

RINGS IN WHICH CERTAIN RIGHT IDEALS ARE DIRECT SUMMANDS OF ANNIHILATORS

YIQIANG ZHOU

(Received 15 May 2000; revised 8 October 2001)

Communicated by Jie Du

Abstract

This paper is a continuation of the study of the rings for which every principal right ideal (respectively, every right ideal) is a direct summand of a right annihilator initiated by Stanley S. Page and the author in [20, 21].

2000 *Mathematics subject classification*: primary 16D50, 16P60.

Introduction

In this paper, we continue the study of left AP-injective and left AGP-injective rings which were introduced and discussed in [20]. Following [20], a ring R is called *left AP-injective* if every principal right ideal is a direct summand of a right annihilator, and the ring R is called *left AGP-injective* if, for any $0 \neq a \in R$, there exists $n > 0$ such that $a^n \neq 0$ and $a^n R$ is a direct summand of $\mathbf{r}l(a^n)$. Recall that a ring R is *left principally injective* (*P-injective*) if every principal right ideal is a right annihilator, and the ring R is *left generalized principally injective* (*GP-injective*) if, for any $0 \neq a \in R$, there exists $n > 0$ such that $a^n \neq 0$ and $a^n R$ is a right annihilator. The detailed discussion of left P-injective and left GP-injective rings can be found in [3, 7, 12, 15, 16, 17, 22, 23, 24, 26]. Clearly, every left AP-injective ring is left P-injective and every left AGP-injective ring is left GP-injective. But there exist left AP-injective rings which are not left GP-injective [20]. In fact, a left AP-injective ring is not necessarily a left mininjective ring. (The ring R is *left mininjective* if,

The author was supported in part by the NSERC grant OGP0194196.

© 2002 Australian Mathematical Society 1446-8107/2000 \$A2.00 + 0.00

for any minimal left ideal Ra , aR is a right annihilator [18], and every left GP-injective ring is left mininjective.) In [20], several results which are known for left P-injective (respectively, left GP-injective) rings were shown to hold for left AP-injective (respectively, left AGP-injective) rings. It has been noted that it is unknown whether there exists a left GP-injective ring that is not left P-injective (see [6, 24]). This may put a bit more weight on our excuse for carrying on the study of the left AGP-injective rings. In this paper, we discuss left AGP-injective rings with various chain conditions.

It is well known that a ring R is quasi-Frobenius (QF) if and only if R is left self-injective and left (or right) noetherian. In [9], Faith proved that any left self-injective ring satisfying the ACC on left annihilators is QF. Björk [2] extended this result from a left self-injective ring to a left f-injective ring, and then Rutter [23] further proved that, if R satisfies the ACC on left annihilators, then R is QF if and only if R is left 2-injective, where the ring R is called *left f-injective* (respectively, *left 2-injective*) if, for any finitely generated (respectively, 2-generated) left ideal I of R , every R -homomorphism from I to R extends to an R -homomorphism from R to R . Note that a left f-injective rings need not be left self-injective, and a left P-injective ring need not be left 2-injective. It was also proved in [23] that any left P-injective ring satisfying the ACC on left annihilators is right artinian. The latter result was extended from a left P-injective ring to a left GP-injective ring in Chen and Ding [7]. It is clear, by Rutter's example in [23], that a left P-injective ring satisfying the ACC on left annihilators need not be left artinian, and hence not be QF. The main result in Section 2 states that a left AGP-injective ring with the ACC on left annihilators is always semiprimary, but is not necessarily right artinian.

A ring is called a *right dual ring* if every right ideal is a right annihilator. The study of right noetherian, right dual rings was initiated by Johns [14], and continued by Faith and Menal in [10, 11] where they gave a counterexample to Johns' result that every right noetherian, right dual ring is right artinian. Recently, Gómez Pardo and Guil Asensio [12] proved that if R is right noetherian and left P-injective, then $J(R)$ is nilpotent and $\mathbf{I}(J(R))$ is essential both as a left and a right ideal of R , and this result allows them to show that every left Kasch, right noetherian and left P-injective ring is right artinian. In Section 2, among other things, we prove that, for a right noetherian and left AGP-injective ring R , $J(R)$ is nilpotent and $\mathbf{I}(J(R))$ is essential both as a left and a right ideal of R . As a corollary of this, we show that every right noetherian, left AGP-injective ring with right (GC2) is right artinian.

In Section 3, we consider right quasi-dual rings. A ring R is called *right quasi-dual* if every right ideal of R is a direct summand of a right annihilator [21]. The right quasi-dual rings form an interesting class of left AP-injective rings. In Section 3, it is proved that, for a right quasi-dual ring, $J(R) = \mathbf{r}(S_r)$, $S_r = \mathbf{r}(Z_r)$ and $\mathbf{I}(J(R))$ is essential in ${}_R R$. Consequently, for a two-sided quasi-dual ring R , the left socle

coincides with the right socle and is essential both as a left and a right ideal of R . We also improve a result of [21] by showing that a ring R is a two-sided PF-ring if and only if every right Goldie torsion R -module is cogenerated by ${}_R R$ and every left Goldie torsion R -module is cogenerated by ${}_R R$.

Throughout, R is an associative ring with identity and modules are unitary. We use M_R (respectively, ${}_R M$) to indicate that M is a right (respectively, left) module over R . For a subset X of R , $\mathbf{l}(X)$ (respectively, $\mathbf{r}(X)$) is the left (respectively, right) annihilator of X in R , and we write $\mathbf{l}(x)$ (respectively, $\mathbf{r}(x)$) for $\mathbf{l}(\{x\})$ (respectively, $\mathbf{r}(\{x\})$) when $x \in R$. The left socle, right socle, left singular ideal, right singular ideal and Jacobson radical of R are denoted by S_l, S_r, Z_l, Z_r and $J(R)$, respectively. For a submodule N of M , we use $N \leq_e M$ to mean that N is essential in M .

1. Left AGP-injective rings with left chain conditions

Following [20], the ring R is *left AP-injective* if, for any $a \in R$, aR is a direct summand of $\mathbf{rl}(a)$, and R is *left AGP-injective* if, for any $0 \neq a \in R$, there exists $n > 0$ such that $a^n \neq 0$ and $a^n R$ is a direct summand of $\mathbf{rl}(a^n)$. Every left P-injective ring is left AP-injective and every left GP-injective ring is left AGP-injective. The rings R in [21, Examples 2.3, 2.4] are commutative AP-injective rings, but not mininjective and hence not GP-injective.

In this section, we prove several results of left AGP-injective rings with some chain conditions on left ideals.

A module M is said to satisfy the *generalized C2-condition* (or (GC2)) if, for any $N \subseteq M$ and $N \cong M$, N is a summand of M . Note that the GC2-condition is the same as the $(*)$ -condition in [20, page 713].

LEMMA 1.1. *Let ${}_R M$ satisfy (GC2). If M is finitely dimensional, then $\text{End}(M)$ is semilocal.*

PROOF. Let $\sigma : M \rightarrow M$ be a monomorphism. Then $M = \sigma(M) \oplus N$ for some $N \subseteq M$. It must be that $N = 0$ since M is finitely dimensional. So, σ is an isomorphism. Therefore, M satisfies the assumptions in Camps-Dicks [5, Theorem 5], and so $\text{End}(M)$ is semilocal. \square

The next corollary extends [21, Proposition 2.12].

COROLLARY 1.2. *Let R be a left AGP-injective ring.*

- (1) *If ${}_R R$ is of finite Goldie dimension, then R is semilocal.*
- (2) *R is left noetherian if and only if R is left artinian.*

PROOF. (1). By [20, Proposition 2.13], ${}_R R$ satisfies (GC2). Since ${}_R R$ has finite Goldie dimension, R is semilocal by Lemma 1.1.

(2). If R is left noetherian, then R is semilocal by (1). By [20, Corollary 2.11], $J(R)$ is nilpotent. So, R is left artinian. □

LEMMA 1.3 ([20]). *If R is a left AGP-injective ring, then $J(R) = Z_l$.*

LEMMA 1.4. *Let R be a left AGP-injective ring and $a \in R$. If $a \notin J(R)$ then there exists $r \in R$ such that the inclusion $\mathbf{l}(a) \subset \mathbf{l}(a - ara)$ is proper.*

PROOF. Let $a \in R$ but $a \notin J(R)$. By Lemma 1.3, $a \notin Z_l$ and hence $\mathbf{l}(a)$ is not essential in ${}_R R$. So, we have $\mathbf{l}(a) \cap I = 0$ for some $0 \neq I \subseteq {}_R R$. Take $0 \neq b \in I$. Then $ba \neq 0$. By the hypothesis, there exists $n > 0$ such that $(ba)^n \neq 0$ and $\mathbf{rl}((ba)^n) = (ab)^n R \oplus X$ where X is a right ideal of R . Since $\mathbf{l}(a) \cap I = 0$, $\mathbf{l}((ba)^n) = \mathbf{l}((ba)^{n-1}b)$. It follows that $(ba)^{n-1}b \in \mathbf{rl}((ba)^{n-1}b) = \mathbf{rl}((ba)^n) = (ba)^n R \oplus X$. Thus, there exists $r \in R$ such that $(ba)^{n-1}b = (ba)^n r + x$ where $r \in R$ and $x \in X$. This gives that $(ba)^{n-1}b(1 - ar) = x$ and hence $(ba)^{n-1}b(a - ara) = xa \in (ba)^n R \cap X$. It follows that $(ba)^{n-1}b(a - ara) = 0$. Let $c = a - ara$. Then $\mathbf{l}(a) \subseteq \mathbf{l}(c)$. Since $(ba)^{n-1}b$ is in $\mathbf{l}(c)$ but not in $\mathbf{l}(a)$, the inclusion $\mathbf{l}(a) \subset \mathbf{l}(c)$ is proper. □

The next result extends [7, Theorem 3.4, Corollary 3.6]. Following [1], a module M is called *finitely projective* (respectively, *singly projective*) if, for each epimorphism $f : N \rightarrow M$ and each finitely generated (respectively, cyclic) submodule M_0 of M , there exists $g \in \text{Hom}_R(M_0, N)$ such that the restriction of $g \circ f$ to M_0 is the identity on M_0 .

THEOREM 1.5. *The following are equivalent for a left AGP-injective ring R :*

- (1) R is a left Perfect ring.
- (2) Every flat left R -module is finitely projective.
- (3) Every flat left R -module is singly projective.
- (4) For any infinite sequence x_1, x_2, x_3, \dots of elements in R , the chain $\mathbf{l}(x_1) \subseteq \mathbf{l}(x_1 x_2) \subseteq \mathbf{l}(x_1 x_2 x_3) \subseteq \dots$ terminates.

PROOF. (1) implies (2) and (2) implies (3) are obvious. (3) implies (4) is by [1, Corollary 25].

(4) implies (1). Firstly, we prove $R/J(R)$ is a von Neumann regular ring. For any $x \in R$, let $\bar{x} = x + J(R)$. Let $a_1 \in R$ but $a_1 \notin J(R)$. We want to show that $\bar{a}_1 = \bar{a}_1 \bar{x} \bar{a}_1$ for some $x \in R$. By Lemma 1.4, there exists $r_1 \in R$ such that $\mathbf{l}(a_1) \subset \mathbf{l}(a_2)$ where $a_2 = a_1 - a_1 r_1 a_1$. If $a_2 \in J(R)$, then $\bar{a}_1 = \bar{a}_1 \bar{r}_1 \bar{a}_1$ and we are done. If $a_2 \notin J(R)$, then, by Lemma 1.4, there exists $r_2 \in R$ such that $\mathbf{l}(a_2) \subset \mathbf{l}(a_3)$ where $a_3 = a_2 - a_2 r_2 a_2$. The induction principle and the hypothesis ensure the existence of a positive integer

n and two sequences $\{a_i : i = 1, \dots, n + 1\}$ and $\{r_i : i = 1, \dots, n\}$ of elements in R such that $a_{n+1} \in J(R)$ and $a_{i+1} = a_i - a_i r_i a_i$ for $i = 1, \dots, n$. Thus, $\bar{a}_n = \bar{a}_n \bar{r}_n \bar{a}_n$. It follows that

$$\begin{aligned} \bar{a}_{n-1} &= \bar{a}_n + \bar{a}_{n-1} \bar{r}_{n-1} \bar{a}_{n-1} \\ &= (\bar{a}_{n-1} - \bar{a}_{n-1} \bar{r}_{n-1} \bar{a}_{n-1}) \bar{r}_n (\bar{a}_{n-1} - \bar{a}_{n-1} \bar{r}_{n-1} \bar{a}_{n-1}) + \bar{a}_{n-1} \bar{r}_{n-1} \bar{a}_{n-1} \\ &= \bar{a}_{n-1} [(\bar{1} - \bar{r}_{n-1} \bar{a}_{n-1}) \bar{r}_n (\bar{1} - \bar{a}_{n-1} \bar{r}_{n-1}) + \bar{r}_{n-1}] \bar{a}_{n-1}, \end{aligned}$$

so \bar{a}_{n-1} is also a regular element. Continuing this process, we see that \bar{a}_1 is a regular element.

Secondly, we prove that Z_l is left T-nilpotent. Let $a_i \in Z_l$ for $i = 1, 2, \dots$. We have a chain $\mathbf{I}(a_1) \subseteq \mathbf{I}(a_1 a_2) \subseteq \dots$. By our assumption, there exists $n > 0$ such that $\mathbf{I}(a_1 \cdots a_n) = \mathbf{I}(a_1 \cdots a_n a_{n+1})$. Thus, $\mathbf{I}(a_{n+1}) \cap R a_1 \cdots a_n = 0$. Since $\mathbf{I}(a_{n+1})$ is essential in ${}_R R$, we have $a_1 \cdots a_n = 0$, so Z_l is left T-nilpotent. Therefore, by Lemma 1.3, we have proved that $R/J(R)$ is a von Neumann regular ring and $J(R)$ is left T-nilpotent. So, it suffices to show that $R/J(R)$ is an artinian semisimple ring. By [13, Corollary 2.16], we only need to show that $R/J(R)$ contains no infinite sets of nonzero orthogonal idempotents. This can be proved by arguing as in [7, page 2107]. \square

COROLLARY 1.6. *If R is a left AGP-injective ring with ACC on left annihilators, then R is semiprimary.*

PROOF. It is well known that Z_l is nilpotent for any ring R with ACC on left annihilators. By Lemma 1.3 and Theorem 1.5, R is semiprimary. \square

COROLLARY 1.7. *Let R be a left AGP-injective ring with ACC on left annihilators and $S_r \subseteq S_l$. Then R is right artinian if and only if S_r is a finitely generated right ideal of R .*

PROOF. By Corollary 1.6, R is semiprimary. By [20, Corollary 2.7], $S_l \subseteq S_r$, and so $S = S_l = S_r$ by the hypothesis. Now the result follows from [4, Lemma 6]. \square

A left GP-injective ring with the ACC on left annihilators is always right artinian [7, Theorem 3.7]. The ring R [21, Example 2.4] is a commutative AP-injective ring with the ACC on annihilators, but R is not artinian.

Recall that a ring R is called left Kasch if $\mathfrak{r}(K) \neq 0$ for every maximal left ideal K of R .

COROLLARY 1.8. *Let R be a left AGP-injective ring with ACC on left annihilators. If every minimal right ideal is a right annihilator, then R is right artinian. Moreover, R is left artinian if and only if S_l is finitely generated as a left ideal of R .*

PROOF. By Corollary 1.6, R is semiprimary. By [18, Corollary 3.15], R is right finite dimensional with $S_r = S_l$. Now, by [4, Lemma 6], R is right artinian. The last assertion follows from [4, Lemma 6] again. \square

Now the following result, [7, Theorem 3.7], is an immediate corollary of the above:

COROLLARY 1.9 ([7]). *Every left GP-injective ring with ACC on left annihilators is right artinian.*

PROOF. If R is a left GP-injective ring, then every minimal right ideal is a right annihilator. For, if I is a minimal right ideal of R , then $I = eR$ where $e^2 = e \in R$ or $I^2 = 0$. If $I = eR$, clearly I is an annihilator. On the other hand, if $I = xR$ for some $x \in R$ with $I^2 = 0$, it follows from the definition of left GP-injectivity that $I = xR = \mathbf{rl}(I)$. Now the result follows from Corollary 1.8. \square

2. Left AGP-injective rings with right chain conditions

In this section, we first consider right noetherian, left AGP-injective rings. We prove that, for a right noetherian, left AGP-injective ring R , $J(R)$ is nilpotent and $\mathbf{l}(J(R))$ is essential as a left and as a right ideal of R . As a corollary of this, we prove that every right noetherian, left AGP-injective ring R such that R_R satisfies (GC2) is right artinian. We next prove that every maximal left (respectively, right) annihilator of a semiprime left AGP-injective ring is a maximal left (respectively, right) ideal generated by an idempotent.

The next result extends [12, Theorem 2.7] from a left P-injective ring to a left AGP-injective ring.

THEOREM 2.1. *Let R be a right noetherian, and left AGP-injective ring. Then $J(R)$ is nilpotent and $\mathbf{l}(J(R))$ is essential both as a left and as a right ideal of R .*

PROOF. Let $J = J(R)$. First we prove that $\mathbf{l}(J) \leq_e {}_R R$. Let $0 \neq x \in R$. Since R is right noetherian, the non-empty set $\mathcal{F} = \{\mathbf{r}((ax)^k) : a \in R, k > 0 \text{ such that } (ax)^k \neq 0\}$ has a maximal element, say $\mathbf{r}((yx)^n)$.

We claim that $(yx)^n J = 0$. If not, then there exists $t \in J$ such that $(yx)^n t \neq 0$. Since R is left AGP-injective, there exists $m > 0$ such that $((yx)^n t)^m \neq 0$ and $((yx)^n t)^m R$ is a direct summand of $\mathbf{rl}(((yx)^n t)^m)$. Write $((yx)^n t)^m = (yx)^n s$ where $s = t((yx)^n t)^{m-1} \in J$. Then $\mathbf{rl}((yx)^n s) = (yx)^n s R \oplus X$ for some right ideal X of R . We proceed with the following two cases.

Case 1. $\mathbf{rl}((yx)^n) = \mathbf{rl}((yx)^n s)$. Then $(yx)^n \in \mathbf{rl}((yx)^n) = (yx)^n s R \oplus X$. Write $(yx)^n = (yx)^n s v + z$, where $v \in R$ and $z \in X$. Then $(yx)^n s = (yx)^n s v s + z s$ and

so $zs \in (yx)^n s R \cap X$. Thus, $zs = 0$ and hence $(yx)^n s = (yx)^n s vs$. It follows that $(yx)^n s(1 - vs) = 0$. Since $s \in J$, $1 - vs$ is a unit in R . So, we have $(yx)^n s = 0$. This is a contradiction.

Case 2. $\mathbf{rl}((yx)^n) \neq \mathbf{rl}((yx)^n s)$. Then $\mathbf{I}((yx)^n) \neq \mathbf{I}((yx)^n s)$. It follows that there exists $u \in \mathbf{I}((yx)^n s)$ but $u \notin \mathbf{I}((yx)^n)$. Thus, $u(yx)^n s = 0$ and $u(yx)^n \neq 0$. This gives that $s \in \mathbf{r}(u(yx)^n)$ and $s \notin \mathbf{r}((yx)^n)$. So, the inclusion $\mathbf{r}((yx)^n) \subset \mathbf{r}(u(yx)^n)$ is proper. This is a contradiction because $0 \neq u(yx)^n = (u(yx)^{n-1} y)x$ and $\mathbf{r}(u(yx)^n) \in \mathcal{F}$.

We have proved that $(yx)^n J = 0$, and so $Rx \cap \mathbf{I}(J) \neq 0$. Therefore, $\mathbf{I}(J)$ is an essential left ideal of R .

Next we prove that J is nilpotent. Since R is right noetherian, there exists $k > 0$ such that $\mathbf{I}(J^k) = \mathbf{I}(J^{k+n})$ for all $n > 0$. Suppose J is not nilpotent. Then $J^k \neq 0$ and so $M_R = R/\mathbf{I}(J^k)$ is a nonzero R -module. Since R is right noetherian, the set $\{\mathbf{r}_R(m) : 0 \neq m \in M\}$ has a maximal element, $\mathbf{r}_R(m_1)$ say. Write $m_1 = x + \mathbf{I}(J^k)$ where $x \in R$. Then $xJ^k \neq 0$. Since $\mathbf{I}(J^{2k}) = \mathbf{I}(J^k)$, we see $xJ^k \not\subseteq \mathbf{I}(J^k)$. So, there exists $b \in J^k$ such that $xb \notin \mathbf{I}(J^k)$. Since $\mathbf{I}(J) \leq_e {}_R R$, $Rxb \cap \mathbf{I}(J^k) \neq 0$. So, we have $0 \neq axb \in \mathbf{I}(J^k)$ for some $a \in R$. Let $m_2 = ax + \mathbf{I}(J^k) \in M$. Then $m_2 \neq 0$ and $b \in \mathbf{r}_R(m_2)$. But, $b \notin \mathbf{r}_R(m_1)$. So, the inclusion $\mathbf{r}_R(m_1) \subset \mathbf{r}_R(m_2)$ is proper. This contradicts the choice of m_1 .

Finally, for any $0 \neq x \in R$, $xJ = 0$, or $xJ^n \neq 0$ and $xJ^{n+1} = 0$ for some $n > 0$. It follows that $xR \cap \mathbf{I}(J) \neq 0$. So, $\mathbf{I}(J)$ is an essential right ideal of R . \square

The next result extends [12, Corollary 2.9]. (Note that, if R is left Kasch, then R_R satisfies (C2) (see [25]) and hence satisfies (GC2)).

COROLLARY 2.2. *Every right noetherian, left AGP-injective ring R such that R_R satisfies (GC2) is right artinian.*

PROOF. Since R is right finitely dimensional and R_R satisfies (GC2), R is semilocal by Lemma 1.1. By Theorem 2.1, $J(R)$ is nilpotent. So, R is semiprimary. Since R is right noetherian, R is right artinian. \square

Next, we consider semiprime left AGP-injective rings.

LEMMA 2.3. *Let R be an arbitrary ring and $a \in R$ such that $\mathbf{I}(a)$ is a maximal left annihilator or $\mathbf{r}(a)$ is a maximal right annihilator. Then $\mathbf{I}(at) = \mathbf{I}(a)$ for any $t \notin \mathbf{r}(a)$ and $Z_l \subseteq \mathbf{r}(a)$, and $\mathbf{r}(ta) = \mathbf{r}(a)$ for any $t \notin \mathbf{I}(a)$ and $Z_r \subseteq \mathbf{I}(a)$.*

PROOF. Let $x \in Z_l$. Then $\mathbf{I}(x)$ is essential in ${}_R R$. So, $\mathbf{I}(x) \cap Rr \neq 0$ for any $0 \neq r \in R$. Thus, there exists $y \in R$ such that $0 \neq yr$ and $yrx = 0$. So, the inclusion $\mathbf{I}(r) \subset \mathbf{I}(rx)$ is proper.

Case 1. Let $\mathbf{l}(a)$ be a maximal left annihilator. As above, $\mathbf{l}(a) \subset \mathbf{l}(ax)$ for all $x \in Z_l$. It must be that $ax = 0$. This shows that $a \in \mathbf{l}(Z_l)$. Clearly, in this case $\mathbf{l}(at) = \mathbf{l}(a)$ for any $t \notin \mathbf{r}(a)$.

Case 2. Let $\mathbf{r}(a)$ be a maximal right annihilator. If $t \notin \mathbf{r}(a)$, then $at \neq 0$. For $x \in \mathbf{l}(at)$, $t \in \mathbf{r}(xa)$ and so the inclusion $\mathbf{r}(a) \subset \mathbf{r}(xa)$ is proper. By the maximality of $\mathbf{r}(a)$, $xa = 0$. Thus, $\mathbf{l}(at) = \mathbf{l}(a)$. It follows that $Ra \cap \mathbf{l}(t) = 0$. Thus, $t \notin Z_l$. Therefore, $Z_l \subseteq \mathbf{r}(a)$.

The remaining part is by the left-right symmetry of the hypothesis. □

The next theorem extends [7, Theorem 3.1].

THEOREM 2.4. *Let R be a semiprime left AGP-injective ring. Then every maximal left (respectively, right) annihilator is a maximal left (respectively, right) ideal of R which is generated by an idempotent.*

PROOF. Let L be a maximal left (respectively, right) annihilator. Then $L = \mathbf{l}(a)$ (respectively, $\mathbf{r}(a)$) for some $0 \neq a \in R$. Since R is semiprime, $Z_l \cap \mathbf{l}(Z_l) = 0$. Claim: $a \notin Z_l$. Otherwise, $a \notin \mathbf{l}(Z_l)$, that is, $aZ_l \neq 0$. Take $x \in Z_l$ such that $ax \neq 0$. Since $x \notin \mathbf{r}(a)$, $\mathbf{l}(ax) = \mathbf{l}(a)$ by Lemma 2.3. Thus, $\mathbf{l}(x) \cap Ra = 0$, a contradiction, since $x \in Z_l$. Therefore, $a \notin Z_l$. By Lemma 1.3 and Lemma 1.4, the inclusion $\mathbf{l}(a) \subset \mathbf{l}(a - ara) = \mathbf{l}[a(1 - ra)]$ is proper for some $r \in R$. It follows from Lemma 2.3 that $a - ara = 0$. Therefore, $L = \mathbf{l}(ar)$ (respectively, $L = \mathbf{r}(ra)$) with ar (respectively, ra) an idempotent. So we can assume that $a = e$ is an idempotent. To see L is a maximal left (respectively, right) ideal, we show that Re (respectively, eR) is a minimal left (respectively, right) ideal of R . Since R is semiprime, it suffices to show that eRe is a division ring. Let $0 \neq d \in eRe$. Since R is left AGP-injective, there exists $n > 0$ such that $d^n \neq 0$ and $d^n R$ is a direct summand of $\mathbf{rl}(d^n)$. By Lemma 2.3, $\mathbf{l}(d^n) = \mathbf{l}(e)$ and so $\mathbf{rl}(d^n) = \mathbf{rl}(e) = eR$. Thus, $d^n R$ is a direct summand of eR and hence of R_R . It follows that $d^n R = \mathbf{rl}(d^n) = eR$. Write $e = d^n b$ where $b \in R$. Then $e = d(d^{n-1}be)$ with $d^{n-1}be \in eRe$. So, eRe is a division ring. □

A ring R is a left PP ring if every principal left ideal of R is projective. The next result extends [6, Theorem 2.9] from a left GP-injective ring to a left AGP-injective ring.

PROPOSITION 2.5. *The ring R is a von Neumann regular ring if and only if R is left PP and left AGP-injective.*

PROOF. One direction is obvious. Suppose that R is left PP and left AGP-injective. For any nonzero element $a \in R$, there exists $n > 0$ such that $a^n \neq 0$ and $\mathbf{rl}(a^n) = a^n R \oplus X$ where X is a right ideal of R . Since R is left PP, Ra^n is projective, and

hence $0 \rightarrow \mathbf{l}(a^n) \rightarrow R \rightarrow Ra^n \rightarrow 0$ splits. Thus, $\mathbf{l}(a^n) = Re$ where $e^2 = e \in R$. It follows that $\mathbf{rl}(a^n) = \mathbf{r}(Re) = (1 - e)R$. Thus, $a^n R$ is a direct summand of $(1 - e)R$, and hence a direct summand of R_R . This implies that a^n is a regular element of R . If $a \neq 0$ but $a^2 = 0$, the argument above shows that a is a regular element. So, by [6, Theorem 2.9], R is a regular ring. \square

3. Right quasi-dual rings

Following [21], a ring R is called right quasi-dual if every right ideal of R is a direct summand of a right annihilator. As shown in [21], the ring R is right quasi-dual if and only if every essential right ideal of R is a right annihilator if and only if every singular cyclic right R -module is cogenerated by R . Every right dual ring is certainly right quasi-dual, and every right quasi-dual ring is left AP-injective.

LEMMA 3.1. *Let R be a right quasi-dual ring. For any right ideal I of R and $a \in R$, $\mathbf{r}[Ra \cap \mathbf{l}(I)] = I + (X_{aI} : a)_r$ with $(X_{aI} : a)_r \cap I \subseteq \mathbf{r}(a)$ and $(X_{aI} : a)_r = \{x \in R : ax \in X_{aI}\}$, where X_{aI} is a right ideal of R such that $\mathbf{rl}(aI) = aI \oplus X_{aI}$.*

PROOF. Let $x \in \mathbf{r}[Ra \cap \mathbf{l}(I)]$. Then $\mathbf{l}(aI) \subseteq \mathbf{l}(ax)$, and so $ax \in \mathbf{rl}(ax) \subseteq \mathbf{rl}(aI) = aI \oplus X_{aI}$. Write $ax = at + y$ where $t \in I$ and $y \in X_{aI}$. Then $a(x - t) = y \in X_{aI}$ and thus $x - t \in (X_{aI} : a)_r$. Therefore, $x \in I + (X_{aI} : a)_r$ and $\mathbf{r}[Ra \cap \mathbf{l}(I)] \subseteq I + (X_{aI} : a)_r$. It is easy to see that $(X_{aI} : a)_r \cap I \subseteq \mathbf{r}(a)$ and that $I \subseteq \mathbf{r}[Ra \cap \mathbf{l}(I)]$. Let $y \in (X_{aI} : a)_r$. Then $ay \in X_{aI} \subseteq \mathbf{rl}(aI)$. For any $ra \in Ra \cap \mathbf{l}(I)$, $raI = 0$. This gives that $r \in \mathbf{l}(aI)$. Since $ay \in \mathbf{rl}(aI)$, it follows that $ray = 0$. Thus, $y \in \mathbf{r}[Ra \cap \mathbf{l}(I)]$ and $(X_{aI} : a)_r \subseteq \mathbf{r}[Ra \cap \mathbf{l}(I)]$. \square

THEOREM 3.2. *Let R be a right quasi-dual ring and $J = J(R)$. Then*

- (1) $J = Z_l = \mathbf{r}(S_r)$, $S_r = \mathbf{r}(Z_r)$, and R is right Kasch.
- (2) $\mathbf{l}(J)$ is essential in ${}_R R$.

PROOF. (1). Clearly, $S_r \subseteq \mathbf{r}(Z_r)$. Let K be any essential right ideal of R . Then $\mathbf{l}(K) \subseteq Z_r$ and so $K = \mathbf{rl}(K) \supseteq \mathbf{r}(Z_r)$. It follows that $S_r \supseteq \mathbf{r}(Z_r)$ since S_r is the intersection of all essential right ideals. Thus, $S_r = \mathbf{r}(Z_r)$. By [21, Lemma 2.5 and Lemma 2.6], $J = Z_l$ and R is right Kasch. Since R is right Kasch, $J = \mathbf{r}(S_r)$.

(2). Let $0 \neq a \in R$ and assume that $Ra \cap \mathbf{l}(J) = 0$. Then, by Lemma 3.1, $R = \mathbf{r}[Ra \cap \mathbf{l}(J)] = J + (X_{aJ} : a)_r$ where X_{aJ} is a right ideal of R such that $\mathbf{rl}(aJ) = aJ \oplus X_{aJ}$. Since J is small in R_R , $R = (X_{aJ} : a)_r$. It follows that $aR \subseteq X_{aJ}$ and so $aJ \subseteq aJ \cap X_{aJ} = 0$. Thus, $a \in Ra \cap \mathbf{l}(J) = 0$, a contradiction. \square

COROLLARY 3.3. *Let R be a quasi-dual ring. Then $S = S_r = S_l$ is essential as a left and a right ideal of R .*

PROOF. By [21, Theorem 2.8] and Theorem 3.2. \square

It was proved in [21] that, for a two-sided quasi-dual ring R , every Goldie torsion right R -module is cogenerated by R_R if and only if the second singular right ideal $Z_2(R_R)$ of R is injective. This result can be improved as follows.

THEOREM 3.4. *Consider the following conditions on a ring R :*

- (1) *Every Goldie torsion right R -module is cogenerated by R_R .*
- (2) *$Z_2(R_R)$ is injective and R is right Kasch.*
- (3) *R is right self-injective and right Kasch.*

Then (3) implies (2) and (2) implies (1). In addition (1) implies (3) if R is left quasi-dual.

PROOF. (3) implies (2) is obvious, and (2) implies (1) is by the proof of [21, Theorem 4.1].

Suppose R is left quasi-dual and (1) holds. By [21, Theorem 4.1], $Z_2(R_R)$ is injective. Write $R_R = Z_2(R_R) \oplus K$ where K is right ideal of R . It suffices to show that K_R is injective. Note that R is a two-sided quasi-dual ring, so $Z_l = Z_r$ and $S_r = \mathbf{I}(Z_l)$ by [21, Theorem 2.8]. It follows that $K \subseteq \mathbf{I}((Z_2(R_R))) \subseteq \mathbf{I}(Z_l) = S_r$. So, K_R is semisimple. Thus, to show that K_R is injective, it suffices to show that K is $Z_2(R_R)$ -injective. But, this is clear because K is non-singular and $Z_2(R_R)$ is Goldie torsion. \square

A ring R is right PF if R is an injective cogenerator for $\text{Mod-}R$. It is known that R is right PF if and only if R is right self-injective and right Kasch. The next corollary improved [21, Corollaries 4.4–4.6].

COROLLARY 3.5. *R is a two-sided PF-ring if and only if every Goldie torsion right R -module is cogenerated by R_R and every Goldie torsion left R -module is cogenerated by ${}_R R$.*

Dischinger and Müller [8] constructed a left PF-ring that is not right PF. By Corollary 3.5, the left PF-ring in [8] does not cogenerate every Goldie torsion right R -module. Osofsky [19] constructed a non-injective cogenerator for $\text{Mod-}R$. We note that Osofsky's ring R has the property that $Z_2(R_R) = R$ (since $J(R)^2 = 0$ and $J(R)_R \leq_e R_R$). This shows the conditions (1) and (2) in Theorem 3.4 are not equivalent.

PROPOSITION 3.6. *The following are equivalent for a ring R :*

- (1) *R is right PF.*
- (2) *$Z_2(R_R)$ is injective, R is right Kasch and $R = Z_2(R_R) + S_r$.*

PROOF. (2) implies (1). It suffices to show that R is right self-injective. Since $R = Z_2(R_R) + S_r$, $R = Z_2(R_R) \oplus K$ where K is a non-singular semisimple right ideal of R . Clearly, K_R is $Z_2(R_R)$ -injective and K_R -injective. So, K_R is injective. Thus, R_R is injective.

(1) implies (2). We only need to show that $R = Z_2(R_R) + S_r$. Since $Z_2(R_R)$ is injective, write $R = Z_2(R_R) \oplus K$ where K is a right ideal of R . Since R is right PF, $J(R) = Z_r \subseteq Z_2(R_R)$ and S_r is a finitely generated essential right ideal of R . Thus $\text{Soc}(K_R)$ is finitely generated and essential in K_R . Since every minimal right ideal contained in K is idempotent, $\text{Soc}(K_R)$ is a summand of R_R and hence of K_R . Thus, $K = \text{Soc}(K_R)$ is semisimple. \square

We do not know if the condition that $R = Z_2(R_R) + S_r$ in Proposition 3.6 can be removed.

Acknowledgment

The author is very grateful to the referee for careful reading this article and valuable suggestions, in particular, the comments on weakening the hypothesis in Corollary 1.8.

References

- [1] G. Azumaya, 'Finite splitness and finite projectivity', *J. Algebra* **106** (1987), 114–134.
- [2] J. E. Björk, 'Rings satisfying certain chain conditions', *J. Reine Angew. Math.* **245** (1970), 63–73.
- [3] V. Camillo, 'Commutative rings whose principal ideals are annihilators', *Portugal. Math.* **46** (1989), 33–37.
- [4] V. Camillo and M. F. Yousif, 'Continuous rings with ACC on annihilators', *Canad. Math. Bull.* **34** (1991), 462–464.
- [5] R. Camps and W. Dicks, 'On semi-local rings', *Israel J. Math.* **81** (1993), 203–211.
- [6] J. Chen and N. Ding, 'On regularity of rings', *Algebra Colloq.* **8** (2001) 267–274.
- [7] ———, 'On general principally injective rings', *Comm. Algebra* **27** (1999), 2097–2116.
- [8] F. Dischinger and W. Müller, 'Left PF is not right PF', *Comm. Algebra* **14** (1986), 1223–1227.
- [9] C. Faith, 'Rings with ascending chain conditions on annihilators', *Nagoya Math. J.* **27** (1966), 179–191.
- [10] C. Faith and P. Menal, 'A counter example to a conjecture of Johns', *Proc. Amer. Math. Soc.* **116** (1992), 21–26.
- [11] ———, 'The structure of Johns rings', *Proc. Amer. Math. Soc.* **120** (1994), 1071–1081.
- [12] J. L. Gómez Pardo and P. A. Guil Asensio, 'Torsionless modules and rings with finite essential socle', in: *Abelian groups, module theory, and topology (Padua, 1997)* (eds. D. Dikranjan and L. Salce), Lecture Notes in Pure and Appl. Math. 201 (Dekker, New York, 1998) pp. 261–278.
- [13] K. R. Goodearl, *Von Neumann regular rings* (Pitman, London, 1979).
- [14] B. Johns, 'Annihilator conditions in noetherian rings', *J. Algebra* **30** (1974), 103–121.
- [15] S. B. Nam, N. K. Kim and J. Y. Kim, 'On simple GP-injective modules', *Comm. Algebra* **23** (1995), 5437–5444.

- [16] W. K. Nicholson and M. F. Yousif, 'On a theorem of Camillo', *Comm. Algebra* **23** (1995), 5309–5314.
- [17] ———, 'Principally injective rings', *J. Algebra* **174** (1995), 77–93.
- [18] ———, 'Mininjective rings', *J. Algebra* **187** (1997), 548–578.
- [19] B. L. Osofsky, 'A generalization of quasi-Frobenius rings', *J. Algebra* **4** (1966), 373–387.
- [20] S. Page and Y. Zhou, 'Generalizations of principally injective rings', *J. Algebra* **206** (1998), 706–721.
- [21] ———, 'Quasi-dual rings', *Comm. Algebra* **28** (2000), 489–504.
- [22] G. Puninski, R. Wisbauer and M. F. Yousif, 'On P-injective rings', *Glasgow Math. J.* **37** (1995), 373–378.
- [23] E. A. Rutter, 'Rings with the principal extension property', *Comm. Algebra* **3** (1975), 203–212.
- [24] W. Xue, 'A note on YJ-injectivity', *Riv. Mat. Univ. Parma (6)* **1** (1998), 31–37 (1999).
- [25] M. F. Yousif, 'CS rings and Nakayama permutations', *Comm. Algebra* **25** (1997), 3787–3795.
- [26] R. Yue Chi Ming, 'On injectivity and p-injectivity', *J. Math. Kyoto Univ.* **27** (1987), 439–452.

Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John's A1C 5S7
Canada
e-mail: zhou@math.mun.ca