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A NOTE ON MULTIPLIERS OF $L^p(G, A)$

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Abstract

Let G be a locally compact abelian group, $1 < p < \infty$, and A be a commutative Banach algebra. In this paper, we study the space of multipliers on $L^p(G, A)$ and characterize it as the space of multipliers of certain Banach algebra. We also study the multipliers space on $L^1(G, A) \cap L^p(G, A)$.

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1. Introduction and preliminaries

Let G be a locally compact abelian group with Haar measure, A be a commutative Banach algebra with identity of norm 1. Denote by $L^1(G, A)$ the space of all Bochner integrable A-valued functions defined on G. It is a commutative Banach algebra under convolution and has an approximate identity in $C_c(G, A)$ of norm 1, $L^p(G, A)$ is the set of all strong measurable functions $f : G \to A$ such that $\| f(x)\|_A^p$ is integrable for $1 \le p < \infty$, that is, $\| f(x)\|_A^p \in L^1(G)$. The norm of a function f in $L^p(G, A)$ is defined as

$$\| f \|_{L^p(G,A)} = \left(\int_G \| f(x)\|_A^p \, dx \right)^{1/p} \quad 1 \le p < \infty.$$

It follows that $L^p(G, A)$ is a Banach space for $1 \le p < \infty$ and $L^p(G, A)$ is an essential $L^1(G, A)$-module under convolution such that for $f \in L^1(G, A)$ and $g \in L^p(G, A)$, we have

$$\| f * g\|_{L^p(G,A)} \le \| f \|_{L^1(G,A)} \|g\|_{L^p(G,A)}.$$

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Denote by $C_c(G, A)$ the space of all A-valued continuous functions with compact support. $C_c(G, A)$ is dense in $L^p(G, A)$ (for more details see [2, 6, 7]).

For each $f \in L^1(G, A)$, define the mapping T_f by $T_f(g) = f * g$ whenever $g \in L^p(G, A)$. T_f is an element of $\ell(L^p(G, A))$, Banach algebra of all continuous linear operators from $L^p(G, A)$ to $L^p(G, A)$, and $\|T_f\| \leq \|f\|_{L^1(G,A)}$. Identifying f with T_f, we get an embedding of $L^1(G, A)$ in $\ell(L^p(G, A))$. Let $H_{L^1(G,A)}(L^p(G, A))$ denote the space of all module homomorphisms of $L^1(G, A)$-module $L^p(G, A)$, that is, an operator $T \in \ell(L^p(G, A))$ satisfies $T(f*g) = f*T(g)$ for each $f \in L^1(G, A)$, $g \in L^p(G, A)$.

The module homomorphisms space, called the *multipliers space*

$$H_{L^1(G,A)}(L^p(G, A)),$$

is an essential $L^1(G, A)$-module by $(f \circ T)(g) = f * T(g) = T(f * g)$ for all $g \in L^p(G, A)$.

Let A be a Banach algebra without order, for all $x \in A$, $xA = Ax = \{0\}$ implies $x = 0$. Obviously if A has an identity or an approximate identity then it is without order. A multiplier of A is a mapping $T : A \to A$ such that

$$T(fg) = fT(g) = (Tf)g, \quad (f, g \in A).$$

By $M(A)$ we denote the collection of all multipliers of A. Every multiplier turns out to be a bounded linear operator on A. If A is a commutative Banach algebra without order, $M(A)$ is a commutative operator algebra and $M(A)$ is called the multiplier algebra of A [15].

In this paper we are interested in the relationship between the multipliers $L^1(G, A)$-module and the multipliers on a certain normed (or Banach) algebra. The multipliers of type (p, p) and multipliers of the group L^p-algebras were studied and developed by many authors. Let us mention McKennon [10, 11] Griffin [5], Feichtinger [3] and Fisher [4]. In these studies, a multiplier is defined to be an invariant operator (a bounded linear operator T commutes with translation). In the case of a scalar function space on G, the multipliers are identified with the translation invariant operators. However, in the Banach-valued function spaces, an invariant operator does not need to be a multiplier [8, 14]. Dutry [1] gave a new proof of the identification theorem concerning multipliers of $L^1(G)$-module and of Banach algebra. His ideas are used in this paper for the generalization of the results of McKennon concerning multipliers of type (p, p) to the Banach-valued function spaces.

We briefly describe the content of this paper. In Section 2 we construct the p-temperate functions space for the Banach-valued function spaces whenever $1 < p < \infty$ and study their basic properties. In Section 3 we characterize the multipliers space of $L^p(G, A)$ as a certain Banach algebra and extend the results of McKennon to Banach-valued space. In Section 4 we study the multipliers space of $L^1(G, A) \cap L^p(G, A)$.

2. The $L^t_p(G, A)$ space and its basic properties

Let G be a locally compact abelian group with Haar measure, A a commutative Banach algebra with identity of norm 1.

DEFINITION 2.1. An element $f \in L^p(G, A)$ is called p-temperate function if

$$\|f\|^t_{L^p(G,A)} = \sup\{\|g * f\|_{L^p(G,A)} \mid g \in L^p(G, A), \|g\|_{L^p(G,A)} \leq 1\} < \infty$$

or

$$\|f\|^t_{L^p(G,A)} = \sup\{\|g * f\|_{L^p(G,A)} \mid g \in C_c(G, A), \|g\|_{L^p(G,A)} \leq 1\} < \infty.$$

The space of all such f is denoted by $L^t_p(G, A)$. It is easy to see that

$$\left(L^t_p(G, A), \|\cdot\|^t_{L^p(G,A)}\right)$$

is a normed space. For each $f \in L^t_p(G, A)$, there is precisely one bounded linear operator on $L^p(G, A)$, denoted by W_f, such that

$$W_f(g) = g * f \quad \text{and} \quad \|W_f\| = \|f\|^t_{L^p(G,A)}. \tag{2.1}$$

It is easy to check that $W_f \in H_{L^1(G,A)}(L^p(G, A))$.

PROPOSITION 2.2. *$L^t_p(G, A)$ is a dense subspace of $L^p(G, A)$.*

PROOF. Since each $f \in C_c(G, A)$ belongs to $L^t_p(G, A)$ and $C_c(G, A)$ is dense in $L^p(G, A)$, the proof is completed. □

LEMMA 2.3. *The space $L^t_p(G, A)$ is a normed algebra under the convolution.*

PROOF. By (2.1) we get

$$\begin{aligned}\|f * g\|^t_{L^p(G,A)} &= \sup_{\|h\|_{L^p(G,A)} \leq 1} \|h * (f * g)\|_{L^p(G,A)} = \sup_{\|h\|_{L^p(G,A)} \leq 1} \|W_g(h * f)\|_{L^p(G,A)} \\ &\leq \|W_g\| \sup_{\|h\| \leq 1} \|h * f\|_{L^p(G,A)} = \|g\|^t_{L^p(G,A)} \|f\|^t_{L^p(G,A)}\end{aligned}$$

for all f and g in $L^t_p(G, A)$. Hence $(L^t_p(G, A), \|\cdot\|^t_{L^p(G,A)})$ is a normed algebra.

Let us notice that

$$W_{f*g} = W_f \circ W_g = W_g \circ W_f \tag{2.2}$$

for all f and g in $L^t_p(G, A)$. Moreover, the closed linear subspace of $\ell(L^p(G, A))$ spanned by $\{W_{f*g} \mid f \in L^t_p(G, A), g \in C_c(G, A)\}$ is denoted by $\Lambda_{L^p(G,A)}$. □

THEOREM 2.4. *The space* $\Lambda_{L^p(G,A)}$ *is a complete subalgebra of* $H_{L^1(G,A)}(L^p(G,A))$ *and it has a minimal approximate identity, that is, a net* (T_α) *such that* $\overline{\lim}_\alpha \|T_\alpha\| \leq 1$ *and* $\lim_\alpha \|T_\alpha \circ T - T\| = 0$ *for all* $T \in \Lambda_{L^p(G,A)}$.

PROOF. Let $f \in L_p^t(G,A)$, then $W_f \in \ell(L^p(G,A))$. Since $L^p(G,A)$ is a $L^1(G,A)$-module we have

$$W_f(g * h) = g * h * f = g * W_f(h)$$

for all $g \in L^1(G,A)$ and $h \in L^p(G,A)$.

Thus W_f belongs to $H_{L^1(G,A)}(L^p(G,A))$. Since $H_{L^1(G,A)}(L^p(G,A))$ is a Banach algebra under the usual operator norm, $\Lambda_{L^p(G,A)}$ is a complete subalgebra of $H_{L^1(G,A)}(L^p(G,A))$.

Now we only need to prove the existence of minimal approximate identity of $\Lambda_{L^p(G,A)}$. Let (Φ_{U_α}) be a minimal approximate identity for $L^1(G,A)$ [2]. If (Φ_α) denotes the product net of (Φ_{U_α}) with itself, then (Φ_α) is again minimal approximate identity for $L^1(G,A)$. It is easy to see that the net W_{Φ_α} is in $\Lambda_{L^p(G,A)}$ and $\overline{\lim}_\alpha \|W_{\Phi_\alpha}\| \leq 1$.

Let $f \in L_p^t(G,A)$ and $g \in C_c(G,A)$. Since (2.2) and (Φ_α) is a minimal approximate identity for $L^1(G,A)$, we get

$$\begin{aligned}\overline{\lim_\alpha}\|W_{\Phi_\alpha} \circ W_{f*g} - W_{f*g}\| &= \overline{\lim_\alpha}\|(W_{\Phi_\alpha} \circ W_g - W_g) \circ W_f\| \leq \overline{\lim_\alpha}\|W_{g*\Phi_\alpha - g}\|\,\|W_f\| \\ &\leq \overline{\lim_\alpha}\|g * \Phi_\alpha - g\|_{L^1(G,A)}\|W_f\| = 0.\end{aligned}$$

Consequently, we have $\overline{\lim}_\alpha \|W_{\Phi_\alpha} \circ T - T\| = 0$ for all $T \in \Lambda_{L^p(G,A)}$. □

PROPOSITION 2.5. *The space* $\Lambda_{L^p(G,A)}$ *is an essential* $L^1(G,A)$*-module.*

PROOF. Let $g \in L^1(G,A)$, $W_f \in \Lambda_{L^p(G,A)}$. Define $g \circ W_f : L^p(G,A) \to L^p(G,A)$ by letting $(g \circ W_f)(h) = W_f(h * g) = W_f(g * h)$ for each $h \in L^p(G,A)$.

$$\|g \circ W_f\| = \sup_{\|h\|_{L^p(G,A)} \leq 1} \|W_f(g * h)\|_{L^p(G,A)} \leq \|f\|^t_{L^p(G,A)} \|g\|_{L^1(G,A)}.$$

Consequently, $\Lambda_{L^p(G,A)}$ is a $L^1(G,A)$-module. On the other hand, since $L^1(G,A)$ has a minimal approximate identity (Φ_α), $(\Phi_\alpha \geq 0)$ with a compact support such that it is also an approximate identity in $L^p(G,A)$, [2].

For any $W_f \in \Lambda_{L^p(G,A)}$, we have

$$\begin{aligned}\|\Phi_\alpha \circ W_f - W_f\| &= \sup_{\|h\|_{L^p(G,A)} \leq 1} \|(\Phi_\alpha \circ W_f - W_f)(h)\|_{L^p(G,A)} \\ &= \sup_{\|g\|_{L^p(G,A)} \leq 1} \|W_f(\Phi_\alpha * h - h)\|_{L^p(G,A)} \\ &\leq \|f\|^t_{L^p(G,A)} \|\Phi_\alpha * h - h\|_{L^p(G,A)} = 0\end{aligned}$$

for all $h \in L^p(G, A)$. Using [13, Proposition 3.4] we have that $\Lambda_{L^p(G,A)}$ is an essential $L^1(G, A)$-module. Moreover, $\Lambda_{L^p(G,A)}$ contains $L^1(G, A)$. □

3. Identification for the multipliers spaces of $L^1(G, A)$-module with the multipliers space of certain normed algebra

In this section, we obtain the generalization of the results of McKennon [10, 11, 12] to the Banach-valued spaces.

PROPOSITION 3.1. *Let T be in $H_{L^1(G,A)}(L^p(G, A))$ and $f, g \in L^p(G, A)$. Then,*

(i) *if $f \in L^t_p(G, A)$, $T(f) \in L^t_p(G, A)$;*
(ii) *if $g \in L^t_p(G, A)$, $T(f * g) = f * T(g)$.*

PROOF. (i) Let f be in $L^t_p(G, A)$. By the definition $T \in H_{L^1(G,A)}(L^p(G, A))$,

$$\begin{aligned} \|T(f)\|^t_{L^p(G,A)} &= \sup\{\|h * T(f)\|_{L^p(G,A)} \mid h \in C_c(G, A), \|h\|_{L^p(G,A)} \leq 1\} \\ &= \sup\{\|T(h * f)\|_{L^p(G,A)} \mid h \in C_c(G, A), \|h\|_{L^p(G,A)} \leq 1\} \\ &\leq \|T\| \|f\|^t_{L^p(G,A)} < \infty. \end{aligned}$$

Hence we get $T(f) \in L^t_p(G, A)$.

To prove (ii), let g be in $L^t_p(G, A)$. Since $C_c(G, A)$ is dense in $L^p(G, A)$, for each $f \in L^p(G, A)$ there exists $(f_n) \subset C_c(G, A)$ such that $\lim_n \|f_n - f\|_{L^p(G,A)} = 0$.

From (2.1) we get $\lim_n \|f_n * g - f * g\|_{L^p(G,A)} = 0$. By (i) we have

$$\lim_n \|f_n * T(g) - f * T(g)\|_{L^p(G,A)} = 0$$

and $f * T(g) = \lim_n f_n * T(g) = \lim_n T(f_n * g) = T(f * g)$. □

DEFINITION 3.2. For the space $\Lambda_{L^p(G,A)}$, the space $(\Lambda_{L^p(G,A)})$ is defined by

$$(\Lambda_{L^p(G,A)}) = \{T \in H_{L^1(G,A)}(L^p(G, A)) \mid T \circ W \in \Lambda_{L^p(G,A)}, \text{for all } W \in \Lambda_{L^p(G,A)}\}.$$

LEMMA 3.3. *The space $(\Lambda_{L^p(G,A)})$ is equal to the space $H_{L^1(G,A)}(L^p(G, A))$.*

PROOF. Let $T \in H_{L^1(G,A)}(L^p(G, A))$. For any $S \in \Lambda_{L^p(G,A)}$, we have $S = W_{f*g}$, for each $f \in L^t_p(G, A)$, $g \in C_c(G, A)$. By Proposition 3.1 we get

$$(T \circ W_{f*g})(h) = T(h * f * g) = h * T(f * g) = W_{T(f*g)}(h) = W_{g*T(f)}(h)$$

for all $h \in L^p(G, A)$. Thus $T \circ S \in \Lambda_{L^p(G,A)}$. Consequently,

$$(\Lambda_{L^p(G,A)}) = H_{L^1(G,A)}(L^p(G, A)).$$

□

Let us note that we have the inclusion $M(\Lambda_{L^p(G,A)}) \subset H_{L^1(G,A)}(\Lambda_{L^p(G,A)})$.

THEOREM 3.4. *Let G be a locally compact abelian group, $1 < p < \infty$, and A be a commutative Banach algebra with identity of norm 1. The space of multipliers on Banach algebra $\Lambda_{L^p(G,A)}$, $M(\Lambda_{L^p(G,A)})$, is isometrically isomorphic to the space* $(\Lambda_{L^p(G,A)})$.

PROOF. Define the mapping $\mathcal{F} : \Lambda_{L^p(G,A)} \to M(\Lambda_{L^p(G,A)})$ by letting $\mathcal{F}(T) = \rho_T$ for each $T \in \Lambda_{L^p(G,A)}$, where $\rho_T(S) = T \circ S$ for all $S \in \Lambda_{L^p(G,A)}$. Note that $\mathcal{F}$ is well defined and moreover if $\rho_T(S \circ K) = T \circ S \circ K = \rho_T(S) \circ K$ for all $S, K \in \Lambda_{L^p(G,A)}$, $\rho_T \in M(\Lambda_{L^p(G,A)})$.

It is obvious that the mapping $T \to \rho_T$ is linear. We now show that it is an isometry. We obtain easily $\|T\| \geq \|\rho_T\|$. Since W_{Φ_α} is a minimal approximate identity for the space $\Lambda_{L^p(G,A)}$, we have

$$\|\rho_T\| = \sup_{S \in \Lambda_{L^p(G,A)}} \frac{\|T \circ S\|}{\|S\|} \geq \sup_{\alpha} \frac{\|T \circ W_{\Phi_\alpha}\|}{\|W_{\Phi_\alpha}\|} \geq \|T\|.$$

Therefore, $\|\rho_T\| = \|T\|$.

Finally, we show that the mapping $T \to \rho_T$ is onto. It is sufficient to show that if ρ is an element of $M(\Lambda_{L^p(G,A)})$, the limit of $\rho\Phi_\alpha$ exists for the strong operator topology and this limit T satisfies $\rho_T = \rho$. Let ρ be in $M(\Lambda_{L^p(G,A)})$ and $(\Phi_\alpha) \subset L^1(G, A)$. By $\rho\Phi_\alpha(f * g) = \rho(\Phi_\alpha * f)g$, we have

$$\lim_{\alpha}(\rho\Phi_\alpha)(f * g) = \rho f(g) \tag{3.1}$$

for all $f \in L^1(G, A)$, $g \in L^p(G, A)$. Since $L^p(G, A)$ is an essential $L^1(G, A)$-module, the limit of $(\rho\Phi_\alpha)(f * g)$ exists in $L^p(G, A)$ and is denoted by Tg, and $Tg \in H_{L^1(G,A)}(L^p(G, A))$. From (3.1) we get, for all $f \in L^1(G, A)$,

$$f \circ T = \rho f. \tag{3.2}$$

So for all $W \in \Lambda_{L^p(G,A)}$ we have

$$T \circ \Phi_\alpha \circ W = (\rho\Phi_\alpha) \circ W = \rho(\Phi_\alpha \circ W). \tag{3.3}$$

Since $\Lambda_{L^p(G,A)}$ is an essential $L^1(G, A)$-module, we have $T \circ W = \rho(W)$ and also $\rho_T(W) = \rho(W)$ for all $W \in \Lambda_{L^p(G,A)}$. So $\rho_T = \rho$. □

COROLLARY 3.5. *The following spaces of multipliers are isometrically isomorphic*: $M(\Lambda_{L^p(G,A)}) \cong H_{L^1(G,A)}(L^p(G, A))$.

REMARK 3.6. (i) Let $p = 1$. Since $L_1^t(G, A)$ is a Banach algebra, it follows that $L_1^t(G, A) = L^1(G, A)$ and $\Lambda_{L^p(G,A)}$ is isomorphic to $L^1(G, A)$ as a Banach algebra. Thus by [14] we get $H_{L^1(G,A)}(L^p(G, A)) = M(L^1(G, A)) = M(G, A)$. Here $M(G, A)$ denotes A-valued bounded measure space.

(ii) If $A = \not{c}$ we have the case of the scalar valued function space in [10, 11].

4. The identification for the space $L^1(G, A) \cap L^p(G, A)$

Before starting the identification, let us mention some properties of the space $L^1(G, A) \cap L^p(G, A)$.

If $1 < p < \infty$, then the space $L^1(G, A) \cap L^p(G, A)$ is a Banach space with the norm $|||f||| = \|f\|_{L^1(G,A)} + \|f\|_{L^p(G,A)}$ for $f \in L^1(G, A) \cap L^p(G, A)$.

LEMMA 4.1. *For* $L^1(G, A) \cap L^p(G, A)$,

(i) $L^1(G, A) \cap L^p(G, A)$ *is dense in* $L^1(G, A)$ *with respect to the norm* $\|\cdot\|_{L^1(G,A)}$.

(ii) *For every* $f \in L^1(G, A) \cap L^p(G, A)$ *and* $x \in G$, $x \to L_x f$ *is continuous, where* $L_x f(y) = f(x^{-1}y)$ *for all* $y \in G$.

PROOF. (i) Since $C_c(G, A)$ is dense in $L^1(G, A)$ with respect to the norm $\|\cdot\|_{L^1(G,A)}$ and $C_c(G, A) \subset L^1(G, A) \cap L^p(G, A) \subset L^1(G, A)$ it is obtained.

(ii) Let $f \in L^1(G, A) \cap L^p(G, A)$. It is easy to see that $|||L_x f||| = |||f|||$. By [2] the function $x \to L_x f$ is continuous, $G \to L^p(G, A)$, where $1 \le p < \infty$. Therefore for any $x_\circ \in G$ and $\epsilon > 0$, there exists $U_1 \in \vartheta_{(x_\circ)}$ and $U_2 \in \vartheta_{(x_\circ)}$ such that for every $x \in U_1$

$$\|L_x f - L_{x_\circ} f\|_{L^p(G,A)} < \epsilon/2$$

and for every $x \in U_2$

$$\|L_x f - L_{x_\circ} f\|_{L^1(G,A)} < \epsilon/2.$$

Set $V = U_1 \cap U_2$, then for all $x \in V$, we have $|||L_x f - L_{x_\circ} f||| < \epsilon$. □

PROPOSITION 4.2. *The space* $L^1(G, A) \cap L^p(G, A)$ *has a minimal approximate identitiy in* $L^1(G, A)$.

LEMMA 4.3. *The space* $L^1(G, A) \cap L^p(G, A)$ *is an essential* $L^1(G, A)$*-module.*

PROOF. Let $f \in L^1(G, A)$ and $g \in L^1(G, A) \cap L^p(G, A)$. Since $L^p(G, A)$ is an $L^1(G, A)$-module, we have

$$|||f * g||| = |||f * g|||_{L^1(G,A)} + |||f * g|||_{L^p(G,A)} \le |||f|||\ |||g|||.$$

By [13, Proposition 3.4] we get that $L^1(G, A) \cap L^p(G, A)$ is an essential $L^1(G, A)$-module. □

PROPOSITION 4.4. *$L^1(G, A) \cap L^p(G, A)$ is a Banach ideal in $L^1(G, A)$.*

PROPOSITION 4.5. *$L^1(G, A) \cap L^p(G, A)$ is a Banach algebra with the norm $|||\cdot|||$.*

PROOF. For any $f, g \in L^1(G, A) \cap L^p(G, A)$, using the inequality

$$\|\cdot\|_{L^1(G,A)} \leq |||\cdot|||,$$

we get that $|||f * g||| \leq |||f|||\,|||g|||$. □

COROLLARY 4.6. *The space $L^1(G, A) \cap L^p(G, A)$ is a Segal algebra.*

PROOF. By Lemma 4.1 and Proposition 4.5 we obtain that $L^1(G, A) \cap L^p(G, A)$ is a Segal algebra. □

We now return to Section 3 to mention the multipliers of $L^1(G, A) \cap L^p(G, A)$. Since $L^1(G, A) \cap L^p(G, A)$ is an $L^1(G, A)$-module and a Banach algebra, then we get easily $M(L^1(G, A) \cap L^p(G, A)) \cong H_{L^1(G,A)}(L^1(G, A) \cap L^p(G, A))$.

PROPOSITION 4.7. *$H_{L^1(G,A)}(L^1(G, A) \cap L^p(G, A))$ is an essential Banach module over $L^1(G, A)$.*

PROOF. Let $f \in L^1(G, A)$ and $T \in H_{L^1(G,A)}(L^1(G, A) \cap L^p(G, A))$. Define the operator fT on $L^1(G, A) \cap L^p(G, A)$ by $(fT)(g) = T(f * g)$ for all $f \in L^1(G, A) \cap L^p(G, A)$. By Proposition 4.5 T is well defined. Then $H_{L^1(G,A)}(L^1(G, A) \cap L^p(G, A))$ is an $L^1(G, A)$-module. Let (Φ_α) be a minimal approximate identity for $L^1(G, A)$ and T be in $H_{L^1(G,A)}(L^1(G, A) \cap L^p(G, A))$. We have

$$\lim_\alpha \|\Phi_\alpha \circ T - T\| = 0.$$

By [13, Proposition 3.4], we have that $H_{L^1(G,A)}(L^1(G, A) \cap L^p(G, A))$ is an essential Banach module over $L^1(G, A)$. □

Define $\wp$ to be the closure of $L^1(G, A)$ in $H_{L^1(G,A)}(L^1(G, A) \cap L^p(G, A))$ for the operator norm. Evidently,

$$H_{L^1(G,A)}(L^1(G, A) \cap L^p(G, A)) = (H_{L^1(G,A)}(L^1(G, A) \cap L^p(G, A)))_e = \wp = (\wp)_e,$$

where $(\cdot)_e$ denotes the essential part and we have

$$H_{L^1(G,A)}(L^1(G, A) \cap L^p(G, A)) = (\wp).$$

Here $(\wp)$ is defined as the space of the elements $T \in H_{L^1(G,A)}(L^1(G, A) \cap L^p(G, A))$ such that $T \circ \wp \subset \wp$.

Using the same method as in Theorem 3.4 we get the following lemma.

LEMMA 4.8. *The multipliers space of Banach algebra $\wp$ is isometrically isomorphic to the space $(\wp)$.*

We also get the following corollary.

COROLLARY 4.9. $H_{L^1(G,A)}(L^1(G, A) \cap L^p(G, A)) \cong M(\wp)$.

So the multipliers space of $L^1(G, A) \cap L^p(G, A)$ can be identified with the multipliers space of the closure of $L^1(G, A)$ in $H_{L^1(G,A)}(L^1(G, A) \cap L^p(G, A))$.

REMARK 4.10. (i) It is evident that every measure $\mu \in M(G, A)$ defines multiplier for $L^1(G, A) \cap L^p(G, A)$, $1 < p < \infty$. This is obvious from the fact that $|||\mu * f||| \le \|\mu\| |||f|||$, $f \in L^1(G, A) \cap L^p(G, A)$.
On the other hand, for $\mu \in M(G, A)$, we have $\mu \circ L^1(G, A) \subset L^1(G, A)$, the inclusion in the space $H_{L^1(G,A)}(L^1(G, A) \cap L^p(G, A))$.
Hence, $\mu \circ \wp \subset \wp$, thus $M(G, A)$ can be embeded into $(\wp)$.

(ii) If $A = \not{c}$ and G is a noncompact locally compact abelian, we have the more general result than the Corollary 3.5.1 in Larsen [9].

References

[1] C. Datry, *Multiplicateurs d'un $L^1(G)$-module de Banach consideres comme multiplicateurs d'une certaine algèbre de Banach* (Groupe de travail d'Analyse Harmonique, Institut Fourier, Grenoble, 1981).

[2] N. Dinculeanu, *Integration on locally compact spaces* (Nordhoff, The Netherlands, 1974).

[3] H. Feichtinger, 'Multipliers of Banach spaces of functions on groups', *Math. Z.* **152** (1976), 47–58.

[4] M. J. Fisher, 'Properties of three algebras related to L^p-multipliers', *Bull. Amer. Math. Soc.* **80** (1974), 262–265.

[5] J. Griffin and K. McKennon, 'Multipliers and group L^p algebras', *Pacific J. Math.* **49** (1973), 365–370.

[6] G. P. Johnson, 'Spaces of functions with values in a Banach algebra', *Trans. Amer. Math. Soc.* **92** (1959), 411–429.

[7] H. C. Lai, 'Multipliers of Banach-valued function spaces', *J. Austral. Math. Soc. (Ser. A)* **39** (1985), 51–62.

[8] H. C. Lai and T. K. Chang, 'Multipliers and translation invariant operators', *Tohoku Math. J.* **41** (1989), 31–41.

[9] R. Larsen, *An introduction to the theory of multipliers* (Springer, Berlin, 1971).

[10] K. McKennon, 'Multipliers of type (p, p)', *Pacific J. Math.* **43** (1972), 429–436.

[11] ———, 'Multipliers of type (p, p) and multipliers of group L^p algebras', *Pacific J. Math.* **45** (1972), 297–302.
[12] ———, 'Corrections to [10, 11] and [5]', *Pacific J. Math.* **61** (1975), 603–606.
[13] M. Rieffel, 'Induced Banach representations of Banach algebras and locally compact groups', *J. Funct. Anal.* **1** (1967), 443–491.
[14] U. B. Tewari, M. Dutta and D. P. Vaidya, 'Multipliers of group algebra of vector-valued functions', *Proc. Amer. Math. Soc.* **81** (1981), 223–229.
[15] J. K. Wang, 'Multipliers of commutative Banach algebras', *Pacific J. Math.* **11** (1961), 1131–1149.

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