

TWO WEIGHTED INEQUALITIES FOR MAXIMAL FUNCTIONS RELATED TO CESÀRO CONVERGENCE

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Abstract

We characterize the pairs of weights (u, v) for which the maximal operator

$$M_{\alpha}^{-} f(x) = \sup_{R>0} R^{-1-\alpha} \int_{x-2R}^{x-R} |f(s)|(x-R-s)^{\alpha} ds, \quad -1 < \alpha < 0,$$

is of weak and restricted weak type (p, p) with respect to $u(x) dx$ and $v(x) dx$. As a consequence we obtain analogous results for

$$M_{\alpha} f(x) = \sup_{R>0} R^{-1-\alpha} \int_{R<|x-y|<2R} |f(y)|(|x-y|-R)^{\alpha} dy.$$

We apply the results to the study of the Cesàro- α convergence of singular integrals.

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1. Introduction

Let M_{α} be the maximal operator defined at a measurable function f on the real line by

$$M_{\alpha} f(x) = \sup_{R>0} \frac{1}{R^{1+\alpha}} \int_{R<|x-y|<2R} |f(y)|(|x-y|-R)^{\alpha} dy, \quad -1 < \alpha < 0.$$

This operator occurs in a natural way when one studies the Cesàro- α convergence of singular integrals [2]. Alternatively,

$$M_{\alpha} f(x) = \sup_{R>0} |f| * \varphi_R(x),$$

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where $\varphi_R(x) = R^{-1}\varphi(R^{-1}x)$ and $\varphi(s) = (|s| - 1)^\alpha \chi_{(1,2)}(|s|)$. From this point of view, M_α is a particular case of the operator studied in [5]. It follows from [5, Theorem 1] that M_α is of restricted weak type $(1/(1 + \alpha), 1/(1 + \alpha))$ and that it is not of weak type $(1/(1 + \alpha), 1/(1 + \alpha))$ with respect to the Lebesgue measure when $\alpha < 0$ (notice that M_0 is equivalent to the Hardy-Littlewood maximal operator).

Weighted inequalities for M_α were studied in [2] and [3]. In [3] we obtained a characterization of weighted inequalities for a single weight. The doubling condition plays an essential role in the proof of this characterization; it was also the key reason why we were not able to study the two-weight case in [3].

In this paper we develop a different approach to the study of weighted inequalities for M_α which enables us to obtain a characterization of the two-weighted weak and restricted weak type inequalities for M_α . This new method consists of the study of one-sided versions of M_α

$$M_\alpha^- f(x) = \sup_{R>0} \frac{1}{R^{1+\alpha}} \int_{x-2R}^{x-R} |f(y)|(x - R - y)^\alpha dy$$

and

$$M_\alpha^+ f(x) = \sup_{R>0} \frac{1}{R^{1+\alpha}} \int_{x+R}^{x+2R} |f(y)|(y - x - R)^\alpha dy.$$

These operators are of interest because they naturally appear in the investigation of the Cesàro- α convergence of singular integrals with kernels supported in $(0, \infty)$ and in $(-\infty, 0)$.

The paper is organized as follows. In Section 2 we state and prove a characterization of two-weighted weak and restricted weak type inequalities for M_α^- , M_α^+ and M_α ; in Section 3 we apply these results to the study of the existence of the singular integrals in the Cesàro- α sense.

Throughout the paper, u, v and w are weights, that is, positive measurable functions, $u(A)$ denotes the integral $\int_A u(s) ds$, p' denotes the conjugate exponent of p , $1 < p < \infty$, and the letter C means a positive constant that may change from one line to another.

2. Two-weighted inequalities

We start with the results for M_α^- (analogous results hold for M_α^+).

THEOREM 2.1. *Let u and v be weights on \mathbb{R} and let $-1 < \alpha < 0$. If $1 < p < \infty$, then the following are equivalent:*

- (i) M_α^- is of weak type (p, p) with respect to $u(x) dx$ and $v(x) dx$, that is, there exists C such that $u(\{M_\alpha^- f > \lambda\}) \leq C\lambda^{-p} \int |f|^p v$, for all $\lambda > 0$ and all $f \in L^p(v)$.

(ii) (u, v) satisfies $A_{p,\alpha}^-$, that is, there exists C such that for any three numbers $a < b < c$,

$$\left(\int_b^c u(s) ds \right)^{1/p} \left(\int_a^b v^{1-p'}(s)(b-s)^{\alpha p'} ds \right)^{1/p'} \leq C(c-a)^{1+\alpha}.$$

REMARK. Observe that if $\alpha < 0$ and $\text{ess inf}_{x \in (a,b)} v^{1-p'}(x) > 0$ for some interval (a, b) then the two-weighted weak type (p, p) inequality is not possible for $1 < p \leq 1/(1 + \alpha)$ since (ii) does not hold in this case. However the operator M_α^- is of restricted weak type $(1/(1 + \alpha), 1/(1 + \alpha))$ with respect to the Lebesgue measure. Therefore it is interesting to study the restricted weak type inequalities for pairs of weights.

THEOREM 2.2. Let u and v be weights on \mathbb{R} and let $-1 < \alpha < 0$. If $1 \leq p < \infty$, then the following are equivalent:

(i) M_α^- is of restricted weak type (p, p) with respect to $u(x) dx$ and $v(x) dx$, that is, there exists C such that $u(\{x : M_\alpha^- \chi_E(x) > \lambda\}) \leq C \lambda^{-p} v(E)$ for all $\lambda > 0$ and all measurable $E \subset \mathbb{R}$.

(ii) (u, v) satisfies $RA_{p,\alpha}^-$, that is, there exists C such that for any three numbers $a < b < c$ and all measurable $E \subset \mathbb{R}$

$$\left(\int_b^c u(s) ds \right) \left(\int_a^b \chi_E(s)(b-s)^\alpha ds \right)^p \leq C(c-a)^{(1+\alpha)p} \int_a^b \chi_E(s)v(s) ds.$$

The corresponding results for M_α are obtained immediately from Theorem 2.1 and Theorem 2.2 and from the analogous ones for M_α^+ .

Now we shall state the results for M_α which generalize the weak and restricted weak type inequalities from [3] to the two-weight case.

THEOREM 2.3. Let u and v be weights on \mathbb{R} and let $-1 < \alpha < 0$. If $1 < p < \infty$, then the following are equivalent:

- (i) M_α is of weak type (p, p) with respect to $u(x) dx$ and $v(x) dx$.
- (ii) (u, v) satisfies $A_{p,\alpha}$, that is, there exists C such that for any interval I

$$\left(\int_I u(s) ds \right)^{1/p} \left(\int_{2I \setminus I} v^{1-p'}(s)d(s, I)^{\alpha p'} ds \right)^{1/p'} \leq C|I|^{1+\alpha},$$

where $2I$ is the interval with the same center and double length as I and $d(s, I)$ is the Euclidean distance from s to I .

THEOREM 2.4. Let u and v be weights on \mathbb{R} and let $-1 < \alpha < 0$. If $1 \leq p < \infty$, then the following are equivalent:

- (i) M_α is of restricted weak type (p, p) with respect to $u(x) dx$ and $v(x) dx$.
- (ii) (u, v) satisfies $RA_{p,\alpha}$, that is, there exists C such that for every interval I and all measurable $E \subset \mathbb{R}$

$$\left(\int_I u(s) ds \right) \left(\int_{2I \setminus I} \chi_E(s) d(s, I)^\alpha ds \right)^p \leq C |I|^{(1+\alpha)p} \int_{2I \setminus I} \chi_E(s) v(s) ds.$$

The proofs of Theorem 2.3 and Theorem 2.4 are omitted since they are immediate corollaries of the previous results.

In order to prove Theorem 2.1 and Theorem 2.2 we use a noncentred maximal operator which is pointwise equivalent to M_α^- . In what follows we define this operator and state the pointwise equivalence.

DEFINITION 2.5. For each $x \in \mathbb{R}$, let us consider the family of intervals $\mathcal{A}_x = \{(a, b) : b < x \text{ and } b - a \geq x - b\}$. We define the noncentred maximal operator N_α^- associated with M_α^- as

$$N_\alpha^- f(x) = \sup_{(a,b) \in \mathcal{A}_x} \frac{1}{(b-a)^{1+\alpha}} \int_a^b |f(s)|(b-s)^\alpha ds.$$

PROPOSITION 2.6. Let $-1 < \alpha < 0$. There exists a constant C depending only on α such that $M_\alpha^- f \leq N_\alpha^- f \leq C M_\alpha^- f$, for all measurable functions f .

PROOF. The first inequality is obvious. Let $(a, b) \in \mathcal{A}_x$, $R = x - a$ and let N be the natural number such that $x - 2^{-N}R \leq b < x - 2^{-N-1}R$. Then

$$\begin{aligned} & \int_a^b |f(s)|(b-s)^\alpha ds \\ &= \sum_{i=0}^{N-1} \int_{x-R/2^i}^{x-R/2^{i+1}} |f(s)| \left(x - \frac{R}{2^{i+1}} - s\right)^\alpha \left(\frac{b-s}{x - (R/2^{i+1}) - s}\right)^\alpha ds \\ & \quad + \int_{x-R/2^N}^b |f(s)|(b-s)^\alpha ds = \text{I} + \text{II}. \end{aligned}$$

Since $(a, b) \in \mathcal{A}_x$,

$$\text{II} \leq \int_{x-2(x-b)}^b |f(s)|(b-s)^\alpha ds \leq (x-b)^{1+\alpha} M_\alpha^- f(x) \leq (b-a)^{1+\alpha} M_\alpha^- f(x).$$

On the other hand, since the function $s \rightarrow [(b-s)/(x - 2^{-i-1}R - s)]^\alpha$ is decreasing

on $(x - 2^{-i}R, x - 2^{-i-1}R)$, $0 \leq i \leq N - 1$,

$$\begin{aligned} \text{I} &\leq \left(\sum_{i=0}^{N-1} \left(b - \left(x - \frac{R}{2^i} \right) \right)^\alpha \frac{R}{2^{i+1}} \right) M_\alpha^- f(x) \\ &\leq M_\alpha^- f(x) \sum_{i=0}^{N-1} \int_{x-R/2^i}^{x-R/2^{i+1}} (b-s)^\alpha ds \leq C(b-a)^{1+\alpha} M_\alpha^- f(x), \end{aligned}$$

and we are done. \square

PROOF OF THEOREM 2.1. By Proposition 2.6, (i) is equivalent to the weighted weak type (p, p) inequality for N_α^- . Let $a < b < c$ and let $\bar{a} < a$ be such that $b - \bar{a} = c - a$. If we consider the function $f(s) = v^{1-p'}(s)(b-s)^{\alpha(p'-1)}\chi_{(a,b)}(s)$, then for all $x \in (b, c)$

$$N_\alpha^- f(x) \geq \frac{1}{(b-\bar{a})^{1+\alpha}} \int_a^b v^{1-p'}(s)(b-s)^{\alpha p'} ds \equiv \lambda.$$

This means that $(b, c) \subset \{N_\alpha^- f \geq \lambda\}$. Then (ii) follows from (i) (with N_α^-) by a standard argument. \square

The implication (ii) implies (i) follows from the following proposition and the fact that the maximal operator $M_u^- g(x) = \sup_{h < x} \left(\int_h^x |g|u / \int_h^x u \right)$ is of weak type $(1, 1)$ with respect to the measure $u(x) dx$.

PROPOSITION 2.7. Let $-1 < \alpha < 0$ and $p > 1$. If (u, v) satisfies $A_{p,\alpha}^-$, then there exists $C > 0$ such that for every measurable function f

$$N_\alpha^- f \leq C[M_u^-(|f|^p v u^{-1})]^{1/p}.$$

PROOF. Let $x \in \mathbb{R}$ and $(a, b) \in \mathcal{I}_x$. First, let us assume that $4 \int_b^x u > \int_a^x u$. Since the pair (u, v) satisfies $A_{p,\alpha}^-$, we have

$$\begin{aligned} \int_a^b |f(s)|(b-s)^\alpha ds &\leq \left(\int_a^b |f|^p(s)v(s) ds \right)^{1/p} \left(\int_a^b v^{-p'/p}(s)(b-s)^{\alpha p'} ds \right)^{1/p'} \\ &\leq C \left(\int_a^x |f|^p(s)v(s) ds \right)^{1/p} \left(\int_b^x u(s) ds \right)^{-1/p} (x-a)^{1+\alpha} \\ &\leq C[M_u^-(|f|^p v u^{-1})]^{1/p}(x)(b-a)^{1+\alpha}. \end{aligned}$$

Assume now that $4 \int_b^x u \leq \int_a^x u$. Let $\{x_i\}$ be the increasing sequence in $[a, x]$ defined by $x_0 = a$ and $\int_{x_i}^x u = \int_{x_{i+1}}^{x_{i+1}} u = \frac{1}{2} \int_{x_i}^x u$. Let N be such that $x_N \leq b < x_{N+1}$ (observe

that $N \geq 2$). Then we have

$$\int_a^b |f(s)|(b-s)^\alpha ds = \sum_{i=0}^{N-2} \int_{x_i}^{x_{i+1}} |f(s)|(b-s)^\alpha ds + \int_{x_{N-1}}^b |f(s)|(b-s)^\alpha ds \\ = \text{I} + \text{II}.$$

By the $A_{p,\alpha}^-$ condition and the fact that $\int_{x_{N-1}}^x u \leq 4 \int_b^x u$, we have

$$\text{II} \leq \left(\int_{x_{N-1}}^b |f|^p(s)v(s) ds \right)^{1/p} \left(\int_{x_{N-1}}^b v^{-p'/p}(s)(b-s)^{\alpha p'} ds \right)^{1/p'} \\ \leq C[M_u^-(|f|^p v u^{-1})]^{1/p}(x)(b-a)^{1+\alpha}.$$

On the other hand, since the function $s \rightarrow [(b-s)/(x_{i+1}-s)]^\alpha$ is decreasing in the interval (x_i, x_{i+1}) , $0 \leq i \leq N-2$, we obtain

$$\int_{x_i}^{x_{i+1}} |f(s)|(b-s)^\alpha ds \leq \left(\frac{b-x_i}{x_{i+1}-x_i} \right)^\alpha \int_{x_i}^{x_{i+1}} |f(s)|(x_{i+1}-s)^\alpha ds \\ \leq \left(\frac{b-x_i}{x_{i+1}-x_i} \right)^\alpha \left(\int_{x_i}^{x_{i+1}} |f|^p(s)v(s) ds \right)^{1/p} \left(\int_{x_i}^{x_{i+1}} v^{-p'/p}(s)(x_{i+1}-s)^{\alpha p'} ds \right)^{1/p'} \\ \leq C \left(\frac{b-x_i}{x_{i+1}-x_i} \right)^\alpha \left(\int_{x_i}^{x_{i+1}} |f|^p(s)v(s) ds \right)^{1/p} \left(\int_{x_{i+1}}^{x_{i+2}} u(s) ds \right)^{-1/p} (x_{i+2}-x_i)^{1+\alpha} \\ \leq C(b-x_i)^\alpha (x_{i+2}-x_i) \left(\frac{\int_{x_i}^x |f|^p(s)v(s) ds}{\int_{x_i}^x u(s) ds} \right)^{1/p} \\ \leq C[M_u^-(|f|^p v u^{-1})]^{1/p}(x) \int_{x_i}^{x_{i+2}} (b-s)^\alpha ds.$$

Now, summing up in i , we get

$$\text{I} \leq C[M_u^-(|f|^p v u^{-1})]^{1/p}(x) \int_a^{x_N} (b-s)^\alpha ds \leq C[M_u^-(|f|^p v u^{-1})]^{1/p}(x)(b-a)^{1+\alpha}.$$

Finally, putting together the estimates of I and II, we are done. \square

PROOF OF THEOREM 2.2. The proof is similar to that of Theorem 2.1. We give just a sketch. First, (ii) follows from (i) on applying the standard argument to $\chi_{E \cap (a,b)}$. The converse follows from the fact that (ii) implies $N_\alpha^- \chi_E(x) \leq C[M_u^-(\chi_E v u^{-1})]^{1/p}(x)$, for some constant C independent of the measurable subset E . To prove the above inequality, let $x \in \mathbb{R}$, $(a, b) \in \mathcal{A}_x$ and assume first that $4 \int_b^x u > \int_a^x u$. Since (u, v)

satisfies $RA_{p,\alpha}^-$ we obtain

$$\begin{aligned} \int_a^b \chi_E(s)(b-s)^\alpha ds &\leq C(x-a)^{1+\alpha} \left(\int_a^b \chi_E(s)v(s) ds \right)^{1/p} \left(\int_b^x u(s) ds \right)^{-1/p} \\ &\leq C(b-a)^{1+\alpha} \left(\int_a^x \chi_E(s)v(s) ds \right)^{1/p} \left(\int_a^x u \right)^{-1/p} \\ &\leq C(b-a)^{1+\alpha} [M_u^-(\chi_E v u^{-1})]^{1/p}(x). \end{aligned}$$

If $4 \int_b^x u \leq \int_a^x u$, we proceed as in the proof of Proposition 2.7. □

3. Singular integrals in the Cesàro sense

Let K be a Calderón-Zygmund kernel on \mathbb{R} , that is, a function $K \in L^1_{loc}(\mathbb{R} \setminus \{0\})$ such that

- (1) $|K(x)| \leq C|x|^{-1}, |x| > 0$,
- (2) $|K(x-y) - K(x)| \leq C|y||x|^{-2}$, if $|x| > 2|y| > 0$,
- (3) $|\int_{\epsilon < |x| < N} K(x) dx| \leq C$ for all ϵ and all N with $0 < \epsilon < N$.

If the limit $\lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |y| < 1} K(y) dy$ exists, then the principal-value singular integral

$$Tf(x) = \lim_{\epsilon \rightarrow 0^+} \int_{|x-y|>\epsilon} K(x-y)f(y) dy$$

exists for $f \in L^p(wdx)$ with w in the Muckenhoupt class A_p (see for instance [4]). When the kernel K has support in $(0, \infty)$ (or in $(-\infty, 0)$), then, as proved in [1], the same result holds for a wider class of weights, more precisely for weights in the Sawyer class $A_{p,0}^- \equiv A_{p,0}^-$ ([7]).

Recently, in [2], we studied the existence in the Cesàro- α sense of the singular integral associated with K for $-1 < \alpha < 0$, that is, the existence of the limit

$$\lim_{\epsilon \rightarrow 0^+} T_{\epsilon,\alpha} f(x) = \lim_{\epsilon \rightarrow 0^+} \int_{|x-y|>\epsilon} f(y)K(x-y) \left(1 - \frac{\epsilon}{|x-y|}\right)^\alpha dy,$$

in the setting of weighted L^p -spaces. The aim in this section is to obtain sharper results on singular integrals in the Cesàro- α sense for kernels with support in $(0, \infty)$ (or in $(-\infty, 0)$). We shall show, using the results of Section 2, that, for these kernels, the results in [2] are true for a wider class of weights.

One of the key steps in [2] is the pointwise estimate from above of the maximal operator $T_\alpha^* f = \sup_{\epsilon > 0} |T_{\epsilon,\alpha} f|$ by $C(M_\alpha f + T_0^* f)$. If the support of K is contained in $(0, \infty)$, then we can improve this estimate by replacing M_α with a smaller operator M_α^- .

PROPOSITION 3.1. *Let $-1 < \alpha < 0$ and let K be a Calderón-Zygmund kernel with support contained in $(0, \infty)$. If f is a measurable function such that $T_{\epsilon,\alpha}f(x)$ is defined for every $\epsilon > 0$, then there exists $C > 0$ independent of f such that*

$$T_{\alpha}^*f(x) \leq C[M_{\alpha}^-f(x) + T_0^*f(x)].$$

The proof is similar to that of [2, Proposition 2.5], and is therefore omitted. This proposition together with Theorem 2.1 in this paper and [1, Theorem 2.1] enables us to prove the following result.

THEOREM 3.2. *Let $-1 < \alpha < 0$ and let K be a Calderón-Zygmund kernel with support contained in $(0, \infty)$ such that the limit*

$$\lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^1 K(y) \left(1 - \frac{\epsilon}{y}\right)^{\alpha} dy$$

exists. Then the singular integral exists a.e. in the Cesàro- α sense if $f \in L^p(w dx)$ with $p(1 + \alpha) > 1$ and $w \in A_{p,\alpha}^-$ (the pair (w, w) satisfies $A_{p,\alpha}^-$).

To prove the theorem we have to show first that the truncations $T_{\epsilon,\alpha}f$ are well defined for $f \in L^p(wdx)$, $w \in A_{p,\alpha}^-$, $p(1 + \alpha) > 1$. This can be proved as in [2, Theorem 2.7]. The rest of the proof is a consequence of the following facts: (i) the existence of the limit $\lim_{\epsilon \rightarrow 0^+} T_{\epsilon,\alpha}f$ for f in a dense class and (ii) the weak type (p, p) boundedness with respect to $w(x) dx$ of the maximal operator T_{α}^* . The former is clear since $L^p(wdx) \cap L^p(dx)$ is dense in $L^p(wdx)$ and the convergence holds for $f \in L^p(wdx) \cap L^p(dx)$ by [2, Theorem 2.7]. The latter immediately follows from Proposition 3.1, Theorem 2.1, [1, Theorem 2.1] and the easy implication $w \in A_{p,\alpha}^- \Rightarrow w \in A_{p,0}^- \equiv A_p^-$.

REMARK. In particular, the result holds if w belongs to the Sawyer’s class [7] $A_{p(1+\alpha)}^-$ since $A_{p(1+\alpha)}^- \subset A_{p,\alpha}^-$. This inclusion follows from $A_r^- \subset A_{p,\alpha}^-$, $1 < r < p(1 + \alpha)$, which is true by Hölder’s inequality and the implication $w \in A_{p(1+\alpha)}^- \Rightarrow w \in A_r^-$ for some $r < p(1 + \alpha)$ (see [7] or [6]).

We do not know whether $A_{p(1+\alpha)}^-$ is equal to $A_{p,\alpha}^-$ for $\alpha < 0$ and $p > 1/(1 + \alpha)$ but in the endpoint $p = 1/(1 + \alpha)$ it is possible to see that $RA_{1/(1+\alpha),\alpha}^-$ equals the Sawyer’s class A_1^- . The proof of this fact is similar to the proof of [3, Proposition 6.5]. Then, following the steps in the proof of [2, Theorem 2.7] and using the corresponding results in this paper and in [1] we have our next result.

THEOREM 3.3. *Let α and K be as in Theorem 3.2. If f belongs to the Lorentz space $L_{1/(1+\alpha),1}(\omega dx) = \{f : \int_0^{\infty} [\omega(\{x : |f(x)| > t\})]^{1+\alpha} dt < \infty\}$ and $\omega \in A_1^-$, then the singular integral exists a.e. in the Cesàro- α sense.*

EXAMPLE. Observe that the Calderón-Zygmund kernel

$$K(x) = \frac{1}{x} \frac{\sin(\log x)}{\log x} \chi_{(0,\infty)}(x)$$

given in [1] satisfies the condition in Theorem 3.2, that is,

$$\lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^1 K(y) \left(1 - \frac{\epsilon}{y}\right)^{\alpha} dy$$

exists. In fact, for any $0 < \epsilon < 1/2$, if $\Omega(x) = \sin x/x$, then

$$\begin{aligned} \int_{\epsilon}^1 K(y) \left(1 - \frac{\epsilon}{y}\right)^{\alpha} dy &= \int_{\epsilon}^1 \frac{\Omega(\log y)}{y} \left(1 - \frac{\epsilon}{y}\right)^{\alpha} dy \\ &= \int_{\epsilon}^{2\epsilon} \cdots dy + \int_{2\epsilon}^1 \cdots dy = I + II. \end{aligned}$$

Applying the Hölder inequality to I with $p > 1/(1 + \alpha)$ and changing variables we obtain

$$\begin{aligned} |I| &\leq \left(\int_{\epsilon}^{2\epsilon} \frac{|\Omega(\log y)|^p}{y} dy \right)^{1/p} \left(\int_{\epsilon}^{2\epsilon} \left(1 - \frac{\epsilon}{y}\right)^{\alpha p'} \frac{1}{y} dy \right)^{1/p'} \\ &\leq C \left(\int_{\log \epsilon}^{\log 2\epsilon} |\Omega(t)|^p dt \right)^{1/p} \leq C \left(\int_{\log \epsilon}^{\log 2\epsilon} \frac{1}{|t|^p} dt \right)^{1/p} \end{aligned}$$

and therefore $\lim_{\epsilon \rightarrow 0^+} I = 0$. On the other hand,

$$II = \int_{2\epsilon}^1 \frac{\Omega(\log y)}{y} \left[\left(1 - \frac{\epsilon}{y}\right)^{\alpha} - 1 \right] dy + \int_{2\epsilon}^1 \frac{\Omega(\log y)}{y} dy = III + IV.$$

Clearly, by changing the variables, we see that $\lim_{\epsilon \rightarrow 0^+} IV$ exists. In order to estimate III, we apply the mean value theorem to get

$$|III| \leq |\alpha| \int_{2\epsilon}^1 |\Omega(\log y)| \left(1 - \frac{\epsilon}{y}\right)^{\alpha-1} \frac{\epsilon}{y^2} dy.$$

Changing variables again, we obtain

$$|III| \leq |\alpha| \int_{\epsilon}^{1/2} |\Omega(\log(\epsilon/t))| (1-t)^{\alpha-1} dt.$$

Finally, $\lim_{\epsilon \rightarrow 0^+} III = 0$, applying the dominated convergence theorem and the facts that Ω is bounded and $\lim_{\epsilon \rightarrow 0^+} \Omega(\log(\epsilon/t)) = 0$.

REMARK. If we do not assume anything about the support of K , then Theorem 3.2 is valid for weights w in $A_{p,\alpha}$. The proof is similar to the proof of [2, Theorem 2.7] but using Theorem 2.3 instead of [2, Theorem 2.6].

An analogous comment can be written about Theorem 3.3, that is, Theorem 3.3 is valid for Calderón-Zygmund kernels and weights w in the Muckenhoupt Class A_1 (in fact, notice that this result is contained in [2, Theorem 2.7]).

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