PASCHKE'S CONJECTURE FOR THE ENDPOINT ANISOTROPIC SERIES REPRESENTATIONS OF THE FREE GROUP

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Abstract

Let Γ be a free noncommutative group with free generating set A_+. Let $\mu \in \ell^1(\Gamma)$ be real, symmetric, nonnegative and suppose that $\mathrm{supp}(\mu) = A_+ \cup A_+^{-1}$. Let λ be an endpoint of the spectrum of μ considered as a convolver on $\ell^2(\Gamma)$. Then $\lambda - \mu$ is in the left kernel of exactly one pure state of the reduced $C^*_{\mathrm{reg}}(\Gamma)$; in particular, Paschke's conjecture holds for $\lambda - \mu$.

1. Introduction

Let Γ be a noncommutative free group on finitely many generators. Choose a basis for Γ and denote by $|\cdot|$ the length with respect to this basis. Let A consist of the basis elements and of their inverses. Each $x \in \Gamma$ can be uniquely represented as a reduced word, that is, a product $a_1 a_2 \cdots a_n$ of elements of A with $a_j a_{j+1} \neq e$. Let $q + 1 = |A|$.

The reduced C^*-algebra of Γ is the C^*-algebra generated by the left regular representation π_{reg}, that is, the completion of the convolution algebra of finitely supported functions with respect to the norm

$$\|f\|^2_{C^*_{\mathrm{reg}}(\Gamma)} = \|\pi_{\mathrm{reg}}(f)\|^2 = \sup_{\|g\|_{\ell^2(\Gamma)}=1} \langle f * g, f * g \rangle.$$

A unitary representation π of Γ is weakly contained in π_{reg} if for every finitely supported function f on Γ one has $\|\pi(f)\| \leq \|\pi_{\mathrm{reg}}(f)\|$, where $\| \cdot \|$ denotes the

operator norm of $\pi(f)$; in other words π is weakly contained in π_{reg} if and only if π extends to a representation of $C^*_{\mathrm{reg}}(\Gamma)$. In our case, using Power's result [8] that $C^*_{\mathrm{reg}}(\Gamma)$ is simple, the above condition is equivalent to saying $\|\pi(f)\| = \|\pi_{\mathrm{reg}}(f)\|$. From this point on every representation is assumed to be unitary and weakly contained in π_{reg}.

Fix a nonzero, finitely supported function $\mu' \in \ell^2(\Gamma)$. In [6] Paschke conjectures that μ' lies in the left kernel of at most finitely many pure states of $C^*_{\mathrm{reg}}(\Gamma)$. In terms of representations this means that

 (a) there are only finitely many equivalence classes of irreducible representations π, weakly contained in π_{reg}, such that $\pi(\mu')$ has nontrivial kernel, and

 (b) for any such π, $\dim \ker \pi(\mu') = 1$.

It is convenient for our exposition to fix $\mu \in \ell^2(\Gamma)$ and let $\mu' = \lambda - \mu = \lambda\delta_e - \mu$ for $\lambda \in \mathbb{C}$.

Let $A_+ \subseteq A$ be a set of free generators of Γ. Let $n = \sharp(A_+) = (q+1)/2$. Paschke treats the case $\mu = \mathbf{1}_{A_+} = \sum_{a \in A_+} \delta_a$. For each λ, $0 \leq \lambda \leq \sqrt{n}$, he constructs an irreducible representation π_λ so that $\dim \ker \pi_\lambda(\lambda - \mu) = 1$. No such representation can occur for $\lambda > \sqrt{n}$ because $\sqrt{n}$ is the spectral radius of μ. In the case $\lambda = \sqrt{n}$, but only in that case, he proves that π_λ is, up to equivalence, the unique irreducible representation such that $\pi(\lambda - \mu)$ has nontrivial kernel. Thus, for $\lambda = \sqrt{n}$, the conjecture is proved. In [7] Paschke extends his results to the analogous anisotropic situation.

In this paper we shall prove that the conjecture is true in the case

$$\mu = \sum_{a \in A} p_a \delta_a, \quad \lambda = \mu_0$$

where p_a ($a \in A$) are positive numbers with $p_a = p_{a^{-1}}$, $\sum_{a \in A} p_a = 1$ and μ_0 is the right endpoint of the $\ell^2(\Gamma)$ spectrum of μ. The case of the left endpoint, $-\mu_0$, reduces trivially to the case of $+\mu_0$ by taking tensor products with the character χ_0 such that $\chi_0(a) = -1$ for each generator $a \in A$. For analogous reasons, Paschke's results hold not only for $\lambda = \sqrt{n}$, but also for $\lambda = \sqrt{n}e^{i\theta}$.

In fact we shall prove a stronger result, like Paschke's — there is, up to equivalence, *only one* irreducible representation for which $\pi(\mu)v = \mu_0 v$ for some nonzero vector v, namely the one in the *anisotropic principal series* of Figà-Talamanca and Steger [2]. The case $p_a = 1/(q+1)$ for all $a \in A$ corresponds to the *isotropic principal series* of Figà-Talamanca and Picardello [1].

In the present case, as in Paschke's case, the conjecture is proved only for the extreme value of λ. Referring to our paper [5] for the definition of *perfect boundary realization*, observe that Paschke's representation π_λ for $\lambda = \sqrt{n}$ admits exactly one boundary realization, which is perfect. The same holds for the anisotropic principal series representation corresponding to the endpoint, $\lambda = \mu_0$. In neither case does this

remain true as one passes to the interior of the spectrum. In spite of these notable points in common, our techniques are quite different from Paschke's. Ours definitely don't apply to his case, and it is not clear how his might apply to our case.

The techniques developed here are based on the growth of the matrix coefficients of π and give a positive answer to the question every time that matrix coefficients grow as fast as they can. This happens for the endpoint representations of the isotropic or anisotropic principal series. A central role is played by Haagerup's inequality [3].

THEOREM 1.1. *Let* (π, H) *be a unitary representation of* Γ. *Suppose that* $v \in H$ *is cyclic for* π *and consider the positive definite function* $\phi(x) = \langle \pi(x)v, v \rangle$. *The following conditions are equivalent*

(a) *π is weakly contained in the regular representation.*
(b) $\sum_{|x|=n} |\phi(x)|^2 \leq (n+1)^2 \|v\|^4$.
(c) *For every positive ϵ the function* $x \rightarrow \phi(x)e^{-\epsilon|x|}$ *belongs to* $\ell^2(\Gamma)$.

Observe first that, by polarization, every matrix coefficient $\rho(x) = \langle \pi(x)v, v' \rangle$ can be written as a finite linear combination of positive definite functions so that if we replace $\phi(x)$ with $\langle \pi(x)v, v' \rangle$ (c) remains true while (with the same arguments used in [3]) (b) becomes

$$\sum_{|x|=n} |\langle \pi(x)v, v' \rangle|^2 \leq C(n+1)^2 \|v\|^2 \|v'\|^2.$$

Let (π_1, H_1), (π_2, H_2) be two irreducible representations of Γ weakly contained in the regular representation. Fix vectors v in H_1 and w in H_2 with $\|v\| = \|w\| = 1$. Define a sesquilinear form on $H_2 \times H_1$ by letting

$$B_\epsilon(w', v') = \sum_{x \in \Gamma} \langle \pi_1(x)v, v' \rangle \overline{\langle \pi_2(x)w, w' \rangle} e^{-\epsilon|x|}.$$

By the Cauchy-Schwartz inequality

$$\sum_{|x|=n} |\langle \pi_1(x)v, v' \rangle \overline{\langle \pi_2(x)w, w' \rangle}| e^{-\epsilon|x|}$$

$$\leq \left(\sum_{|x|=n} |\langle \pi_1(x)v, v' \rangle|^2 e^{-\epsilon|x|} \right)^{1/2} \left(\sum_{|x|=n} |\langle \pi_2(x)w, w' \rangle|^2 e^{-\epsilon|x|} \right)^{1/2}$$

$$\leq C(n+1)^2 e^{-\epsilon n} \|v'\| \|w'\|.$$

Thus B_ϵ is well defined.

Moreover, since $\sum_{n=0}^{\infty} (n+1)^2 x^n = (1+x)/(1-x)^3$ for $|x| < 1$ we have

$$|B_\epsilon(w', v')| \leq C \sum_{n=0}^{\infty} (n+1)^2 e^{-\epsilon n} \|v'\| \|w'\| = C(1 + e^{-\epsilon}) \frac{\|v'\| \|w'\|}{(1 - e^{-\epsilon})^3}.$$

Hence $\epsilon^3 B_\epsilon$ is bounded as ϵ goes to 0. See Corollary 3.2 below for a proof that if

(1)
$$\limsup_{\epsilon \to 0} \epsilon^3 \, |B_\epsilon(w', v')| > 0,$$

then the two representations are equivalent and moreover a version of Schur orthogonality holds for limits of normalized sums of products of matrix coefficients.

The problem is that in many known cases the limit in (1) is zero. For example, using the most natural choices for the vectors, B_ϵ grows like $1/\epsilon$ in the case of a non-endpoint isotropic ([1]) or anisotropic ([2]) spherical series representation. Growth like $1/\epsilon^2$ is to be found for Paschke's endpoint representation ($\lambda = \sqrt{n}$) and for the representations of our 1996 paper [4].

There are a few known cases in which the growth of the quantity $B_\epsilon(v', w')$ is of the magnitude of $1/\epsilon^3$, leading to a nonzero limit for $\epsilon^3 B_\epsilon$. These cases are

- when $\pi = \pi_{z_0}$ is an endpoint representation of the isotropic spherical series (corresponding to the value $z_0 = 1/2 + ik\pi$ in the notation of [1])
- when $\pi = \pi_{\pm\mu_0}$ is an endpoint representation of the anisotropic principal series of [2].

In these cases the spherical functions $\langle \pi(x)v, v \rangle$ grow as fast as Haagerup's results allow them to and we can get precise and interesting information about the eigenspace of the operator $\pi(\mu - \lambda_0)$.

2. The spherical functions

Fix once and for all a free set A_+ of generators and let $A = A_+ \cup A_-$ consist of the generators and of their inverses. The Cayley graph of Γ with respect to A is a homogeneous tree of degree $q+1 = |A|$. Each vertex is labelled with a group element and the (unoriented) edges are given by the pairs $\{x, xa\}$ where $a \in A$.

The anisotropic series representations were defined by Figà-Talamanca and Steger in the memoir [2] for all discrete groups whose Cayley graph is a tree. Those groups are free product of M copies of $\mathbb{Z}$ and N copies of $\mathbb{Z}_2$, where $2M + N = q + 1$. In the paper mentioned above, the case $N = q + 1$ was considered in detail. Nonetheless, formulas and results hold unchanged if we pass from a free product of $q + 1$ copies of $\mathbb{Z}_2$ to a free group on $(q + 1)/2$ generators.

Choose positive numbers p_a $(a \in A)$ with $\sum_{a \in A} p_a = 1$ and $p_a = p_{a^{-1}}$. Let $\mu = \sum_{a \in A} p_a \delta_a$. Consider the operator of right convolution by μ acting on $\ell^2(\Gamma)$. Its spectrum is a real interval $[-\mu_0, \mu_0]$ and, for $\lambda \notin [-\mu_0, \mu_0]$, the resolvent is given by right convolution with the Green function $g_\lambda(x) = (\lambda - \mu)^{-1}(x)$ which is positive for λ real and greater than μ_0.

Let $x = a_1 a_2 \cdots a_n$ be the reduced word expression for x. Set

$$g_\lambda(e) = \frac{1}{2w(\lambda)}.$$

Then

(2) $$g_\lambda(x) = \frac{1}{2w(\lambda)} \xi_{a_1}(\lambda) \xi_{a_2}(\lambda) \cdots \xi_{a_n}(\lambda)$$

where $\{\xi_{a_j}(\lambda)\}_{j=1}^{q+1}$ and $w(\lambda)$ are algebraic functions of λ which are positive for λ real and greater than μ_0 and where $\xi_{a_j} = \xi_{a_j^{-1}}$.

It is more convenient to turn $w(\lambda)$ and $\xi_{a_j}(\lambda)$ into functions of w: the formulas below can be found in [2, page 10] and are valid for large positive values of w.

$$\lambda = -(q-1)w + \sum_{a \in A} \sqrt{w^2 + p_a}$$

$$\xi_a = \left(\sqrt{w^2 + p_a} - w\right)/p_a$$

There is a unique positive w_0 for which we have $d\lambda/dw|_{w_0} = 0$. The point $\lambda(w_0) = \mu_0$ is the (right) endpoint of the spectrum of μ and it is also a branch point for the analytic function $w(\lambda)$ (see the discussion in [2, pages 22–27]).

At this point we also have $d^2\lambda/dw^2|_{w_0} = L > 0$ so that

$$\lambda = \mu_0 + \frac{L}{2}(w - w_0)^2 + \cdots .$$

Set $\sqrt{\lambda - \mu_0} = \epsilon$ and use ϵ as a parameter for the formulae in [2]. In a neighbourhood of $\lambda = \mu_0$ we have

(3) w is an analytic function of ϵ, $w(\mu_0) > 0$, $dw/d\epsilon|_{\mu_0} = l > 0$,

(4) ξ_a is an analytic function of ϵ, $\xi_a(\mu_0) > 0$, $d\xi_a/d\epsilon|_{\mu_0} = -k_a < 0$,

(5) $$g_\lambda^2(e) = (\lambda - \mu)^{-2}(e) = \frac{-d(\lambda - \mu)^{-1}}{d\lambda}(e) = \frac{-d}{d\lambda} \frac{1}{2w(\lambda)}$$

$$= \frac{l}{2(w(\mu_0))^2} \frac{1}{2\epsilon} + O(1).$$

Let π_λ be the representation of the anisotropic principal series for μ which corresponds to some $\lambda \in [-\mu_0, \mu_0]$. Note that π_λ is irreducible. There is a special vector $\mathbf{1}$ in the representation space of π_λ. Actually, $\mathbf{1}$ is the function identically 1 in the realization of π_λ acting on $L^2(\Omega, dv)$ given in [2]. One has $\pi_\lambda(\mu)\mathbf{1} = \lambda\mathbf{1}$. The spherical functions are first defined as usual by $\phi_\lambda(x) = \langle \pi_\lambda(x)\mathbf{1}, \mathbf{1}\rangle$ for $\lambda \in (-\mu_0, \mu_0)$. When $\lambda = \mu_0$ the spherical function can be computed as a limit, see [2, page 30]

$$\phi_{\mu_0}(x) = \lim_{\lambda \to \mu_0} \langle \pi_\lambda(x)\mathbf{1}, \mathbf{1}\rangle = \frac{\frac{d}{d\epsilon}|_{\lambda=\mu_0}(\lambda - \mu)^{-1}(x)}{\frac{d}{d\epsilon}|_{\lambda=\mu_0}(\lambda - \mu)^{-1}(e)}$$

and analogously for $\lambda = -\mu_0$.

3. The results

In order to prove our results we need to modify a little bit the factor $e^{-\epsilon|x|}$ which guarantees the convergence of $\sum_{x\in\Gamma}\langle\pi_1(x)v, v'\rangle\overline{\langle\pi_2(x)w, w'\rangle}e^{-\epsilon|x|}$. In the case of the *isotropic* principal series, no modification is necessary.

LEMMA 3.1. *Let $\psi_\epsilon(x)$ be any family of complex functions having the properties*:

(a) $|\psi_\epsilon(x)| \le Ce^{-c\epsilon|x|}$ *for some positive constants C and c.*
(b) $|\psi_\epsilon(x) - \psi_\epsilon(xa)| \le K(\epsilon)\, e^{-c\epsilon|x|}$ *for each $a \in A$ where $K(\epsilon) \to 0$ as $\epsilon \to 0^+$.*

Assume that (π_1, H_1) and (π_2, H_2) are representations weakly contained in the regular representation. Suppose that for some $v_1, w_1 \in H_1$ and $v_2, w_2 \in H_2$ we have that

$$(6) \qquad \limsup_{\epsilon\to 0}\epsilon^3\left|\sum_{x\in\Gamma}\langle\pi_1(x)v_1, w_1\rangle\overline{\langle\pi_2(x)v_2, w_2\rangle}\psi_\epsilon(x)\right| > 0.$$

Then there is a Γ-intertwiner $J : H_2 \to H_1$ so that $\langle Jw_2, w_1\rangle \ne 0$.

PROOF. We prove first that $\limsup_{\epsilon\to 0^+}\epsilon^3\big|\sum_{x\in\Gamma}\langle\pi_1(x)v_1, v_1'\rangle\overline{\langle\pi_2(x)v_2, v_2'\rangle}\psi_\epsilon(x)\big|$ is finite and that we can form the analogue of $B_\epsilon(w', v')$ by replacing $e^{-\epsilon|x|}$ with $\psi_\epsilon(x)$. In fact condition (a) together with Haagerup's inequality says that for any π_1 and π_2 weakly contained in π_{reg} we have

$$\sum_{|x|=n}\left|\langle\pi_1(x)v_1, v_1'\rangle\overline{\langle\pi_2(x)v_2, v_2'\rangle}\psi_\epsilon(x)\right|$$

$$\le \left(\sum_{|x|=n}|\langle\pi_1(x)v_1, v_1'\rangle|^2|\psi_\epsilon(x)|\right)^{1/2}\left(\sum_{|x|=n}|\langle\pi_2(x)v_2, v_2'\rangle|^2|\psi_\epsilon(x)|\right)^{1/2}$$

$$\le C(n+1)^2 e^{-c\epsilon n}\|v_1'\|\,\|v_2'\|\,\|v_1\|\,\|v_2\|.$$

Adding up over n we get

$$(7) \qquad \left|\sum_{x\in\Gamma}\langle\pi_1(x)v_1, v_1'\rangle\overline{\langle\pi_2(x)v_2, v_2'\rangle}\psi_\epsilon(x)\right| \le 2C\frac{\|v_1'\|\,\|v_2'\|\,\|v_1\|\,\|v_2\|}{(1 - e^{-c\epsilon})^3}.$$

Consider v_1 and v_2 from (6) to be fixed. For every $\epsilon > 0$, define $J_\epsilon : H_2 \to H_1$ by letting $\langle J_\epsilon v_2', v_1'\rangle = \epsilon^3\sum_{x\in\Gamma}\langle\pi_1(x)v_1, v_1'\rangle\overline{\langle\pi_2(x)v_2, v_2'\rangle}\psi_\epsilon(x)$.

The inequality (7) bounds the operator norm of J_ϵ as $\epsilon \to 0$. Choose a sequence $\{\epsilon_j\}$ tending to zero such that $\langle J_{\epsilon_j}w_2, w_1\rangle$ has a nonzero limit, then extract a weak operator convergent subsequence of $\{J_{\epsilon_j}\}$ with limit J. This is possible since norm closed balls are compact in the weak operator topology and H_i are separable.

For some suitable sequence $\{\epsilon_j\}$, we have

$$\langle J(v_2'), v_1' \rangle = \lim_j \epsilon_j^3 \sum_{x \in \Gamma} \langle \pi_1(x)v_1, v_1' \rangle \overline{\langle \pi_2(x)v_2, v_2' \rangle} \psi_{\epsilon_j}(x).$$

By a construction $\langle Jw_2, w_1 \rangle \neq 0$. Now we show that J intertwines π_2 to π_1. It is enough to check that $J\pi_2(a) = \pi_1(a)J$ when a is a generator of Γ. Let us compute $\langle J(\pi_2(a)v_2'), v_1' \rangle - \langle \pi_1(a)J(v_2'), v_1' \rangle$:

$$|\langle J\pi_2(a)(v_2'), v_1' \rangle - \langle \pi_1(a)J(v_2'), v_1' \rangle|$$
$$= |\langle J\pi_2(a)(v_2'), v_1' \rangle - \langle J(v_2'), \pi_1(a^{-1})v_1' \rangle|$$
$$= \lim_j \epsilon_j^3 \left| \sum_{x \in \Gamma} \langle \pi_1(x)v_1, v_1' \rangle \overline{\langle \pi_2(x)v_2, \pi_2(a)v_2' \rangle} \psi_{\epsilon_j}(x) \right.$$
$$\left. - \sum_{x \in \Gamma} \langle \pi_1(x)v_1, \pi_1(a^{-1})v_1' \rangle \overline{\langle \pi_2(x)v_2, v_2' \rangle} \psi_{\epsilon_j}(x) \right|$$
$$= \lim_j \epsilon_j^3 \left| \sum_{x \in \Gamma} \langle \pi_1(ax)v_1, v'_1 \rangle \overline{\langle \pi_2(x)v_2, v_2' \rangle} \psi_{\epsilon_j}(ax) \right.$$
$$\left. - \sum_{x \in \Gamma} \langle \pi_1(ax)v_1, v_1' \rangle \overline{\langle \pi_2(x)v_2, v_2' \rangle} \psi_{\epsilon_j}(x) \right|$$
$$\leq \lim_j \sup \epsilon_j^3 K(\epsilon_j) \sum_{x \in \Gamma} \left| \langle \pi_1(x)v_1, \pi_1(a^{-1})v_1' \rangle \overline{\langle \pi_2(x)v_2, v_2' \rangle} \right| e^{-c\epsilon_j|x|} = 0. \qquad \square$$

COROLLARY 3.2. *Let ψ_ϵ be a family of functions satisfying conditions* (a) *and* (b) *of Lemma* 3.1. *Let (π, H) be an irreducible representation. Then there exist a sequence $\epsilon_j \to 0$ and a constant $C_0 \geq 0$ so that for any $v_1, v_1', v_2, v_2' \in H$*

$$\lim_j \epsilon_j^3 \sum_{x \in \Gamma} \langle \pi_1(x)v_1, v_1' \rangle \overline{\langle \pi_2(x)v_2, v_2' \rangle} \psi_{\epsilon_j}(x) = C_0 \langle v_1, v_2 \rangle \overline{\langle v_1', v_2' \rangle}.$$

PROOF. This is a corollary to the demonstration rather than to the statement of Lemma 3.1. If the lim sup of (6) is zero for all choices of the vectors, then Corollary 3.2 is true with $C_0 = 0$. Otherwise, we fix some v_1 and v_2 and construct $J : H \to H$ as in the proof of Lemma 3.1. Since π is irreducible, J is necessarily scalar, say $J = C_0' = C_0'(v_1, v_2)$. Then for all v_1' and v_2' we have

$$C_0'(v_1, v_2)\overline{\langle v_1', v_2' \rangle} = \langle Jv_2', v_1' \rangle = \lim_j \epsilon_j^3 \sum_{x \in \Gamma} \langle \pi_1(x)v_1, v_1' \rangle \overline{\langle \pi_2(x)v_2, v_2' \rangle} \psi_{\epsilon_j}(x).$$

The rest of the proof proceeds as for the usual Schur orthogonality relations. Exchanging the roles of (v_1, v_2) and (v_1', v_2') we find that C_0' is proportional to $\langle v_1, v_2 \rangle$,

say $C_0'(v_1, v_2) = \langle v_1, v_2 \rangle C_0$. Then for all $v_1, v_2, v_1', v_2' \in H$

$$C_0 \langle v_1, v_2 \rangle \overline{\langle v_1', v_2' \rangle} = \lim_j \epsilon_j^3 \sum_{x \in \Gamma} \langle \pi_1(x)v_1, v_1' \rangle \overline{\langle \pi_2(x)v_2, v_2' \rangle} \psi_{\epsilon_j}(x).$$

Taking $v_1 = v_1'$, $v_2 = v_2'$ we find that $C_0 > 0$. $\qquad\square$

THEOREM 3.3. *Suppose that (π, H) is an irreducible representation weakly contained in π_{reg}. Assume that there exists a nonzero vector $v \in H$ such that $\pi(\mu)v = \mu_0$. Then π is equivalent to the representation (π_{μ_0}, H_{μ_0}) corresponding to the endpoint of the anisotropic principal series of [2]. Moreover $\dim \ker \pi_{\mu_0}(\mu_0 - \mu) = 1$.*

PROOF. Fix any nonzero eigenvector v of $\pi(\mu)$ corresponding to the eigenvalue μ_0. Let $\mathbf{1}$ be the nonzero μ_0 eigenvector of $\pi_{\mu_0}(\mu)$ as described previously. We shall construct a function ψ_ϵ satisfying (a) and (b) of Lemma 3.1 for which

$$\lim_{\epsilon \to 0} \epsilon^3 \sum_{x \in \Gamma} \langle \pi_{\mu_0}(x)\mathbf{1}, \mathbf{1} \rangle \overline{\langle \pi(x)v, v \rangle} \psi_\epsilon(x) \neq 0.$$

Let $g_\lambda(x) = (\lambda - \mu)^{-1}(x)$ be the resolvent of μ. Choose λ positive and greater than μ_0 and set $\epsilon = \sqrt{\lambda - \mu_0}$. Define

$$\psi_\epsilon(x) = \frac{(\lambda - \mu)^{-2}(x)}{(\lambda - \mu)^{-2}(e) \langle \pi_{\mu_0}(x)\mathbf{1}, \mathbf{1} \rangle} = \frac{g_\lambda^2(x)}{g_\lambda^2(e)} \frac{1}{\phi_{\mu_0}(x)}.$$

Since $\pi(\mu)v = \mu_0 v$ the functional calculus gives $\pi\big((\lambda - \mu)^{-2}\big)v = v/(\lambda - \mu_0)^2$. One computes

$$\sum_{x \in \Gamma} \langle \pi_{\mu_0}(x)\mathbf{1}, \mathbf{1} \rangle \overline{\langle \pi(x)v, v \rangle} \psi_\epsilon(x) = \sum_{x \in \Gamma} \langle \pi(x)v, v \rangle \frac{(\lambda - \mu)^{-2}(x)}{(\lambda - \mu)^{-2}(e)}$$

$$= \frac{1}{g_\lambda^2(e)} \langle \pi((\lambda - \mu)^{-2})v, v \rangle = \frac{\langle v, v \rangle}{g_\lambda^2(e)(\lambda - \mu_0)^2}.$$

So,

$$\lim_{\epsilon} \epsilon^3 \sum_{x \in \Gamma} \langle \pi_{\mu_0}(x)\mathbf{1}, \mathbf{1} \rangle \overline{\langle \pi(x)v, v \rangle} \psi_\epsilon(x) = \lim_{\lambda \to \mu_0} \frac{\big(\sqrt{\lambda - \mu_0}\big)^3}{g_\lambda^2(e)(\lambda - \mu_0)^2} \langle v, v \rangle.$$

By (5) the quantity $(\lambda - \mu)^{-2}(e) = g_\lambda^2(e)$ behaves like $l/\big(4\sqrt{\lambda - \mu_0}\, w^2(\mu_0)\big)$ so that the above limit is

$$\frac{4w^2(\mu_0)}{l} \langle v, v \rangle \neq 0.$$

Taking for granted that ψ_ϵ satisfies conditions (a) and (b) of Lemma 3.1, said lemma guarantees the existence of a Γ-intertwiner $J : H_{\mu_0} \to H$ such that $\langle Jv, \mathbf{1} \rangle \neq 0$. As π and π_{μ_0} are irreducible, π is unitarily equivalent to π_{μ_0}.

Suppose that $\dim \ker \pi_{\mu_0}(\mu_0 - \mu) > 1$. In the above argument, choose $\pi = \pi_{\mu_0}$ and choose $v \in \ker \pi_{\mu_0}(\mu_0 - \mu)$ so that $\langle v, \mathbf{1} \rangle = 0$. Then J intertwines H_{μ_0} to itself so that $\langle Jv, \mathbf{1} \rangle \neq 0$. As J is necessarily a scalar, this is a contradiction. (Alternatively, [2] contains a direct, computational proof that $\dim \ker \pi_{\mu_0}(\mu_0 - \mu) = 1$.)

Now we prove that ψ_ϵ satisfies (a) and (b) of Lemma 3.1. First note that for $\lambda > \mu_0$

$$(8) \qquad \frac{(\lambda - \mu)^{-2}(x)}{(\lambda - \mu)^{-2}(e)} = \frac{d((\lambda - \mu)^{-1}(x))/d\lambda}{d((\lambda - \mu)^{-1}(e))/d\lambda} = \frac{d((\lambda - \mu)^{-1}(x))/d\epsilon}{d((\lambda - \mu)^{-1}(e))/d\epsilon} = \frac{dg_\lambda(x)/d\epsilon}{dg_\lambda(e)/d\epsilon},$$

where $\epsilon = \sqrt{\lambda - \mu_0}$. Fix $x = a_1 \cdots a_n$, an element of length n in Γ. Define $\xi(x)$, a function of λ, by $\xi(x) = \xi_{a_1}(\lambda) \ldots \xi_{a_n}(\lambda)$. Using (8) and (2) we compute

$$\frac{(\lambda - \mu)^{-2}(x)}{(\lambda - \mu)^{-2}(e)} = \xi(x) \left(1 + \frac{w}{dw/d\epsilon} \sum_{j=1}^{n} \frac{-d\xi_{a_j}/d\epsilon}{\xi_{a_j}} \right).$$

Since $\phi_{\mu_0}(x)$ is given by the right hand side of (8) evaluated at $\lambda = \mu_0$,

$$(9) \qquad \psi_\epsilon(x) = \frac{(\lambda - \mu)^{-2}(x)}{(\lambda - \mu)^{-2}(e)\phi_{\mu_0}(x)} = \frac{\xi(x)|_\lambda}{\xi(x)|_{\mu_0}} \left(\frac{1 + \frac{w}{dw/d\epsilon}\big|_\lambda \sum_{j=1}^{n} \frac{-d\xi_{a_j}/d\epsilon}{\xi_{a_j}}\big|_\lambda}{1 + \frac{w}{dw/d\epsilon}\big|_{\mu_0} \sum_{j=1}^{n} \frac{-d\xi_{a_j}/d\epsilon}{\xi_{a_j}}\big|_{\mu_0}} \right).$$

Using (3) and (4) one sees that for λ in a neighbourhood of μ_0 the functions w, $dw/d\epsilon$, ξ_a, and $-d\xi_a/d\epsilon$ are positive and differentiable with respect to ϵ. It follows that there exist constants $h, \epsilon_0 > 0$ so that

$$1 - h\epsilon \leq \frac{w(\lambda)}{w(\mu_0)} \cdot \frac{dw/d\epsilon|_{\mu_0}}{dw/d\epsilon|_\lambda} \cdot \frac{\xi_a(\mu_0)}{\xi_a(\lambda)} \cdot \frac{-d\xi_a/d\epsilon|_\lambda}{-d\xi_a/d\epsilon|_{\mu_0}} \leq 1 + h\epsilon$$

for $0 < \epsilon \leq \epsilon_0$ and for all $a \in A$. Hence

$$(10) \qquad 1 - h\epsilon \leq \frac{1 + \frac{w}{dw/d\epsilon}\big|_\lambda \sum_{j=1}^{n} \frac{-d\xi_{a_j}/d\epsilon}{\xi_{a_j}}\big|_\lambda}{1 + \frac{w}{dw/d\epsilon}\big|_{\mu_0} \sum_{j=1}^{n} \frac{-d\xi_{a_j}/d\epsilon}{\xi_{a_j}}\big|_{\mu_0}} \leq 1 + h\epsilon.$$

According to (4) $d\xi_a/d\epsilon|_{\mu_0} = -k_a < 0$. Hence there exist constants $H, c > 0$ so that

$$(11) \qquad 1 - H\epsilon \leq \frac{\xi_a(\lambda)}{\xi_a(\mu_0)} \leq e^{-\epsilon c}$$

for all $0 < \epsilon \leq \epsilon_0$ and for all $a \in A$. Consequently,

$$(12) \qquad (\xi(x)|_\lambda)/(\xi(x)|_{\mu_0}) \leq e^{-c\epsilon|x|}.$$

Together with (9) and (10), this proves condition (a) in Lemma 3.1.

Now we prove condition (b). Fix $a \in A$ so that $|ax| = n + 1$. The opposite case, $|ax| = n - 1$, easily reduces to this one. From (9) we obtain

$$\psi_\epsilon(ax) - \psi_\epsilon(x)$$

$$= \frac{\xi(x)|_\lambda}{\xi(x)|_{\mu_0}} \left(\frac{\xi_a(\lambda)}{\xi_a(\mu_0)} \frac{1 + \frac{w}{dw/d\epsilon}\big|_\lambda \left(\frac{\xi_a}{-d\xi_a/d\epsilon}\big|_\lambda + \sum_{j=1}^n \frac{\xi_{a_j}}{-d\xi_{a_j}/d\epsilon}\big|_\lambda \right)}{1 + \frac{w}{dw/d\epsilon}\big|_{\mu_0} \left(\frac{\xi_a}{-d\xi_a/d\epsilon}\big|_{\mu_0} + \sum_{j=1}^n \frac{\xi_{a_j}(\mu_0)}{-d\xi_{a_j}/d\epsilon}\big|_{\mu_0} \right)} \right.$$

$$\left. - \frac{1 + \frac{w}{dw/d\epsilon}\big|_\lambda \left(\sum_{j=1}^n \frac{\xi_{a_j}}{-d\xi_{a_j}/d\epsilon}\big|_\lambda \right)}{1 + \frac{w}{dw/d\epsilon}\big|_{\mu_0} \left(\sum_{j=1}^n \frac{\xi_{a_j}}{-d\xi_{a_j}/d\epsilon}\big|_{\mu_0} \right)} \right).$$

By (10) and (11)

$$((1 - H\epsilon)(1 - h\epsilon) - (1 + h\epsilon)) \frac{\xi(x)|_\lambda}{\xi(x)|_{\mu_0}} \leq \psi_\epsilon(ax) - \psi_\epsilon(x)$$

$$\leq ((1 + h\epsilon) - (1 - h\epsilon)) \frac{\xi(x)|_\lambda}{\xi(x)|_{\mu_0}}$$

which means that

$$-(2h + H)\epsilon \frac{\xi(x)|_\lambda}{\xi(x)|_{\mu_0}} \leq \psi_\epsilon(ax) - \psi_\epsilon(x) \leq 2h\epsilon \frac{\xi(x)|_\lambda}{\xi(x)|_{\mu_0}}.$$

Together with (12), this completes the proof of condition (b).   □

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