{\theta_\delta(\tilde{a}_1\tilde{\eta}^{-1}, \tilde{a}_2\tilde{\eta}^{-1})}$. The result follows. □

LEMMA 5.2. *Let $\delta \in \Delta_k$ and $\varepsilon \in \Delta_l$, where $k, l \in \mathbb{N}$. Let $\psi : A_k \to A_l$ be a homomorphism which induces a homomorphism of K-forms from $(\tilde{A}_k, \theta_\delta)$ to $(\tilde{A}_l, \theta_\varepsilon)$ (that is, $\theta_\delta(\tilde{a}_1, \tilde{a}_2) = \theta_\varepsilon(\tilde{a}_1\tilde{\psi}, \tilde{a}_2\tilde{\psi})$ for all $a_1, a_2 \in A_k$). Then $(e_\delta \psi)e_\varepsilon = e_\varepsilon$.*

PROOF. For all $a_1, a_2 \in A_k$,

$$\chi_\delta([a_1, a_2]) = \omega^{\theta_\delta(\tilde{a}_1, \tilde{a}_2)} = \omega^{\theta_\varepsilon(\tilde{a}_1\tilde{\psi}, \tilde{a}_2\tilde{\psi})} = \chi_\varepsilon([a_1\psi, a_2\psi]) = \chi_\varepsilon([a_1, a_2]\psi).$$

It follows that $\chi_\delta(w) = \chi_\varepsilon(w\psi)$ for all $w \in \mathbb{F}A_k'$. Therefore, by (5.2) and (5.3),

$$(e_\delta\psi)e_\varepsilon = \chi_\varepsilon(e_\delta\psi)e_\varepsilon = \chi_\delta(e_\delta)e_\varepsilon = e_\varepsilon. \qquad \square$$

For each k, we consider $\mathbb{F}(A_k \times A_k)$, identified with $\mathbb{F}A_k \otimes_{\mathbb{F}} \mathbb{F}A_k$. If $\psi : A_k \to A_l$ is a homomorphism, then ψ yields homomorphisms $\psi : A_k \times A_k \to A_l \times A_l$ and $\psi : \mathbb{F}(A_k \times A_k) \to \mathbb{F}(A_l \times A_l)$. For $\delta, \delta' \in \Delta_k$, we write $e_\delta \otimes e_{\delta'}$ as $e_{\delta\delta'}$ and $I_\delta \otimes I_{\delta'}$ as $I_{\delta\delta'}$. Thus, by (5.6),

$$\mathbb{F}(A_k \times A_k) = \bigoplus_{\delta,\delta' \in \Delta_k} I_{\delta\delta'}. \tag{5.8}$$

Also, $I_{\delta\delta'}$ is the ideal of $\mathbb{F}(A_k \times A_k)$ generated by the central idempotent $e_{\delta\delta'}$, and $\sum_{\delta,\delta'} e_{\delta\delta'} = 1$.

For $\delta, \delta' \in \Delta_k$, let $\theta_{\delta\delta'}$ be the alternating $K \oplus K$-form on $\tilde{A}_k$ determined by $\theta_{\delta\delta'}(\tilde{x}_i, \tilde{x}_j) = (\theta_\delta(\tilde{x}_i, \tilde{x}_j), \theta_{\delta'}(\tilde{x}_i, \tilde{x}_j))$ for all $(i, j) \in Q_k$. Every alternating $K \oplus K$-form on $\tilde{A}_k$ arises in this way from some δ, δ'.

The following two results are easily deduced from Lemma 5.1 and Lemma 5.2, respectively.

LEMMA 5.3. *Let $\delta, \delta' \in \Delta_k$ and $\eta \in \mathrm{Aut}(A_k)$. Then $e_{\delta\delta'}\eta = e_{\varepsilon\varepsilon'}$ where $\varepsilon, \varepsilon' \in \Delta_k$ and $\theta_{\varepsilon\varepsilon'}(\tilde{a}_1, \tilde{a}_2) = \theta_{\delta\delta'}(\tilde{a}_1\tilde{\eta}^{-1}, \tilde{a}_2\tilde{\eta}^{-1})$ for all $a_1, a_2 \in A_k$.*

LEMMA 5.4. *Let $\delta, \delta' \in \Delta_k$ and $\varepsilon, \varepsilon' \in \Delta_l$, where $k, l \in \mathbb{N}$. Let $\psi : A_k \to A_l$ be a homomorphism which induces a homomorphism of $K \oplus K$-forms from $(\tilde{A}_k, \theta_{\delta\delta'})$ to $(\tilde{A}_l, \theta_{\varepsilon\varepsilon'})$. Then $(e_{\delta\delta'}\psi)e_{\varepsilon\varepsilon'} = e_{\varepsilon\varepsilon'}$.*

Let $N = n_0(2n_0^4 + n_0^2)$. By Lemma 3.3, every $K \oplus K$-form is N-regular with respect to some basis. For $\delta, \delta' \in \Delta_k$, we say that $\theta_{\delta\delta'}$ is *regular* if it is N-regular with respect to the basis $\{\tilde{x}_1, \dots, \tilde{x}_k\}$ of $\tilde{A}_k$.

LEMMA 5.5. *Let $\delta, \delta' \in \Delta_k$. Then there exists $\eta \in \mathrm{Aut}(A_k)$ such that $e_{\delta\delta'}\eta = e_{\varepsilon\varepsilon'}$ where $\varepsilon, \varepsilon' \in \Delta_k$ and $\theta_{\varepsilon\varepsilon'}$ is regular.*

PROOF. By Lemma 3.3, there is a basis $\{\tilde{a}_1, \dots, \tilde{a}_k\}$ of $\tilde{A}_k$ such that $(\tilde{A}_k, \theta_{\delta\delta'})$ is N-regular with respect to this basis. It is easily verified that there exists a generating set $\{y_1, \dots, y_k\}$ of A_k such that $\tilde{y}_i = \tilde{a}_i$ for $i = 1, \dots, k$. Since A_k is a finite relatively free group of rank k, it follows that $\{y_1, \dots, y_k\}$ is a free generating set. Let η be the automorphism of A_k satisfying $y_i\eta = x_i$ for $i = 1, \dots, k$. By Lemma 5.3, $e_{\delta\delta'}\eta = e_{\varepsilon\varepsilon'}$ where $\theta_{\varepsilon\varepsilon'}(\tilde{x}_i, \tilde{x}_j) = \theta_{\delta\delta'}(\tilde{y}_i, \tilde{y}_j)$ for all $(i, j) \in Q_k$. Thus $\theta_{\varepsilon\varepsilon'}$ is regular. $\square$

LEMMA 5.6. *Let $\delta, \delta' \in \Delta_k$ and $\varepsilon, \varepsilon' \in \Delta_l$, where $k, l \in \mathbb{N}$, and consider $I_{\delta\delta'}$ and $I_{\varepsilon\varepsilon'}$ as subsets of $\mathbb{F}(A \times A)$. Then $I_{\delta\delta'} \cap I_{\varepsilon\varepsilon'} = \{0\}$ unless $k = l$, $\delta = \delta'$ and $\varepsilon = \varepsilon'$.*

PROOF. Suppose that $k < l$. It is easily verified that $([x_{l-1}, x_l] \otimes 1)w \notin \mathbb{F}(A_k \times A_k)$ for all $w \in \mathbb{F}(A_k \times A_k) \setminus \{0\}$. On the other hand, for all $v \in I_{\varepsilon\varepsilon'}$, the element $([x_{l-1}, x_l] \otimes 1)v$ is a scalar multiple of v by (5.2). Thus $I_{\delta\delta'} \cap I_{\varepsilon\varepsilon'} = \{0\}$. If $k = l$ then $I_{\delta\delta'} \cap I_{\varepsilon\varepsilon'} \neq \{0\}$ implies $\delta = \delta'$ and $\varepsilon = \varepsilon'$ by (5.8). □

A non-zero element w of $\mathbb{F}(A \times A)$ will be called *regular* if $w \in I_{\delta\delta'}$ for some k and some $\delta, \delta' \in \Delta_k$ such that $\theta_{\delta\delta'}$ is regular. (By Lemma 5.6, k, δ and δ' are then unique.)

LEMMA 5.7. *Every Ψ-closed left ideal of $\mathbb{F}(A \times A)$ is generated, as a Ψ-closed vector space, by regular elements.*

PROOF. Let J be a Ψ-closed left ideal of $\mathbb{F}(A \times A)$ and let J_0 be the vector space spanned by all elements $v\psi$ where v is a regular element of J and $\psi \in \Psi$. It suffices to show that $J = J_0$. Clearly $J_0 \subseteq J$. Let $w \in J$. Then $w \in \mathbb{F}(A_k \times A_k)$ for some k, and we have $w = \left(\sum_{\delta,\delta' \in \Delta_k} e_{\delta\delta'}\right) w = \sum_{\delta,\delta' \in \Delta_k} (e_{\delta\delta'} w)$, where $e_{\delta\delta'} w \in J \cap I_{\delta\delta'}$. It suffices to show that $e_{\delta\delta'} w \in J_0$. Clearly we may assume that $e_{\delta\delta'} w \neq 0$. By Lemma 5.5, there exists $\eta \in \mathrm{Aut}(A_k)$ such that $(e_{\delta\delta'} w)\eta$ is regular. But $(e_{\delta\delta'} w)\eta \in J$, since η extends to an automorphism of A. Thus $e_{\delta\delta'} w = (e_{\delta\delta'} w)\eta\eta^{-1} \in J_0$. □

Let $\delta, \delta' \in \Delta_k$. Since the elements $x_1^{\alpha_1} \cdots x_k^{\alpha_k} e_\delta$ with $\alpha_i \in \mathbb{Z}/n\mathbb{Z}$ form a basis of I_δ, the elements

$$(5.9) \qquad \left(x_1^{\alpha_1} \cdots x_k^{\alpha_k} \otimes x_1^{\alpha'_1} \cdots x_k^{\alpha'_k}\right) e_{\delta\delta'},$$

with $\alpha_i, \alpha'_i \in \mathbb{Z}/n\mathbb{Z}$, form a basis of $I_{\delta\delta'}$.

An element of $\mathbb{F}(A \times A)$ will be called a *monomial* if it has the form (5.9) for some k and some $\delta, \delta' \in \Delta_k$, and a *regular monomial* if $\theta_{\delta\delta'}$ is regular. We write $\mathcal{M}$ for the set of all monomials, $\mathcal{M}^*$ for the set of all regular monomials, and $\mathcal{M}_{\delta\delta'}$ for the set of all monomials of $I_{\delta\delta'}$.

Let $T = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, that is, the Cartesian square of the set $\mathbb{Z}/n\mathbb{Z}$. With the monomial (5.9) we associate the k-tuple $(t_1, t_2, \dots, t_k)$ where $t_i = (\alpha_i, \alpha'_i) \in T$ for $i = 1, \dots, k$. Let $\leq$ be a total order on T which is arbitrary except that $(0, 0) \leq t$ for all $t \in T$. Then the set of all k-tuples of elements of T can be ordered lexicographically from the right: if $\mathbf{t} = (t_1, \dots, t_k)$ and $\mathbf{t}' = (t'_1, \dots, t'_k)$ are two such k-tuples, we set $\mathbf{t} < \mathbf{t}'$ if there exists $q \in \{1, \dots, k\}$ such that $t_q < t'_q$ but $t_i = t'_i$ for $i = q + 1, \dots, k$. Hence, for $\delta, \delta' \in \Delta_k$, we obtain an order $\leq$ on the finite set $\mathcal{M}_{\delta\delta'}$.

Each non-zero element f of $I_{\delta\delta'}$ can be written (uniquely) in the form $f = \lambda_1 w_1 + \cdots + \lambda_r w_r$, where $w_1, \dots, w_r \in \mathcal{M}_{\delta\delta'}$, $w_1 > \cdots > w_r$, and $\lambda_1, \dots, \lambda_r \in \mathbb{F} \setminus \{0\}$. The largest monomial w_1 is called the *leading monomial* of f, and we write $w_1 = \mathrm{lead}(f)$.

We shall now define a quasi-order on $\mathscr{M}$. Let $\delta, \delta' \in \Delta_k$ and $\varepsilon, \varepsilon' \in \Delta_l$. Let $v \in \mathscr{M}_{\delta\delta'}$ and $w \in \mathscr{M}_{\varepsilon\varepsilon'}$, where

$$v = \left(x_1^{\alpha_1} \cdots x_k^{\alpha_k} \otimes x_1^{\alpha'_1} \cdots x_k^{\alpha'_k}\right) e_{\delta\delta'}, \quad w = \left(x_1^{\beta_1} \cdots x_l^{\beta_l} \otimes x_1^{\beta'_1} \cdots x_l^{\beta'_l}\right) e_{\varepsilon\varepsilon'}.$$

We write $v \preccurlyeq w$ if there is a one-one order-preserving map $\phi : \{1, \ldots, k\} \to \{1, \ldots, l\}$ together with a homomorphism $\psi : A_k \to A_l$ with the following three properties.

(i) For $i = 1, \ldots, k$, we have $x_i\psi = z_i x_{i\phi}$ for some $z_i \in \langle x_1, \ldots, x_{i\phi-1}\rangle$.
(ii) ψ induces a homomorphism of $K \oplus K$-forms from $(\tilde{A}_k, \theta_{\delta\delta'})$ to $(\tilde{A}_l, \theta_{\varepsilon\varepsilon'})$.
(iii) For $i = 1, \ldots, k$, we have $\alpha_i = \beta_{i\phi}$ and $\alpha'_i = \beta'_{i\phi}$.

It is straightforward to check that $(\mathscr{M}, \preccurlyeq)$ is a quasi-ordered set. Thus $(\mathscr{M}^*, \preccurlyeq)$ is quasi-ordered. Let $\mathscr{Z}$ be the set of all N-regular $(K \oplus K, T)$-forms as defined in Section 3 with $S = K \oplus K$. Thus, by Proposition 3.7, $(\mathscr{Z}, \preccurlyeq)$ is well-quasi-ordered, where $\preccurlyeq$ is as defined in Section 3. Let $v \in \mathscr{M}^*$, where

$$v = \left(x_1^{\alpha_1} \cdots x_k^{\alpha_k} \otimes x_1^{\alpha'_1} \cdots x_k^{\alpha'_k}\right) e_{\delta\delta'},$$

with $\delta, \delta' \in \Delta_k$ and $\theta_{\delta\delta'}$ regular. Then we can define $Z(v) \in \mathscr{Z}$ by

$$Z(v) = \left(\tilde{A}_k, \theta_{\delta\delta'}, \{\tilde{x}_1, \ldots, \tilde{x}_k\}, \mathbf{t}\right),$$

where $\mathbf{t} = ((\alpha_1, \alpha'_1), \ldots, (\alpha_k, \alpha'_k))$. It is straightforward to verify that if v and w are elements of $\mathscr{M}^*$ such that $Z(v) \preccurlyeq Z(w)$ then $v \preccurlyeq w$. Hence Proposition 3.7 gives the following result.

PROPOSITION 5.8. *The set $(\mathscr{M}^*, \preccurlyeq)$ is well-quasi-ordered.*

If S is any set of elements of $\mathbb{F}(A \times A)$ we write $L_\Psi(S)$ for the Ψ-closed left ideal generated by S.

LEMMA 5.9. *Let $f \in I_{\delta\delta'} \setminus \{0\}$ and $g \in I_{\varepsilon\varepsilon'} \setminus \{0\}$ where $\delta, \delta' \in \Delta_k$ and $\varepsilon, \varepsilon' \in \Delta_l$. Suppose that $\mathrm{lead}(f) \preccurlyeq \mathrm{lead}(g)$. Then there exists $g^* \in I_{\varepsilon\varepsilon'}$ such that $L_\Psi\{f, g\} = L_\Psi\{f, g^*\}$ and either $g^* = 0$ or $\mathrm{lead}(g^*) < \mathrm{lead}(g)$.*

PROOF. Write $f = \lambda_1 v_1 + \cdots + \lambda_r v_r$, where $v_i \in \mathscr{M}_{\delta\delta'}$ and $\lambda_i \in \mathbb{F} \setminus \{0\}$ for all i, and where $v_i < v_1$ for all $i \geq 2$. Similarly, write $g = \mu_1 w_1 + \cdots + \mu_s w_s$, where $w_i \in \mathscr{M}_{\varepsilon\varepsilon'}$ and $\mu_i \in \mathbb{F} \setminus \{0\}$ for all i, and where $w_i < w_1$ for all $i \geq 2$. Write $v = v_1 = \mathrm{lead}(f)$ and $w = w_1 = \mathrm{lead}(g)$. Thus $v \preccurlyeq w$. We use the notation for v and w given in the definition of $\preccurlyeq$. Let ϕ and ψ be as in that definition. Let h_1 and h_2 be the elements of $\mathbb{F}(A_l \times A_l)$ defined by

$$h_1 = \left(x_{1\phi}^{\alpha_1} \cdots x_{k\phi}^{\alpha_k} \otimes x_{1\phi}^{\alpha'_1} \cdots x_{k\phi}^{\alpha'_k}\right) \left(\left(x_1^{\alpha_1} \cdots x_k^{\alpha_k} \otimes x_1^{\alpha'_1} \cdots x_k^{\alpha'_k}\right)^{-1} \psi\right)$$

and $h_2 = \prod_{j\in C} x_j^{\beta_j} \otimes \prod_{j\in C} x_j^{\beta'_j}$, where $C = \{1, \dots, l\} \setminus \{1\phi, \dots, k\phi\}$. Then

$$h_1(v\psi) = \big(x_{1\phi}^{\alpha_1} \cdots x_{k\phi}^{\alpha_k} \otimes x_{1\phi}^{\alpha'_1} \cdots x_{k\phi}^{\alpha'_k}\big)(e_{\delta\delta'}\psi),$$

and so, by Lemma 5.4,

$$h_1(v\psi)e_{\varepsilon\varepsilon'} = \big(x_{1\phi}^{\alpha_1} \cdots x_{k\phi}^{\alpha_k} \otimes x_{1\phi}^{\alpha'_1} \cdots x_{k\phi}^{\alpha'_k}\big)e_{\varepsilon\varepsilon'} = \big(x_{1\phi}^{\beta_{1\phi}} \cdots x_{k\phi}^{\beta_{k\phi}} \otimes x_{1\phi}^{\beta'_{1\phi}} \cdots x_{k\phi}^{\beta'_{k\phi}}\big)e_{\varepsilon\varepsilon'}.$$

Therefore $h_2h_1(v\psi)e_{\varepsilon\varepsilon'} = \big(x_1^{\beta_1} \cdots x_l^{\beta_l} \otimes x_1^{\beta'_1} \cdots x_l^{\beta'_l}\big)(a \otimes a')e_{\varepsilon\varepsilon'}$, where $a, a' \in A'_l$. By (5.2), $(a \otimes a')e_{\varepsilon\varepsilon'} = \lambda e_{\varepsilon\varepsilon'}$ where $\lambda \in \mathbb{F} \setminus \{0\}$. Hence

$$h_2h_1(v\psi)e_{\varepsilon\varepsilon'} = \lambda\big(x_1^{\beta_1} \cdots x_l^{\beta_l} \otimes x_1^{\beta'_1} \cdots x_l^{\beta'_l}\big)e_{\varepsilon\varepsilon'} = \lambda w.$$

Now let u be an element of $\mathscr{M}_{\delta\delta'}$ such that $u < v$. Write $u = \big(x_1^{\gamma_1} \cdots x_k^{\gamma_k} \otimes x_1^{\gamma'_1} \cdots x_k^{\gamma'_k}\big)e_{\delta\delta'}$. Thus there exists $q \in \{1, \dots, k\}$ such that $(\gamma_q, \gamma'_q) < (\alpha_q, \alpha'_q)$ but $(\gamma_i, \gamma'_i) = (\alpha_i, \alpha'_i)$ for $i = q+1, \dots, k$. We can write

$$\begin{aligned}\big(x_1^{\alpha_1} \cdots x_k^{\alpha_k} \otimes x_1^{\alpha'_1} \cdots x_k^{\alpha'_k}\big)^{-1}&\big(x_1^{\gamma_1} \cdots x_k^{\gamma_k} \otimes x_1^{\gamma'_1} \cdots x_k^{\gamma'_k}\big)\\ &= \big(x_1^{\gamma_1-\alpha_1} \cdots x_k^{\gamma_k-\alpha_k} \otimes x_1^{\gamma'_1-\alpha'_1} \cdots x_k^{\gamma'_k-\alpha'_k}\big)(b \otimes b')\end{aligned}$$

where $b, b' \in A'_k$. By (5.2), $(b \otimes b')\psi e_{\varepsilon\varepsilon'} = \nu e_{\varepsilon\varepsilon'}$ where $\nu \in \mathbb{F} \setminus \{0\}$. Hence

$$\begin{aligned}h_1(u\psi)e_{\varepsilon\varepsilon'} &= \nu(x_{1\phi}^{\alpha_1} \cdots x_{k\phi}^{\alpha_k} \otimes x_{1\phi}^{\alpha'_1} \cdots x_{k\phi}^{\alpha'_k})((x_1^{\gamma_1-\alpha_1} \cdots x_k^{\gamma_k-\alpha_k} \otimes x_1^{\gamma'_1-\alpha'_1} \cdots x_k^{\gamma'_k-\alpha'_k})\psi)e_{\varepsilon\varepsilon'}\\ &= \nu(x_{1\phi}^{\alpha_1} \cdots x_{k\phi}^{\alpha_k} \otimes x_{1\phi}^{\alpha'_1} \cdots x_{k\phi}^{\alpha'_k})((x_1^{\gamma_1-\alpha_1} \cdots x_q^{\gamma_q-\alpha_q} \otimes x_1^{\gamma'_1-\alpha'_1} \cdots x_q^{\gamma'_q-\alpha'_q})\psi)e_{\varepsilon\varepsilon'}.\end{aligned}$$

From the properties of ψ we calculate that

$$\begin{aligned}\Big(\big(x_1^{\gamma_1-\alpha_1} \cdots x_q^{\gamma_q-\alpha_q} &\otimes x_1^{\gamma'_1-\alpha'_1} \cdots x_q^{\gamma'_q-\alpha'_q}\big)\psi\Big)\, e_{\varepsilon\varepsilon'}\\ &= \nu'\big(x_1^{\rho_1} \cdots x_{q\phi-1}^{\rho_{q\phi-1}} x_{q\phi}^{\gamma_q-\alpha_q} \otimes x_1^{\rho'_1} \cdots x_{q\phi-1}^{\rho'_{q\phi-1}} x_{q\phi}^{\gamma'_q-\alpha'_q}\big)e_{\varepsilon\varepsilon'}\end{aligned}$$

where $\nu' \in \mathbb{F} \setminus \{0\}$ and $\rho_1, \dots, \rho_{q\phi-1}, \rho'_1, \dots, \rho'_{q\phi-1} \in \mathbb{Z}/n\mathbb{Z}$. Hence $h_1(u\psi)e_{\varepsilon\varepsilon'}$ has the form

$$\nu''\big(x_1^{\sigma_1} \cdots x_{q\phi-1}^{\sigma_{q\phi-1}} x_{q\phi}^{\gamma_q} x_{(q+1)\phi}^{\alpha_{q+1}} \cdots x_{k\phi}^{\alpha_k} \otimes x_1^{\sigma'_1} \cdots x_{q\phi-1}^{\sigma'_{q\phi-1}} x_{q\phi}^{\gamma'_q} x_{(q+1)\phi}^{\alpha'_{q+1}} \cdots x_{k\phi}^{\alpha'_k}\big)e_{\varepsilon\varepsilon'}$$

where $\nu'' \in \mathbb{F} \setminus \{0\}$ and $\sigma_1, \dots, \sigma_{q\phi-1}, \sigma'_1, \dots, \sigma'_{q\phi-1} \in \mathbb{Z}/n\mathbb{Z}$. Therefore $h_2h_1(u\psi)e_{\varepsilon\varepsilon'}$ is a non-zero scalar multiple of a monomial of the form

$$\big(x_1^{\tau_1} \cdots x_{q\phi-1}^{\tau_{q\phi-1}} x_{q\phi}^{\gamma_q} x_{q\phi+1}^{\beta_{q\phi+1}} \cdots x_l^{\beta_l} \otimes x_1^{\tau'_1} \cdots x_{q\phi-1}^{\tau'_{q\phi-1}} x_{q\phi}^{\gamma'_q} x_{q\phi+1}^{\beta'_{q\phi+1}} \cdots x_l^{\beta'_l}\big)e_{\varepsilon\varepsilon'}$$

where $\tau_1, \dots, \tau_{q\phi-1}, \tau'_1, \dots, \tau'_{q\phi-1} \in \mathbb{Z}/n\mathbb{Z}$. Since $(\gamma_q, \gamma'_q) < (\alpha_q, \alpha'_q) = (\beta_{q\phi}, \beta'_{q\phi})$, this monomial is smaller than w.

Since $h_2h_1(f\psi)e_{\varepsilon\varepsilon'} = \lambda_1h_2h_1(v_1\psi)e_{\varepsilon\varepsilon'} + \cdots + \lambda_rh_2h_1(v_r\psi)e_{\varepsilon\varepsilon'}$, we see that $h_2h_1(f\psi)e_{\varepsilon\varepsilon'}$ has leading monomial w with coefficient $\lambda_1\lambda$. Also, since ψ extends to an element of Ψ, we have $h_2h_1(f\psi)e_{\varepsilon\varepsilon'} \in L_\Psi\{f\}$. Let $g^* = g - \mu_1\lambda_1^{-1}\lambda^{-1}h_2h_1(f\psi)e_{\varepsilon\varepsilon'}$. Then g^* has the required properties. □

Now we are in a position to complete the proof of Theorem C. Let J be a Ψ-closed left ideal of $\mathbb{F}(A \times A)$. It suffices to prove that J is finitely generated as a Ψ-closed left ideal. By Proposition 5.8, there exists a finite set S of regular elements of J such that for every regular element g of J there exists $f \in S$ such that $\text{lead}(f) \preccurlyeq \text{lead}(g)$. We claim that $J = L_\Psi(S)$. By Lemma 5.7, it suffices to show that every regular element of J belongs to $L_\Psi(S)$. Suppose, in order to get a contradiction, that this is not so, and let g be a regular element of J such that $g \notin L_\Psi(S)$. Suppose $g \in I_{\varepsilon\varepsilon'}$. Choose g with the given properties such that $\text{lead}(g)$ is as small as possible in the finite set $(\mathscr{M}_{\varepsilon\varepsilon'}, \leq)$. There exists $f \in S$ such that $\text{lead}(f) \preccurlyeq \text{lead}(g)$. By Lemma 5.9, there exists $g^* \in I_{\varepsilon\varepsilon'}$ such that $L_\Psi\{f, g\} = L_\Psi\{f, g^*\}$ and either $g^* = 0$ or $\text{lead}(g^*) < \text{lead}(g)$. Since $g \notin L_\Psi\{f\}$, we have $g^* \neq 0$. Since $g^* \in L_\Psi\{f, g\} \subseteq J$, the choice of g gives that $g^* \in L_\Psi(S)$. Hence $g \in L_\Psi\{f, g^*\} \subseteq L_\Psi(S)$ and we have the required contradiction.

6. Proof of Theorem D

Let n, A, Ψ, $\mathbb{F}$ and R be as in Section 1, where $\mathbb{F}$ is a field of characteristic not dividing n. Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. The subalgebra $\overline{\mathbb{F}} \otimes_{\mathbb{F}} R$ of $\overline{\mathbb{F}}(A \times A)$ corresponds to R in $\mathbb{F}(A \times A)$. If M is an (R, Ψ)-submodule of $\mathbb{F}(A \times A)$ which contains R, then $\overline{\mathbb{F}} \otimes_{\mathbb{F}} M$ is an $(\overline{\mathbb{F}} \otimes_{\mathbb{F}} R, \Psi)$-submodule of $\overline{\mathbb{F}}(A \times A)$ which contains $\overline{\mathbb{F}} \otimes_{\mathbb{F}} R$, and $M = \mathbb{F}(A \times A) \cap \overline{\mathbb{F}} \otimes_{\mathbb{F}} M$. Therefore, to prove Theorem D, we may assume that $\overline{\mathbb{F}} = \mathbb{F}$.

We shall use the notation of Section 5. If $n \leq 2$, then Theorem D follows from Proposition 4.5. Thus, as in Section 5, we assume that $n > 2$, so that $n_0 > 1$.

Let P be the subgroup of $A \times A$ defined by $P = \{(c, c^{-1}) : c \in A'\}$ and let $H = (A \times A)/P$. Note that P is a Ψ-closed subgroup of $A \times A$, so each element of Ψ induces endomorphisms of H and $\mathbb{F}H$. For $i, j \in \mathbb{N}$, let c_{ij} be the element of H given by $c_{ij} = ([x_i, x_j], 1)P = (1, [x_i, x_j])P$.

For each positive integer k, let H_k be the subgroup of H generated by the elements $(x_i, 1)P$ and $(1, x_i)P$ for $i = 1, \ldots, k$. It is easily verified that H_k' is a free abelian group of exponent n_0 with basis $\{c_{ij} : 1 \leq i < j \leq k\}$. Furthermore, there are isomorphisms from A_k' to H_k' and from $\mathbb{F}A_k'$ to $\mathbb{F}H_k'$ given by $[x_i, x_j] \mapsto c_{ij}$ for all i, j. If $\psi : A_k \to A_l$ is a homomorphism, where $k, l \in \mathbb{N}$, then the associated homomorphism $\psi : A_k \times A_k \to A_l \times A_l$ yields homomorphisms $\overline{\psi} : H_k \to H_l$ and $\overline{\psi} : \mathbb{F}H_k \to \mathbb{F}H_l$.

Let $K = \mathbb{Z}/n_0\mathbb{Z}$, and let Q_k and Δ_k be as in Section 5. For each $\delta \in \Delta_k$, let χ_δ and e_δ be defined as in Section 5, but with respect to H_k' rather than A_k'. Thus χ_δ is a character of H_k' and e_δ is an idempotent of $\mathbb{F}H_k'$. Results (5.2)–(5.5) apply just as before. For $\delta \in \Delta_k$ we define $J_\delta = (\mathbb{F}H_k)e_\delta$. Thus J_δ is the ideal of $\mathbb{F}H_k$ generated by e_δ, and we have $\mathbb{F}H_k = \bigoplus_{\delta \in \Delta_k} J_\delta$.

For each k we write $Q_k^0 = Q_k \setminus \{(1,2),(3,4),\dots\}$. An element δ of Δ_k will be called *standard* if $\delta(i,j) = 0$ (equivalently, $\chi_\delta(c_{ij}) = 1$) for all $(i,j) \in Q_k^0$. We write Δ_k^* for the set of all standard elements of Δ_k and $\Delta_k^0 = \Delta_k \setminus \Delta_k^*$.

For each $\delta \in \Delta_k$, let θ_δ be the alternating K-form on $\tilde{A}_k$ defined as in Section 5. Thus $(\tilde{A}_k, \theta_\delta)$ is standard with respect to $\{\tilde{x}_1, \dots, \tilde{x}_k\}$ (in the terminology of Section 3) if and only if δ is standard, that is, $\delta \in \Delta_k^*$.

LEMMA 6.1. *Let $\delta \in \Delta_k$. Then there exists $\eta \in \mathrm{Aut}(A_k)$ such that, for the induced automorphism $\overline{\eta} : \mathbb{F}H_k \to \mathbb{F}H_k$, we have $e_\delta \overline{\eta} = e_\varepsilon$ where $\varepsilon \in \Delta_k^*$.*

PROOF. By Lemma 3.8 there is a basis $\{\tilde{a}_1, \dots, \tilde{a}_k\}$ of $\tilde{A}_k$ such that $(\tilde{A}_k, \theta_\delta)$ is standard with respect to this basis. As in the proof of Lemma 5.5, there is a free generating set $\{y_1, \dots, y_k\}$ of A_k such that $\tilde{y}_i = \tilde{a}_i$ for $i = 1, \dots, k$. Let η be the automorphism of A_k satisfying $y_i \eta = x_i$ for $i = 1, \dots, k$. Note that η acts on A_k' just as $\overline{\eta}$ acts on H_k'. Thus Lemma 5.1 shows that $e_\delta \overline{\eta} = e_\varepsilon$, where $\varepsilon \in \Delta_k$ and $\theta_\varepsilon(\tilde{x}_i, \tilde{x}_j) = \theta_\delta(\tilde{y}_i, \tilde{y}_j)$ for all i, j. Thus $(\tilde{A}_k, \theta_\varepsilon)$ is standard with respect to $\{\tilde{x}_1, \dots, \tilde{x}_k\}$, that is, $\varepsilon \in \Delta_k^*$. □

Since $\mathbb{F}H'$ is a subalgebra of $\mathbb{F}H$, we may regard $\mathbb{F}H$ as a left $\mathbb{F}H'$-module. Following the terminology of Section 1, we shall consider $(\mathbb{F}H', \Psi)$-submodules of $\mathbb{F}H$. A non-zero element w of $\mathbb{F}H$ will be called *standard* if $w \in J_\delta$ for some k and some $\delta \in \Delta_k^*$.

LEMMA 6.2. *Every $(\mathbb{F}H', \Psi)$-submodule of $\mathbb{F}H$ is generated, as a Ψ-closed vector space, by standard elements.*

PROOF. This is similar to the proof of Lemma 5.7, with Lemma 6.1 taking the place of Lemma 5.5. □

Let C be the subgroup of H generated by all elements c_{ij} for which $i < j$ and $(i,j) \notin \{(1,2),(3,4),\dots\}$. Let ρ be the natural homomorphism $\rho : H \to H/C$. We also denote by ρ the associated homomorphisms $H_k \to H/C$ and $\mathbb{F}H_k \to \mathbb{F}(H/C)$. Clearly the kernel of $\rho : H_k \to H/C$ is the subgroup of H_k generated by all c_{ij} for which $(i,j) \in Q_k^0$. Thus the kernel of $\rho : \mathbb{F}H_k \to \mathbb{F}(H/C)$ is the ideal generated by the elements $c_{ij} - 1$ for $(i,j) \in Q_k^0$. We write $(\mathbb{F}H_k)^* = \bigoplus_{\delta \in \Delta_k^*} J_\delta$ and $(\mathbb{F}H_k)^0 = \bigoplus_{\delta \in \Delta_k^0} J_\delta$.

LEMMA 6.3. *The kernel of $\rho : \mathbb{F}H_k \to \mathbb{F}(H/C)$ is $(\mathbb{F}H_k)^0$.*

PROOF. Let $\delta \in \Delta_k^0$. Then $\chi_\delta(c_{ij}) \neq 1$ for some $(i,j) \in Q_k^0$. By (5.2), $(c_{ij} - 1)e_\delta$ is a non-zero scalar multiple of e_δ. But clearly $(c_{ij} - 1)e_\delta \in \ker(\rho)$. Thus $e_\delta \in \ker(\rho)$. It follows that $J_\delta \subseteq \ker(\rho)$ and so $(\mathbb{F}H_k)^0 \subseteq \ker(\rho)$.

Let $(i, j) \in Q_k^0$. Then, for $\varepsilon \in \Delta_k^*$, we have $(c_{ij} - 1)e_\varepsilon = (\chi_\varepsilon(c_{ij}) - 1)e_\varepsilon = 0$. Hence $c_{ij} - 1 = (c_{ij} - 1)\sum_{\delta\in\Delta_k} e_\delta = (c_{ij} - 1)\sum_{\delta\in\Delta_k^0} e_\delta$. Hence $c_{ij} - 1$ belongs to the ideal $(\mathbb{F}H_k)^0$. Since this holds for all $(i, j) \in Q_k^0$ we obtain $\ker(\rho) \subseteq (\mathbb{F}H_k)^0$. □

For $k \in \mathbb{N}$, let ψ_k be the endomorphism of A defined by $x_i\psi_k = 1$ for $i > k$ and $x_i\psi_k = x_i$ for $i \leq k$. Also write ψ_k for the induced endomorphisms of H and $\mathbb{F}H$.

LEMMA 6.4. *Let $u \in (\mathbb{F}H_k)^*$ and let $l \geq k$. Then there exists $v \in (\mathbb{F}H_l)^*$ such that $v\psi_k = u$ and $v\rho = u\rho$.*

PROOF. Let B be the subgroup of H_l' generated by all elements c_{ij} for $(i, j) \in Q_l^0 \setminus Q_k^0$. Let $v = u(|B|^{-1}\sum_{h\in B} h)$. Clearly $v\psi_k = u$ and $v\rho = u\rho$. To prove that $v \in (\mathbb{F}H_l)^*$ it is enough to show that $ve_\varepsilon = 0$ for all $\varepsilon \in \Delta_l^0$.

Let $\varepsilon \in \Delta_l^0$. Then there exists $(i, j) \in Q_l^0$ such that $\chi_\varepsilon(c_{ij}) \neq 1$. We consider two cases. Suppose first that $(i, j) \in Q_k^0$. Then the restriction of χ_ε to H_k' has the form $\chi_{\delta'}$ for some $\delta' \in \Delta_k^0$. Then for all $\delta \in \Delta_k^*$ we have $e_\delta e_\varepsilon = \chi_{\delta'}(e_\delta)e_\varepsilon = 0$, by (5.2) and (5.4). Hence $ue_\varepsilon = 0$ and so $ve_\varepsilon = 0$. Suppose secondly that $(i, j) \in Q_l^0 \setminus Q_k^0$. Then $\sum_{h\in B} h$ can be written as $w(1 + c_{ij} + \cdots + c_{ij}^{n_0})$ for some $w \in \mathbb{F}H_l'$. Since $\chi_\varepsilon(c_{ij})$ is a non-trivial n_0-th root of unity, $\chi_\varepsilon(1+c_{ij}+\cdots+c_{ij}^{n_0}) = 0$. Thus $(1+c_{ij}+\cdots+c_{ij}^{n_0})e_\varepsilon = 0$ and so $ve_\varepsilon = 0$. □

LEMMA 6.5. *Suppose that M_1 and M_2 are $(\mathbb{F}H', \Psi)$-submodules of $\mathbb{F}H$ such that $M_1\rho = M_2\rho$. Then $M_1 = M_2$.*

PROOF. Suppose, in order to get a contradiction, that $M_1 \neq M_2$. Without loss of generality we may assume that $M_1 \not\subseteq M_2$. By Lemma 6.2 there exist k and $\delta \in \Delta_k^*$ such that $M_1 \cap J_\delta \not\subseteq M_2$. Hence there exists $u \in (\mathbb{F}H_k)^*$ such that $u \in M_1 \setminus M_2$. By hypothesis there exists $w \in M_2$ such that $u\rho = w\rho$. Choose $l \geq k$ such that $w \in \mathbb{F}H_l$. Then $w = w^* + w^0$ where $w^* \in (\mathbb{F}H_l)^*$ and $w^0 \in (\mathbb{F}H_l)^0$. Since M_2 is an $\mathbb{F}H'$-module, $w^* \in M_2$. Also $u\rho = w\rho = w^*\rho$. By Lemma 6.4 there exists $v \in (\mathbb{F}H_l)^*$ such that $v\psi_k = u$ and $v\rho = u\rho$. Thus $v\rho = w^*\rho$. By Lemma 6.3, this gives $v = w^* \in M_2$. Hence $u = v\psi_k \in M_2$, which is a contradiction. □

Now we return to the group H/C. Recall that $H = (A \times A)/P$. For each $i \in \mathbb{N}$, let G_i be the subgroup of H/C generated by the four elements $((x_{2i-1}, 1)P)\rho$, $((1, x_{2i-1})P)\rho$, $((x_{2i}, 1)P)\rho$ and $((1, x_{2i})P)\rho$. Write $G = G_1$. Thus G is a finite group. It is easily verified that H/C is the direct product of the groups G_i, and, for each i, there is an obvious isomorphism from G to G_i. Thus we may identify H/C with the direct power D of G considered in Section 4. Let Ξ be the set of endomorphisms of D defined in Section 4.

LEMMA 6.6. *Let M be a Ψ-closed subspace of $\mathbb{F}H$. Then $M\rho$ is a Ξ-closed subspace of $\mathbb{F}D$.*

PROOF. Let $\xi \in \Xi$ and suppose that ξ is associated with ϕ, X and σ in the notation of Section 4. It suffices to show that there exists an endomorphism ψ of A such that the induced endomorphism of H leaves C invariant and induces ξ on H/C. To simplify the notation we rewrite the generators of A by setting $y_i = x_{2i-1}$ and $z_i = x_{2i}$ for all $i \in \mathbb{N}$. We define a homomorphism $\psi : A \to A$ by

$$y_i\psi = y_{i\phi} \prod_{\substack{j \in X \\ j\sigma = i\phi}} y_j, \quad z_i\psi = z_{i\phi} \prod_{\substack{j \in X \\ j\sigma = i\phi}} z_j,$$

for each i. The products are taken over all those values of j, if any, which lie in X and satisfy $j\sigma = i\phi$, and the terms y_j and z_j are taken in increasing order of j (this is an arbitrary choice). It is straightforward to verify that ψ has the required properties. $\square$

By Proposition 4.4 together with Lemma 6.5 and Lemma 6.6 we obtain

LEMMA 6.7. *The maximal condition holds for $(\mathbb{F}H', \Psi)$-submodules of $\mathbb{F}H$.*

Consider the natural homomorphism $\pi : A \times A \to H$ with kernel P, and let I be the kernel of the corresponding homomorphism $\pi : \mathbb{F}(A \times A) \to \mathbb{F}H$.

LEMMA 6.8. *The maximal condition holds for (R, Ψ)-submodules of $\mathbb{F}(A \times A)$ which contain I.*

PROOF. By Lemma 6.7 it suffices to show that if M is an (R, Ψ)-submodule of $\mathbb{F}(A \times A)$ which contains I then $M\pi$ is an $(\mathbb{F}H', \Psi)$-submodule of $\mathbb{F}H$. It is clear that $M\pi$ is Ψ-closed, by definition of the action of Ψ on $\mathbb{F}H$. Also, $M\pi$ is a left $R\pi$-submodule of $\mathbb{F}H$. Thus it suffices to show that $H' \subseteq R\pi$. Since $R\pi$ is an algebra, it suffices to show that $c_{ij} \in R\pi$ for all i, j. Note that $([x_i, x_j] \otimes [x_i, x_j])\pi = c_{ij}^2$ and $([x_i, x_j] \otimes 1 + 1 \otimes [x_i, x_j])\pi = 2c_{ij}$. Hence $c_{ij}^2 \in R\pi$ and $2c_{ij} \in R\pi$. If n_0 is odd then $c_{ij}^2 \in R\pi$ gives $c_{ij} \in R\pi$. But if n_0 is even then $\mathbb{F}$ does not have characteristic 2 and $2c_{ij} \in R\pi$ gives $c_{ij} \in R\pi$. $\square$

In the notation of Section 5, we can write $\mathbb{F}(A_k \times A_k) = \bigoplus_{\delta, \delta' \in \Delta_k} I_{\delta\delta'}$. Note that

$$(e_\delta \otimes 1)\pi = (1 \otimes e_\delta)\pi = e_\delta \in \mathbb{F}H_k'.$$

Hence, for $\delta \neq \delta'$, we have $e_{\delta\delta'}\pi = e_\delta e_{\delta'} = 0$ and so $I_{\delta\delta'} \subseteq \ker(\pi) = I$. It is easily checked that $\bigoplus_{\delta \in \Delta_k} I_{\delta\delta}$ and $\mathbb{F}H_k$ have the same dimension. Hence

$$I \cap \mathbb{F}(A_k \times A_k) = \bigoplus_{\substack{\delta, \delta' \in \Delta_k \\ \delta \neq \delta'}} I_{\delta\delta'}. \tag{6.1}$$

LEMMA 6.9. *Let M be an (R,Ψ)-submodule of $\mathbb{F}(A\times A)$ such that $R\cap I\subseteq M\subseteq I$, and let T be the largest Ψ-closed left ideal of $\mathbb{F}(A\times A)$ contained in M. Then $M=T+(R\cap I)$.*

PROOF. Let L be the subspace of M spanned by all elements of M which have the form $we_{\delta\delta'}$ where $w\in\mathbb{F}(A\times A)$ and $\delta,\delta'\in\Delta_k$ for some k, with $\delta\neq\delta'$. Let $we_{\delta\delta'}$ be such an element of M. Let $\psi\in\Psi$ and $a,a'\in A$. Choose $l\geq k$ so that $e_\delta\psi, e_{\delta'}\psi\in\mathbb{F}A_l'$. Since $e_\delta\psi$ and $e_{\delta'}\psi$ are idempotents, we can write $e_\delta\psi=\sum_{\lambda\in\Lambda}e_\lambda$ and $e_{\delta'}\psi=\sum_{\lambda'\in\Lambda'}e_{\lambda'}$ where $\Lambda,\Lambda'\subseteq\Delta_l$. But $(e_\delta\psi)(e_{\delta'}\psi)=(e_\delta e_{\delta'})\psi=0$. Thus Λ and Λ' are disjoint. For $\varepsilon\in\Lambda$ and $\varepsilon'\in\Lambda'$,

$$((a\otimes a')e_{\varepsilon\varepsilon'}+(a'\otimes a)e_{\varepsilon'\varepsilon})(w\psi)(e_{\delta\delta'}\psi)\in M,$$

because M is an (R,Ψ)-module. However, $e_{\delta\delta'}\psi=\sum_{\lambda,\lambda'}e_{\lambda\lambda'}$. Hence $e_{\varepsilon'\varepsilon}(e_{\delta\delta'}\psi)=0$ and $e_{\varepsilon\varepsilon'}(e_{\delta\delta'}\psi)=e_{\varepsilon\varepsilon'}$. Therefore $(a\otimes a')(w\psi)e_{\varepsilon\varepsilon'}\in M$, and so $(a\otimes a')(w\psi)e_{\varepsilon\varepsilon'}\in L$. Since this holds for all ε,ε', we have $(a\otimes a')(w\psi)(e_{\delta\delta'}\psi)\in L$. Therefore L is a Ψ-closed left ideal of $\mathbb{F}(A\times A)$. We next prove that $M=L+(R\cap I)$, which will give the required result.

Let $u\in M$ and choose k so that $u\in\mathbb{F}(A_k\times A_k)$. Since $M\subseteq I$ we can use (6.1) to write $u=\sum w_{\delta\delta'}e_{\delta\delta'}$, where the sum is over all $\delta,\delta'\in\Delta_k$ with $\delta\neq\delta'$ and each $w_{\delta\delta'}$ belongs to $\mathbb{F}(A_k\times A_k)$. Let $\delta,\delta'\in\Delta_k$ with $\delta\neq\delta'$. Since M is an R-module,

$$(e_{\delta\delta'}+e_{\delta'\delta})u=w_{\delta\delta'}e_{\delta\delta'}+w_{\delta'\delta}e_{\delta'\delta}\in M.$$

Write $v=w_{\delta\delta'}$ and $v'=w_{\delta'\delta}$. Then it suffices to show that $ve_{\delta\delta'}+v'e_{\delta'\delta}\in L+(R\cap I)$.

Let τ be the involutory automorphism of $\mathbb{F}(A\times A)$ satisfying $(a\otimes a')\tau=a'\otimes a$ for all $a,a'\in A$. Then $w+w\tau\in R$ for all $w\in\mathbb{F}(A\times A)$. We can write

$$ve_{\delta\delta'}+v'e_{\delta'\delta}=(v-v'\tau)e_{\delta\delta'}+v'e_{\delta'\delta}+(v'\tau)e_{\delta\delta'}. \tag{6.2}$$

Here

$$v'e_{\delta'\delta}+(v'\tau)e_{\delta\delta'}=v'e_{\delta'\delta}+(v'e_{\delta'\delta})\tau\in R\cap I.$$

Since $R\cap I\subseteq M$, (6.2) gives $(v-v'\tau)e_{\delta\delta'}\in M$, and so $(v-v'\tau)e_{\delta\delta'}\in L$. Therefore, by (6.2), $ve_{\delta\delta'}+v'e_{\delta'\delta}\in L+(R\cap I)$, as required. ☐

To complete the proof of Theorem D, let $M_1\subseteq M_2\subseteq\ldots$ be an ascending chain of (R,Ψ)-submodules of $\mathbb{F}(A\times A)$ which contain R. By Lemma 6.8 the chain $M_1+I\subseteq M_2+I\subseteq\cdots$ becomes stationary. Thus it suffices to show that the chain $M_1\cap I\subseteq M_2\cap I\subseteq\cdots$ becomes stationary. For each i, let T_i be the largest Ψ-closed left ideal of $\mathbb{F}(A\times A)$ contained in $M_i\cap I$. By Lemma 6.9 it suffices to show that the chain $T_1\subseteq T_2\subseteq\cdots$ becomes stationary. But this holds by Theorem C.

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