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STRANGE PERMUTATION REPRESENTATIONS OF FREE GROUPS

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Abstract

Certain permutation representations of free groups are constructed by finite approximation. The first is a construction of a cofinitary group with special properties, answering a question of Tim Wall published by Cameron. The second yields, via a method of Kepert and Willis, a totally disconnected locally compact group which is compactly generated and uniscalar but has no compact open normal subgroup. Finally, an oligomorphic group of automorphisms of the random graph is built, all of whose non-trivial subgroups have just finitely many orbits.

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1. Introduction

In this paper we give three constructions of faithful permutation representations, with peculiar properties, of free groups. We collect them in the same paper more because the methods are similar than because the topics are. In each case, we define the permutation representation by finite approximation, expressing the generators as unions of finite partial functions. It is not important that the groups acting are free: indeed, by a theorem of Dixon [3], the set of pairs of permutations which generate a free group is comeagre in the natural topological space on pairs of permutations of $\mathbb{N}$ (the product topology from the usual topology on $\mathrm{Sym}(\mathbb{N})$), and it remains a challenge to build examples like those below which are *not* free.

Our permutation groups will always act on a countable set $\Omega := \{\xi_i : i \in \mathbb{N}\}$. We use lower case Greek letters for elements of Ω, upper case Greek letters for subsets of Ω, and lower case Roman letters for group elements (except that we allow variables

x, y to range through group elements or Ω). Permutations are written on the right of their arguments.

Our first theorem, proved in Section 2, concerns cofinitary permutation groups. Recall that a permutation group G on Ω is *cofinitary* (Cameron [2]) if every non-identity element has just finitely many fixed points. Our theorem answers a question of Wall [2, Section 10], posed as a test of the construction methods available for cofinitary groups.

THEOREM 1.1. *There is a cofinitary permutation group G on the countably infinite set Ω such that G is freely generated by $\{f_i : i \in \mathbb{N}\}$ and for each $i \in \mathbb{N}$*

(a) *f_i fixes ξ_j for $0 \leq j < i$ and acts as a single cycle on $\Omega \setminus \{\xi_j : j < i\}$,*
(b) *the group $\langle f_0, \ldots, f_i \rangle$ is not $(i+2)$-transitive.*

It is evident that by condition (a), $\langle f_0, \ldots f_i \rangle$ is $(i+1)$-transitive for each i.

In Section 3 we construct a permutation group which provides an answer to a question of George Willis, in his work on scale functions for totally disconnected groups.

THEOREM 1.2. *The free group $F_2 = \langle f, g \rangle$ has a faithful transitive action on a countable set Ω such that the following hold, where $\Omega = \Gamma \cup \Delta$ is a partition of Ω into two infinite sets*:

(a) *each cycle of each element of F_2 is finite*;
(b) *for each $x \in F_2$, the symmetric difference $\Gamma \triangle \Gamma x$ is finite.*

It follows that for each $x \in F_2$ there is $\Lambda \subset \Omega$ such that $\Lambda \triangle \Gamma$ is finite and $\Lambda x = \Lambda$, but (by transitivity) there is no G-*invariant* set $\Lambda \subset \Omega$ with $\Lambda \triangle \Gamma$ finite.

The context of this construction is as follows (see [9] or [6] for background). If G is any totally disconnected locally compact group and $x \in G$, then there is a compact open subgroup U of G so that the following hold, where $U_+ := \bigcap(x^n U x^{-n} : n \in \mathbb{N})$ and $U_- := \bigcap(x^{-n} U x^n : n \in \mathbb{N})$.

(1) $U = U_+ U_-$.
(2) $\bigcup(x^n U_+ x^{-n} : n \in \mathbb{N})$ and $\bigcup(x^{-n} U_- x^n : n \in \mathbb{N})$ are both closed subgroups of G.

The index function $s(x) = |x U_+ x^{-1} : U_+|$, the *scale function* of G, is independent of the choice of U, and is a continuous function $s : G \to \mathbb{N}$ such that $s(x) = 1 = s(x^{-1})$ if and only if x normalises some compact open subgroup of G. The group G is called *uniscalar* if s takes value 1 everywhere. Clearly if G has a compact open normal subgroup then G is uniscalar, and the converse is known to be false (see [10] for references). However, it was not previously known if there was a totally disconnected locally compact *compactly generated* uniscalar group with no compact open normal subgroup, but in [6] Kepert and Willis show that such an example can be obtained from the group constructed in Theorem 1.2. For let K be a finite group, and let $H := \sum_{\Delta} K \times \prod_{\Gamma} K$. Let F_2 act on Ω as in Theorem 1.2. Then F_2 acts on H via its action on the indices, and the semidirect product $G := H \rtimes F_2$ will be a totally disconnected locally compact compactly generated uniscalar group with no compact open normal subgroup. As commented at the end of [6], for each $g \in F_2$ the group G even has a basis of neighbourhoods of the identity consisting of compact open subgroups normalised by g. Possible variations on the construction are discussed at the end of Section 3.

We turn in Section 4 to ZTF groups. A permutation group on an infinite set is said to be *ZTF* 'Zimmer torsion-free' if each non-identity element has just finitely many cycles (so each non-trivial subgroup has finitely many orbits). R. Zimmer raised questions about the structure of such groups, in connection with ergodic theory. An easy example of a ZTF group is the infinite cyclic group acting regularly, and at the other extreme, the free group on 2-generators was shown in [7] to have a faithful ZTF action. These examples are in a sense typical, for by a result of Neumann [7, Lemma 3.3], centralisers in a ZTF group must be cyclic-by-finite. A critical question is whether there exists a *highly implausible* Frobenius group, that is, a Frobenius ZTF group in which point stabilisers are infinite cyclic. Recall that a permutation group on a countably infinite set is *oligomorphic* [1] if it has finitely many orbits on k-sets for all $k > 0$. Neumann [7, Proposition 3.6] showed that any non-trivial ZTF group which is not oligomorphic or regular has a subgroup with a faithful highly implausible Frobenius action on some (possibly different) set. It is not known whether there is any highly implausible Frobenius group, but it is easy to see that such a group cannot be free. We remark that by [8] and [5], there is no *2-transitive* permutation group whose one-point stabilisers are infinite cyclic.

The ZTF group constructed in [7] may well be *highly transitive*, that is, k-transitive for all $k > 0$, and certainly the construction there can be modified to yield a highly transitive group. It is more interesting (and relevant to the existence of highly implausible ZTF groups) to consider *non*-highly transitive ZTF groups. As pointed out by Peter Neumann, if (G, Ω) is the permutation group built in [7], then G has a 'diagonal' action on the disjoint union of two copies of Ω which is oligomorphic, ZTF, but not transitive. However, it is not so clear how to obtain a primitive but not highly transitive ZTF group. Below, we build such a group acting on the random graph (defined at the end of the section).

THEOREM 1.3. *Let* $(\Omega, \sim)$ *be the random graph* (*so* $\sim$ *is a binary irreflexive symmetric relation on the domain* Ω), *and let* $\Omega := \{\xi_i : i \in \mathbb{N}\}$. *Then there are* $f, g \in \mathrm{Aut}(\Omega, \sim)$ *such that*

(a) f, g *generate a free subgroup of* $\mathrm{Aut}(\Omega, \sim)$,

(b) *f has a single cycle on Ω, which is infinite,*
(c) *g fixes ξ_0 and has two cycles on $\Omega \setminus \{\xi_0\}$,*
(d) *the group $\langle f, g\rangle$ is a primitive oligomorphic ZTF group.*

We remark that since $\langle f, g\rangle$ is transitive on vertices, edges, and non-edges, by the primitivity criterion of Higman [4] it acts primitively on Ω. By the remarks above, since F_2 is free but does not act regularly, the action is oligomorphic. It seems likely that the proof could be modified to ensure that $\langle f, g\rangle$ is also a *dense* subgroup of $\mathrm{Aut}(\Omega, \sim)$, that is, has the same orbits on finite ordered sets as the whole automorphism group. The proof is rather involved, but it suggests that many structures which are homogeneous (in the sense defined below) admit large ZTF groups of automorphisms. Observe though that $\mathrm{Aut}(\mathbf{Q}, <)$ has no non-trivial ZTF subgroup. Furthermore, if G is any oligomorphic group acting on a set Ω such that the pointwise stabiliser in G of a finite subset of Ω preserves some partial ordering on Ω with an infinite chain, then the action of G on Ω cannot be ZTF.

The method of proof of Theorems 1.1–1.3 is to build a permutation group generated freely by $\{f_i : i \in I\}$, by approximating each permutation f_i by a chain of finite partial functions. In Section 2, $I = \mathbb{N}$, and in Section 3 and Section 4, $I = \{0, 1\}$, with $f := f_0$ and $g := f_1$. We denote by $f_i^{(k)}$ the partial function on Ω constructed after k steps, so $f_i := \bigcup(f_i^{(k)} : k \in \mathbb{N})$ (so we regard each partial function as a set of ordered pairs). If w is a word in the f_i, then $w^{(k)}$ is the partial function on Ω obtained by composing the $f_i^{(k)}$. A *partial $w^{(k)}$-cycle* is a maximal sequence $\gamma_0, \ldots, \gamma_t$ from Ω (denoted $(\ldots, \gamma_0, \ldots, \gamma_t, \ldots)$) such that $\gamma_0(w^{(k)})^t$ is defined and equals γ_t. We use the word *cycle* for *partial cycle*, and *complete cycle* to refer to a cycle as above where $\gamma_t w^{(k)} = \gamma_0$. A *$w^{(k)}$-chain* is a sequence $(\delta_0, \ldots, \delta_t, i, w) \in \Omega$ such that for some subword $u_1 \cdots u_t$ of a power of w (with u_1 the i^{th} symbol of w, and with $u_1, \ldots, u_t \in \{f, g, f^{-1}, g^{-1}\}$), we have $\delta_0 u_1^{(k)} \cdots u_j^{(k)} = \delta_j$ for each $j = 1, \ldots, t$. In practice, we refer to the w-chain $(\delta_1, \ldots, \delta_t)$ and drop the final entries i, w, but formally, two w-chains are equal if they agree in all entries, including the final ones. A *maximal $w^{(k)}$-chain* is a $w^{(k)}$-chain which is not a proper subsequence of any other $w^{(k)}$-chain. The *length* of a maximal $w^{(k)}$-chain $(\delta_0, \ldots, \delta_t)$ is t. At step k, a *new point* is some $\delta \in \Omega$ such that $\delta \notin \{\xi_0, \ldots, \xi_k\}$ and such that $\delta \notin \mathrm{dom}(f_i^{(k-1)}) \cup \mathrm{ran}(f_i^{(k-1)})$ for all $i \in I$. We often regard partial permutations as sets of ordered pairs, and we use the notation $\langle\alpha, \beta\rangle$ for ordered pairs.

A relational structure M is *homogeneous* if its domain is countably infinite and any isomorphism between finite substructures of M extends to an automorphism of M. The standard method of construction of homogeneous structures is Fraïssé's amalgamation theorem. The *random graph*, is a well-known example of a homogeneous structure. It is up to isomorphism the unique countably infinite graph Γ satisfying the following 'extension property': for any two finite disjoint sets U, V of vertices, there is a vertex adjacent to everything in U and to nothing in V. The homogeneous structure constructed in Section 2, though over an infinite language, has a similar characterisation. See [1] for more on homogeneous structures, Fraïssé amalgamation, and the random graph.

2. Proof of Theorem 1.1

The group G will be a group of automorphisms of a countable homogeneous relational structure Ω^* which we first construct. Let L be a first order language, with, for each $n \in \mathbb{N}$, a single relation symbol R_{n+2} of arity $n+2$. Let $\mathscr{C}$ be the class of all finite L-structures in which, for each $n \geq 0$, whenever $R(x_1, \ldots, x_{n+2})$ holds, we have that (a) all the x_i are distinct, and (b) $R(x_{1g}, \ldots, x_{(n+2)g})$ for each g in the symmetric group S_{n+2}. It is routine to check that $\mathscr{C}$ is an amalgamation class, so there is a unique countable homogeneous L-structure Ω^* whose finite substructures are up to isomorphism precisely the members of $\mathscr{C}$. Let Ω denote the domain of Ω^*, and for each $i > 1$ let Ω_i^* be the reduct of Ω^* to the language containing only the relations R_j for $j \geq i$ (so $\Omega_2^* = \Omega^*$). Put $\Omega = \{\xi_i : i \in \mathbb{N}\}$.

We build the permutations f_i so that for each $i \in \mathbb{N}$,

(i) f_i fixes ξ_j for all $j < i$, and acts as a single infinite cycle on $\{\xi_j : j \geq i\}$, and
(ii) $f_i \in \mathrm{Aut}(\Omega_{i+2}^*)$.

Since some but not all ordered $(i+2)$-sets in Ω_{i+2}^* satisfy R_{i+2}, the group $\langle f_0, \ldots, f_i\rangle$ will not be $(i+2)$-transitive.

We construct the permutations in ω many steps, arranging that for each word in the f_i, f_i^{-1}, after a certain stage it acquires no new fixed points. The group $G := \langle f_i : i \in \mathbb{N}\rangle$ must then be cofinitary. Each f_i is constructed as a union of a chain of finite approximations $(f_i^{(j)} : j \geq i)$, where $f_i^{(j)}$ is the approximation of f_i constructed after j steps.

Let $W := \{w_i : i \in \mathbb{N}\}$ be the set of cyclically reduced words in the f_i and f_i^{-1}. To ensure that G is cofinitary, it suffices to arrange that each element of W induces a permutation of Ω with just finitely many fixed points. This ensures also that G is freely generated by the f_i.

At step 0, we put $f_0^{(0)} = (\ldots \xi_1, \xi_0, \xi_i \ldots)$, where $i \in \mathbb{N} \setminus \{0, 1\}$ is least such that $R_2(\xi_1, \xi_0) \leftrightarrow R_2(\xi_0, \xi_i)$. This notation means that $(\ldots \xi_1, \xi_0, \xi_i \ldots)$ is a partial cycle of $f_0^{(0)}$, so $\xi_1 f_0^{(0)} = \xi_0$ and $\xi_0 f_0^{(0)} = \xi_i$, with $f_0^{(0)}$ not defined elsewhere.

Before the n^{th} step, we will have defined $f_j^{(n-1)}$ for all $j < n$. Here, $f_j^{(n-1)}$ fixes ξ_k for $k < j$ and has exactly one other finite partial cycle, which is incomplete and of length greater than one, and $\xi_k \in \mathrm{dom}(f_j^{(n-1)}) \cap \mathrm{ran}(f_j^{(n-1)})$ for all $k \leq n-1$. This last condition guarantees that the f_j will be defined everywhere and surjective.

At the n^{th} step, we ensure that $\xi_n \in \mathrm{dom}(f_j^{(n)}) \cap \mathrm{ran}(f_j^{(n)})$ for $j \leq n$, and that $f_n^{(n)}$ fixes $\xi_0, \ldots, \xi_{n-1}$. Our procedure to put ξ_n into the domain and range of $f_i^{(n)}$

is as follows (we do this for each $i \leq n$). If $\xi_n \in \mathrm{dom}(f_i^{(n-1)}) \cap \mathrm{ran}(f_i^{(n-1)})$, put $f_i^{(n)} := f_i^{(n-1)}$. If $\xi_n \in \mathrm{dom}(f_i^{(n-1)}) \setminus \mathrm{ran}(f_i^{(n-1)})$, choose a 'new' point $\delta \in \Omega$ ('new' as defined in the end of Section 1) and put $f_i^{(n)} := f_i^{(n-1)} \cup \{\langle \delta, \xi_n \rangle\}$ (so $\delta f_i^{(n)} = \xi_n$). We also assume that a 'new' point for $f_i^{(n)}$ cannot be new for any $f_j^{(n)}$ where $j \in \{0, \ldots, n\} \setminus \{i\}$. The restriction on δ is that $f_i^{(n)}$ preserves the relations R_j for $j \geq i+2$. Since the R_j only hold on tuples of distinct elements, and $\mathrm{dom}(f_i^{(n)})$ is finite, only finitely many R_j need be considered (of arity at most $|\mathrm{dom}(f_i^{(n)})|-1$), and so by the homogeneity of Ω^* there are infinitely many possibilities for δ. Similarly, if $\xi_n \in \mathrm{ran}(f_i^{(n-1)}) \setminus \mathrm{dom}(f_i^{(n-1)})$, then choose new $\delta \in \Omega$ as above and put $f_i^{(n)} := f_i^{(n-1)} \cup \{\langle \xi_n, \delta \rangle\}$. Also, to put ξ_n into the domain and range of $f_n^{(n)}$, just choose suitable distinct new δ, ϵ and put $f_n^{(n)} := (\xi_0) \ldots (\xi_{n-1})(\ldots, \delta, \xi_n, \epsilon, \ldots)$.

We must also consider the case when $\xi_n \notin \mathrm{dom}(f_i^{(n-1)}) \cup \mathrm{ran}(f_i^{(n-1)})$, and $i < n$. Suppose that the non-trivial partial cycle of $f_i^{(n-1)}$ is $(\ldots, \alpha_1, \ldots, \alpha_r, \ldots)$. Choose a new point δ and then a set of distinct new points $B_i = \{\beta_1, \ldots, \beta_s\}$ (with $\delta \notin B_i$) where $s = \max\{r-1, n\}$, and put

$$f_i^{(n)} := (\xi_0) \ldots (\xi_{i-1})(\ldots, \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s, \xi_n, \delta, \ldots).$$

The choice of δ is easy, much as in the last paragraph. The choice of the β_i however needs some care, to ensure that $f_i^{(n)}$ preserves R_j for $j \geq i+2$. We can ignore the fixed points $\xi_0, \ldots, \xi_{i-1}$, since each relation R_j and subset of size k of $\{\xi_0, \ldots, \xi_{i-1}\}$ determines a new relation of arity $j-k \geq i+2-k \geq 2$ on $\Omega \setminus \{\xi_0, \ldots, \xi_{i-1}\}$ which must be preserved by f_i, and there are finitely many of these 'new' relations (we only need to consider relations of arity less than $r+s+2$, the length of the non-trivial cycle of $f_i^{(n)}$). We have two kinds of conditions required for the β_j and δ. First, if one of the relations holds of a tuple from $\{\alpha_1, \ldots, \alpha_r\}$ then it must hold for any translates under $f_i^{(n)}$ which involve the β_i. Conditions of this sort have 'span' at most $r-1$, in the sense that they involve points at most $r-1$ apart in the cycle of $f_i^{(n)}$. Second, if a relation holds of a tuple involving ξ_n and some of $\{\alpha_1, \ldots, \alpha_r\}$, then translates of this under $f_i^{(n)}$ impose conditions on the β_i and δ. Conditions of this second sort have span at least $s+1 \geq r$, so there is no clash between conditions of the two sorts. Thus, using the homogeneity of Ω^* the elements of B_i can be found. The sets B_i (for $i < n$) are all chosen to be disjoint.

It remains to verify that in this construction, each word w_i has finitely many fixed points. Consider a word $w \in W$. As usual let $w^{(n)}$ denote the word obtained from w by replacing, for each $i \in \mathbb{N}$, any occurrence of f_i or f_i^{-1} by $f_i^{(n)}$ or $(f_i^{(n)})^{-1}$ respectively. Suppose that at step n, w acquires a fixed point, that is, there is $\epsilon \in M$ such that $\epsilon w^{(n-1)}$ is undefined but $\epsilon w^{(n)} = \epsilon$. We shall show that either $\ell(w) \geq s$ (so $\ell(w) \geq n$), or f_n occurs in w. It follows that there is some step t such that after step t, w acquires no new fixed points. Since $w^{(t)}$ has just finite domain, the word w has just finitely many fixed points, as required.

We may suppose that f_n does not occur in w. Step n really consists of $n+1$ substeps (one for each of $f_0^{(n)}, \dots, f_n^{(n)}$), and for convenience we shall suppose that $w^{(n)}$ becomes defined at ϵ at the 0^{th} substep, when we put ξ_n into the domain and range of $f_0^{(n)}$. (The arguments for $f_0, \dots, f_{n-1}$ are similar, and by the above assumption, we can ignore the substep when $f_n^{(n)}$ is defined as this cannot introduce a fixed point for w.) We shall suppose that $\xi_n \notin \mathrm{dom}(f_0^{(n-1)}) \cup \mathrm{ran}(f_0^{(n-1)})$, this being the hardest case. So $f_0^{(n-1)} = (\dots, \alpha_1, \dots, \alpha_r, \dots)$, and $f_0^{(n)} = (\dots, \alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s, \xi_n, \delta, \dots)$, where $s = \max\{r-1, n\}$.

Clearly f_0 or f_0^{-1} occurs in w. Write $w = u_1 \cdots u_t$ and $w^{(n)} = u_1^{(n)} \cdots u_t^{(n)}$, where $u_i \in \{f_j, f_j^{-1} : j \in \mathbb{N}\}$ and $u_i^{(n)}$ is the approximation of u_i after n steps. There is $j \le t$ such that $u_1^{(n-1)} \cdots u_{j-1}^{(n-1)}$ is defined at ϵ but $u_1^{(n-1)} \cdots u_j^{(n-1)}$ is undefined at ϵ. This means that u_j is f_0 or f_0^{-1} and ϵ', the image of ϵ under $u_1^{(n-1)} \cdots u_{j-1}^{(n-1)}$, is in $\{\alpha_r, \beta_1, \dots, \beta_s, \xi_n\}$ (if $u_j = f_0$) or in $\{\beta_1, \dots, \beta_s, \xi_n, \delta\}$ (if $u_j = f_0^{-1}$).

In the first case, when $u_j = f_0$, there are three possibilities.

(i) $\epsilon' = \xi_n$;
(ii) $\epsilon' = \alpha_r$;
(iii) $\epsilon' \in \{\beta_1, \dots, \beta_s\}$ (in which case $j = 1$ as the β_i are new).

As $\xi_n f_0^{(n)} = \delta$ which is new and w is reduced, if case (i) holds then $j = t$, $\epsilon = \delta$, and $u_1 = f_0^{-1}$, contrary to the assumption that w is cyclically reduced. If case (ii) holds, then as $w^{(n)}$ is defined at ϵ it follows that each of $u_j, \dots, u_{j+s}$ is equal to f_0, so $\ell(w) \ge s$. In case (iii) we have $\epsilon' = \epsilon = \beta_k$, say. Now since w is cyclically reduced and $u_1 = f_0$ and the β_i are new, $u_t = f_0$. From this it again follows easily that $\ell(w) \ge s$.

In the second case, we have $u_j = f_0^{-1}$. Now, one of the following holds.

(i) $\epsilon' = \xi_n$ and $j = 1$ (as δ is new);
(ii) $\epsilon' = \delta$ and $j = 1$ (as δ is new);
(iii) $\epsilon' \in \{\beta_1, \dots, \beta_s\}$ and $j = 1$.

In case (i), each of $u_j, \dots, u_{j+s}$ equals f_0^{-1}, so $\ell(w) \ge s$. In case (ii), it follows that $\epsilon' = \epsilon = \delta$, and $u_t = f_0$, contrary to the assumption that w is cyclically reduced. Finally, in case (iii), as the β_i are new we have $\epsilon' = \epsilon = \beta_k$, say. Now as in the last paragraph, since w is cyclically reduced it follows that $u_t = f_0^{-1}$, and $\ell(w) \ge s$.

We have shown that in all cases, if f_n does not occur in w, then $\ell(w) \ge s \ge n$. Hence, w has just finitely many fixed points, as required. □

3. Proof of Theorem 1.2

Put $\Omega := \{\xi_i : i \in \mathbb{N}\}$, $\Gamma := \{\xi_{2i} : i \in \mathbb{N}\}$, and $\Delta := \Omega \setminus \Gamma$. Fix a surjection $\Phi : \mathbb{N} \to \mathbb{N}^2$. Let F_2 be the free group on generators f, g. Let $W := \{w_i : i \in \mathbb{N}\}$ be the set of non-empty cyclically reduced words in f, g, f^{-1}, g^{-1}. We shall define an action of F_2 on Ω step-by-step, so that after step k the partial isomorphisms $f^{(k)}, g^{(k)}$ will have been defined (and $f := \cup(f^{(k)} : k \in \mathbb{N}), g := \cup(g^{(k)} : k \in \mathbb{N})$). We adopt other notational conventions of Section 1. For each $k \in \mathbb{N}$ there is an equivalence relation $\sim_k$ on Ω: $\alpha \sim_k \beta$ if there is some word w such that $\alpha w^{(k)} = \beta$. The $\sim_k$-classes will be called *k-components*.

At step 0, we put $f^{(0)} := \{\langle \xi_0, \xi_1 \rangle, \langle \xi_1, \xi_0 \rangle\}$ and $g^{(0)} := \{\langle \xi_0, \xi_2 \rangle, \langle \xi_2, \xi_0 \rangle\}$. We shall preserve throughout the construction the following conditions.

(i) For each $i \in \mathbb{N}$, all partial cycles of $w_i^{(k)}$ are finite.

(ii) Γ and Δ are $g^{(k)}$-invariant;

(iii) All $f^{(k)}$-partial cycles other than (ξ_0, ξ_1) lie within Γ or within Δ.

(iv) If Θ is a non-empty k-component, then there is $\xi \in \Theta$ such that not all of $\xi f^{(k)}, \xi g^{(k)}, \xi (f^{(k)})^{-1}, \xi (g^{(k)})^{-1}$ are defined (and if $\xi_0 \in \Theta$ then ξ can be chosen in either Γ or Δ).

Clearly, the above hold after Step 0. We also ensure that for each $i > 0$, w_i moves some element of Ω, and that $\langle f, g \rangle$ acts transitively on Ω. By (i), part (a) of the theorem holds. By (ii) and (iii), if $x \in F_2$ then all but finitely many of the cycles of x lie entirely in Γ or entirely in Δ, and (b) of the theorem follows.

The construction is in the following steps.

Step $k = 4n$. Ensure that $\xi_n \in \mathrm{dom}(f^{(k)}) \cap \mathrm{ran}(f^{(k)}) \cap \mathrm{dom}(g^{(k)}) \cap \mathrm{ran}(g^{(k)})$.

Step $k = 4n + 1$. Ensure that $w_n^{(k)}$ moves some element of Ω (to guarantee that $\langle f, g \rangle$ acts faithfully).

Step $k = 4n + 2$. Arrange that ξ_0 and ξ_n lie in the same k-component (this will yield transitivity of $\langle f, g \rangle$ on Ω).

Step $k = 4n + 3$. Ensure that if $\Phi(n) = \langle r, s \rangle$ then the $w_r^{(k)}$-cycle containing ξ_s is complete (this yields (i) above—the finiteness of all w-cycles).

We now verify that each of these steps can be carried out. It is easily checked that (i)–(iv) are preserved.

Step $k = 4n$. Suppose that $\xi_n \notin \mathrm{dom}(f^{(k-1)}) \cup \mathrm{ran}(f^{(k-1)})$. Find distinct new points ξ, ξ' in Γ (if n is even) or in Δ (if n is odd) and put

$$f^{(k)} := f^{(k-1)} \cup \{\langle \xi, \xi_n \rangle, \langle \xi_n, \xi' \rangle\}.$$

There are other cases (when f is replaced by g, or when ξ_n lies in just one of the domain or range of f or g), and these are handled similarly.

Step $k = 4n + 1$. Let $l := \ell(w_n)$. We extend $f^{(k-1)}, g^{(k-1)}$ to $f^{(k)}, g^{(k)}$ so that there is an $w_n^{(k)}$-chain consisting of distinct new points $\alpha_0, \ldots, \alpha_l \in \Gamma$ such that $\alpha_0 w_n^{(k)} = \alpha_l$.

Step $k = 4n + 2$. We may suppose $\xi_n \in \Gamma$ (as the case $\xi_n \in \Delta$ is essentially the same). Also, we may suppose that ξ_0 and ξ_n are in distinct $(k-1)$-components, as otherwise the result already holds. By (iv), there is $\gamma \in \Gamma$ lying in the $(k-1)$-component of ξ_0 such that for some $h_1 \in \{f, g, f^{-1}, g^{-1}\}$, $\gamma h_1^{(k-1)}$ is undefined. Likewise, there is δ in the $(k-1)$-component of ξ_n such that for some $h_2 \in \{f, g, f^{-1}, g^{-1}\}$, $\delta h_2^{(k-1)}$ is undefined. Let $h_3 \in \{f, g, f^{-1}, g^{-1}\}$, $h_3 \neq h_1^{-1}, h_2$. Choose new points $\epsilon_1, \epsilon_2 \in \Gamma$, and take the least extension of $f^{(k-1)}, g^{(k-1)}$ to $f^{(k)}, g^{(k)}$ so that $\gamma h_1^{(k)} = \epsilon_1$, $\epsilon_1 h_3^{(k)} = \epsilon_2$ and $\epsilon_2 (h_2^{(k)})^{-1} = \delta$. If $w := h_1 h_3 h_2^{-1}$, then $\gamma w^{(k)} = \delta$, so ξ_0, ξ_n are in the same k-component, as required.

Step $k = 4n + 3$. For notational convenience, put $\xi := \xi_s$, $w := w_r$ and $l := \ell(w)$. We may suppose that the $w^{(k-1)}$-cycle containing ξ is incomplete, and has the form $(\ldots, \delta_1, \ldots, \delta_t, \ldots)$. (We do not exclude here the case when ξ is a new point, so $t = 1$.) Let u_1 be a maximal initial segment of w such that $\delta_t u_1^{(k-1)}$ is defined, and likewise let v_1 be a maximal final segment of w such that $\delta_1 (v_1^{(k-1)})^{-1}$ is defined. Put $\epsilon := \delta_t u_1^{(k-1)}$ and $\epsilon' := \delta_1 (v_1^{(k-1)})^{-1}$. There are words u_1', v_1' so that $w = u_1 u_1' = v_1' v_1$ (so u_1', v_1' are non-empty, but possibly equal w).

Case 1. $\epsilon, \epsilon' \in \Gamma$. (The case $\epsilon, \epsilon' \in \Delta$ is similar.)

Let $m := \ell(v_1') + \ell(u_1')$ and put $w' := u_1' v_1'$. Then w' is reduced, as w is cyclically reduced. Suppose first $\epsilon \neq \epsilon'$. Choose new points $\epsilon_1, \ldots, \epsilon_{m-1} \in \Gamma$ and extend $f^{(k-1)}, g^{(k-1)}$ so that there is a $w'^{(k)}$-chain from ϵ to ϵ' of the form $(\epsilon_0, \ldots, \epsilon_m)$, where $\epsilon_0 := \epsilon$ and $\epsilon_m := \epsilon'$. The w-chain containing ξ is now complete.

If $\epsilon = \epsilon'$, slight extra care is needed if some initial segment of u_1' is equal to an initial segment of $v_1'^{-1}$. However, as w is cyclically reduced, we cannot have $u_1' = v_1'^{-1}$, and so essentially the same argument as above works.

Case 2. $\epsilon' \in \Gamma$ and $\epsilon \in \Delta$. (The case $\epsilon' \in \Delta$ and $\epsilon \in \Gamma$ is similar.)

In this case, by (ii), there is at least one occurrence of f or f^{-1} in w.

CLAIM. *There are $\eta \in \Delta$ and $\eta' \in \Gamma$ and a maximal $w^{(k-1)}$-chain beginning at η and ending at η' with $\eta(u_2 w^j v_2)^{(k-1)} = \eta'$, where u_2 is a proper final segment of w and v_2 is a proper initial segment of w (and possibly $j = 0$).*

PROOF OF CLAIM. For each occurrence of f or f^{-1} in w, consider the maximal $w^{(k-1)}$-chain in which that occurrence takes ξ_0 to ξ_1, and the maximal $w^{(k-1)}$-chain in which that occurrence takes ξ_1 to ξ_0. Let $C_1, \ldots, C_t$ list the $w^{(k-1)}$-chains so obtained. Let a_i be the number of (Γ, Δ)-*crossings* of C_i (that is, successive pairs ξ_0, ξ_1 in C_i), and b_i the number of (Δ, Γ)-*crossings* (successive pairs ξ_1, ξ_0). Each occurrence of f (or f^{-1}) in w determines a unique (Γ, Δ)-crossing of some C_i, and a unique (Δ, Γ) of some (distinct) C_j. Also each (oriented) crossing of each C_i comes from a unique occurrence of f or f^{-1} in w. It follows that

$$a_1 + \cdots + a_t = b_1 + \cdots + b_t.$$

If C_i is a chain of a complete $w^{(k-1)}$-cycle, or begins and ends in Γ, or begins and ends in Δ, then $a_i = b_i$. Likewise, if C_i begins in Γ and ends in Δ then $a_i = b_i + 1$, and if

C_i begins in Δ and ends in Γ then $b_i = a_i + 1$. Since the chain from ϵ' to ϵ begins in Γ and ends in Δ, it follows that there is some C_i which begins in Δ and ends in Γ, as required. $\square$

Given the claim, write $w = u_2'u_2 = v_2v_2'$. As in case 1, the words $u_1'u_2'$ and $v_2'v_1'$ are reduced. Suppose first that $\epsilon \neq \eta$ and $\epsilon' \neq \eta'$. As in Case 1, extend $f^{(k-1)}, g^{(k-1)}$ to $f^{(k)}, g^{(k)}$ in a minimal way, entirely using new points, so that $\epsilon(u_1'u_2')^{(k)} = \eta$ and $\eta'(v_2'v_1')^{(k)} = \epsilon'$. As in Case 1, slight extra care is needed if say $\epsilon = \eta$ (and similarly if $\epsilon' = \eta'$). For example, it could happen that $\epsilon = \eta$ and there is an initial segment u of u_1' such that u^{-1} is a final segment of u_2'. In this case, since w is cyclically reduced we cannot have $u_1' = u = (u_2')^{-1}$, and it follows that the extension is still possible. $\square$

REMARK 3.1. There are certain refinements of the construction in the proof of Theorem 1.2. For example, it is possible to arrange that F_2 acts 2-transitively on Ω. One needs to show that the stabiliser of ξ_2 can be made transitive on $\Omega \setminus \{\xi_2\}$. The idea is, for an arbitrary ξ, to fix ξ_2 and map ξ_0 to ξ_i by some very long word. More generally, one can arrange that the action of F_2 on Ω is *highly transitive*, that is, k-transitive for all $k > 0$.

4. Proof of Theorem 1.3

We build automorphisms f, g of the random graph $(\Omega, \sim)$. Let S be the set of non-empty cyclically reduced words in f, g, f^{-1}, g^{-1}. Define an equivalence relation $\equiv$ on S, putting $u \equiv v$ if and only if there are words $w_1, w_2 \in S$ and $r, s \in \mathbb{Z} \setminus \{0\}$ such that

$$w_1^{-1}u^r w_1 = w_2^{-1}v^s w_2.$$

Let $W = \{w_i : i \in \mathbb{N}\}$ consist of exactly one element, chosen of least possible length, from each $\equiv$-class. By the minimality assumption no element of W can be a proper power, and each is reduced, and not conjugate to any shorter word. The construction of f, g is by finite approximation, and after step k we denote by $f^{(k)}, g^{(k)}$ the restrictions of f, g so far defined (likewise, for any word w, $w^{(k)}$ is the restriction defined after step k). Let $l_n := \ell(w_n)$. We suppose that $w_0 = f$ and $w_1 = g$. To ensure that $\langle f, g\rangle$ generate a ZTF group, we shall arrange that each w_i has just finitely many cycles. One of the steps will be to extend $f^{(k)}, g^{(k)}$ so that certain partial cycles of some w_i are 'joined' into a single cycle. This is not always possible: for example, if a partial automorphism h had incomplete cycles $(\ldots, \alpha_1, \ldots, \alpha_r, \ldots)$ and $(\ldots, \beta_1, \ldots, \beta_s, \ldots)$ where $\alpha_1 \sim \alpha_2$ but $\beta_1 \not\sim \beta_2$, then there is no extension of h with a single cycle extending these partial cycles. This problem did not arise in [7] (where there was no invariant relational structure) and makes the proof here more complicated. First, we must formalise the notion of compatibility of partial cycles.

DEFINITION 4.1. Let w be a word, with $\ell(w) = l$. Then two $w^{(k)}$-cycles

$$(\ldots, \epsilon_1, \ldots, \epsilon_r, \ldots) \quad \text{and} \quad (\ldots, \epsilon'_1, \ldots, \epsilon'_s, \ldots)$$

are *compatible*, if the following hold, where the $w^{(k)}$-chains corresponding to the above cycles are $(\delta_1, \ldots, \delta_n)$ and $(\delta'_1, \ldots, \delta'_m)$.

(i) For all $t \in \mathbb{N}$ with $tl \leq \min\{m, n\}$

$$\delta_1 \sim \delta_{tl} \leftrightarrow \delta'_1 \sim \delta'_{tl}.$$

(ii) Suppose the finite complete $w^{(k)}$-cycles are $C_1, \ldots, C_p$, of lengths $r_1, \ldots, r_p$, and that $D_i := \{x \in C_i : x \sim \epsilon_1\}$ and $D'_i := \{x \in C_i : x \sim \epsilon'_1\}$. Then there are $s_1, \ldots, s_p \in \mathbb{N}$ such that $D'_i = D_i(w^{(k)})^{s_i}$ for each i and there is $a \in \mathbb{N}$ such that $a \equiv s_i \pmod{r_i}$ for each $i = 1, \ldots, p$.

We shall say that two $w^{(k)}$-cycles are *weakly compatible* if just condition (ii) above holds.

Because of condition (i), compatibility is not an equivalence relation (since two incompatible long $w^{(k)}$-cycles can each be compatible with a short $w^{(k)}$-cycle). However, weak compatibility is an equivalence relation. The idea of the above definition is that if two $w^{(k)}$-cycles are compatible then it should be possible to extend $w^{(k)}$ so that they are parts of a single cycle and such that the number of new points used to join the two cycles depends on the a obtained in (ii) above. We shall do this explicitly a little later.

We now describe the construction of f, g. First, we fix a surjective function $\Phi : \mathbb{N} \to \mathbb{N}^3$ which takes each value of $\mathbb{N}^3$ infinitely often. Our construction proceeds through steps $5n$ to $5n + 4$. If $k \in \{5n, \ldots, 5n + 4\}$, then step k may be a sequence of substeps. We adopt the general notation that for a word w, the function determined by w before such a substep of step k is written $w^{(k*)}$, and after the substep it is denoted by $w^{(k+)}$.

DEFINITION 4.2. Suppose $a, b \in \mathbb{N}$ with $a, b < k/5$, and that $w_a, w_b \in W$. A $(w_a, w_b, k*, \alpha, \beta)$-*coincidence* consists of a $w_a^{(k*)}$-chain and a distinct $w_b^{(k*)}$-chain from α to β such that there is a common letter x (one of f, g, f^{-1}, or g^{-1}) such that $\beta x^{(k*)}$ is undefined, but in some extension of both the $w_a^{(k*)}$-chain and the $w_b^{(k*)}$-chain, $\beta x^{(k+)}$ would be the next element after β.

Before starting step $5n$, we partition the incomplete cycles of $w_{n-1}^{(5n-1)}$ which lie in maximal chains of length at least $2l_{n-1}$ into finitely many classes, say

$$K_1^{n-1}, \ldots, K_{h(n-1)}^{n-1},$$

so that any two cycles in some K_i^{n-1} are compatible (here $h : \mathbf{N} \to \mathbf{N}$ is some indexing function). We will eventually arrange that w_{n-1} has $h(n-1)$ infinite cycles (so that $h(0) = 1$ and $h(1) = 2$), with partial cycles in a given K_i^{n-1} eventually being joined so they lie in the same cycle. The word w_{n-1} acquires no new finite complete cycles after step $5n - 1$. If a partial cycle lies in K_i^{n-1} then we refer to K_i^{n-1} as its *type*, or *compatibility type*. This is never changed: later extensions of a partial cycle in K_i^{n-1} will still have type K_i^{n-1}, and at later stages, every cycle of $w_{n-1}^{(k)}$ in a maximal chain of length at least $2l_{n-1}$ will have type K_i^{n-1} for some unique $i \in \{1, \ldots, h(n-1)\}$. At each substep after $5n$, as soon as a partial cycle of w_{n-1} lies in a chain of length at least $2l_{n-1}$, we choose some i so that the cycle is compatible with cycles of type K_i^{n-1}, and specify that it has *type* K_i^{n-1}. At any stage any two cycles of a word of the same type will be compatible.

Step $k = 5n$. Ensure that $\xi_n \in \mathrm{dom}(f^{(k)}) \cap \mathrm{ran}(f^{(k)})$.

Step $k = 5n + 1$. Ensure that $\xi_n \in \mathrm{dom}(g^{(k)}) \cap \mathrm{ran}(g^{(k)})$.

Step $k = 5n + 2$. If $\Phi(n) = (p, q, r)$ with $p < n$, and ξ_q, ξ_r lie in $w_p^{(k-1)}$-cycles of the same type, each in chains of length at least $2l_p$, extend $f^{(k-1)}, g^{(k-1)}$ so that ξ_q, ξ_r are in the same $w_p^{(k)}$-cycle.

Step $k = 5n + 3$. We ensure that over the complete cycles of $w_n^{(k-1)}$ there are $w_n^{(k)}$-cycles of each possible weak compatibility class lying in chains of length at least $2l_n$.

Step $k = 5n + 4$. Extend $f^{(k-1)}, g^{(k-1)}$ to arrange that there are no $(w_i, w_n, 5n + 4, \alpha, \beta)$-coincidences for $i \le n$.

Throughout the steps $5n$ to $5n + 4$, we ensure that

(a) up to compatibility there is a unique $f^{(k)}$-cycle and at most 2 incomplete $g^{(k)}$-cycles, and $g^{(k)}$ has a unique complete cycle (ξ_0).

(b) there is no coincidence in which both the words involved are from $w_0, \ldots, w_{n-1}$,

(c) if $i \le n - 1$, then any complete $w_i^{(5n+4)}$-cycle is a complete $w_i^{(5n-1)}$-cycle,

(d) any extension of cycles of w_i^{k*} $(0 \le i \le n-1)$ respects their compatibility type; that is if two cycles of $w_i^{(k*)}$ have the same type, then so do their extensions to cycles of $w_i^{(k+)}$.

We call any extension $f^{(k+)}, g^{(k+)}$ of $f^{(k*)}, g^{(k*)}$ preserving these properties a *good extension*.

LEMMA 4.3. *Suppose that* $n = [k/5]$, *the integer part of* $k/5$, *and that* $\gamma \notin \mathrm{dom}(f^{(k*)})$. *Suppose that after step* $k*$, (a)–(d) *above hold. Then there is* $\delta \in \Omega$ *so that the extension* $f^{(k+)} := f^{(k*)} \cup \{\langle\gamma, \delta\rangle\}$, $g^{(k+)} := g^{(k*)}$ *is good.*

REMARK. The corresponding statements hold with f^{-1}, g, or g^{-1} in place of f.

PROOF OF LEMMA 4.3. We must choose δ, a *new* point of Ω. The requirement that $f^{(k+)}$ is a partial automorphism essentially says that for certain finite disjoint $\Delta_1, \Delta_2 \subseteq \Omega$, δ must be chosen in

$$\{x : \forall y \in \Delta_1(x \sim y) \wedge \forall y \in \Delta_2(x \not\sim y)\}.$$

By the extension property which characterises the random graph, this set is infinite, that is, there are infinitely many choices for such δ. As δ is new and the w_j are cyclically reduced, for any $i \in \mathbb{N}$ any complete cycle of $w_i^{(k+)}$ is a complete cycle of $w_i^{(k*)}$, so in particular (c) above will be satisfied.

We next check that δ can be found so that (d) holds. Condition (ii) in Definition 4.1 plays no role here, because the weak compatibility class of a partial cycle is determined by one of its elements. Essentially, our compatibility requirements merely force us to restrict the choice of δ by increasing Δ_1 and Δ_2 (to ensure that the conditions are satisfied). The only problem is to ensure that Δ_1 and Δ_2 are disjoint, that is, that it doesn't happen that one compatibility requirement puts some $\epsilon \in \Delta_1$, and some other compatibility (or automorphism) condition puts $\epsilon \in \Delta_2$. There could not be a clash between a compatibility requirement and an automorphism condition, for suppose the compatibility requirement forced $\delta \sim \epsilon$ (that is, $\epsilon \in \Delta_1$) and an automorphism requirement forced $\delta \not\sim \epsilon$ (that is, $\epsilon \in \Delta_2$). This means that for some $a \leq n$ there will be a $w_a^{(k+)}$-chain of length tl_a say from ϵ to δ, and further $\eta := \epsilon(f^{(k*)})^{-1} \not\sim \gamma$. However, in this case there is already a $w_a^{(k*)}$-chain of length tl_a from η to γ which conflicts with our compatibility requirements. It can be checked that two compatibility requirements can only clash if there was a $(w_a, w_b, k*, \alpha, \beta)$-coincidence, and by assumption there is none.

To verify (b), suppose that there is a $(w_a, w_b, k+, \alpha, \beta)$ coincidence. Then either $\alpha = \delta$ or $\beta = \delta$. If $\beta = \delta$, then the last letter used in both the $w_a^{(k+)}$-chain and the $w_b^{(k+)}$-chain is f, so there was previously a $(w_a, w_b, k*, \alpha, \gamma)$-coincidence, contrary to (b) at the previous step. Similarly, if $\delta = \alpha$, then there was previously a $(w_a, w_b, k*, \gamma, \beta)$-coincidence, again a contradiction. □

LEMMA 4.4. (i) *Let $w_a, w_b \in W$, with $w_a \not\equiv w_b$, and put $l_a = \ell(w_a)$, $l_b = \ell(w_b)$. Suppose that w_a, w_b have a common chain of length n. Then*

$$n < \max\{l_a(l_a + 1), l_b(l_b + 1)\}.$$

(ii) *Let $w \in W$ have length l and $(\delta_0, \dots, \delta_{2l})$ be a w-chain with $\delta_0 w^2 = \delta_{2l}$. Suppose that for some $i > 0$ there is a w-chain $(\delta_i, \dots, \delta_{i+l})$. Then $i = l$.*

PROOF. (i) Suppose not. We may suppose $l_a \geq l_b$. By the pigeon-hole principle, we may suppose there are distinct ϵ_i, ϵ_j on the w_a-cycle and an initial subword u of w_a such that $\epsilon_i u, \epsilon_j u$ are on the w_b-cycle. (There is another possible case, handled similarly, when u is a final subword of w_a and $\epsilon_i u^{-1}, \epsilon_j u^{-1}$ are on the w_b-cycle.) Hence $\epsilon_i u w_b^k = \epsilon_j u$ for some $k \in \mathbb{N}$. Put $m := j - i$. Then $\epsilon_i w_a^m = \epsilon_j$ (via the same chain), so $w_a^m u = u w_b^k$. Hence $w_a \equiv w_b$, which is a contradiction.

(ii) Suppose $i \neq l$ and suppose u, v are respectively initial and final segments of w such that $\delta_0 u = \delta_i$ and $\delta_{i+l} v = \delta_{2l}$. Then (by considering lengths of words), $\delta_i v = \delta_l$ and $\delta_l u = \delta_{i+l}$. It follows that $uw = wu$, so w is a proper power of u, which is impossible. □

To get started we write down the first 10 steps explicitly (remembering that $w_0 = f$, $w_1 = g$). This will serve to check that the conditions hold early on.

Step 0: Put $f^{(0)} = (\ldots \xi_1, \xi_0, \xi_i \ldots)$, where $i \in \mathbb{N} \setminus \{0, 1\}$ is least such that $\xi_1 \sim \xi_0 \leftrightarrow \xi_0 \sim \xi_i$. As in Section 2 this notation means that $(\ldots \xi_1, \xi_0, \xi_i \ldots)$ is a partial cycle of $f^{(0)}$, so $\xi_1 f^{(0)} = \xi_0$ and $\xi_0 f^{(0)} = \xi_i$, with $f^{(0)}$ not defined elsewhere.

Step 1: Put $g^{(0)} = (\xi_0)$. That is g fixes ξ_0.

It is easy to see that there is nothing to be done in Steps 2, 3 and 4. At this stage we specify that $h(0) = 1$, that is, $w_0 = f$ has a unique compatibility type K_1^0.

Step 5: Put $f^{(5)} = (\ldots \xi_j, \xi_1, \xi_0, \xi_i \ldots)$, where $j \in \mathbb{N} \setminus \{0, 1, i\}$ is least such that $\xi_j \sim \xi_1 \leftrightarrow \xi_1 \sim \xi_0$ and $\xi_j \sim \xi_0 \leftrightarrow \xi_1 \sim \xi_i$.

Step 6: Here we need to put ξ_1 into the domain and range of g. Let us call a point ξ of Ω a *neighbour* of ξ_0 if $\xi \sim \xi_0$ and a *non-neighbour* otherwise. Since g fixes ξ_0 we extend g in such a way that all neighbours will eventually be in one cycle and the non-neighbours in another, thus giving us 3 cycles in all. Put $g^{(5)} = (\xi_0)(\ldots \xi_n, \xi_1, \xi_m \ldots)$ where ξ_n, ξ_1, ξ_m are either all neighbours or all non-neighbours of ξ_0 and

$$\xi_n \sim \xi_1 \leftrightarrow \xi_1 \sim \xi_m.$$

Step 7: Nothing need be done, as $g^{(6)}$ has a unique incomplete cycle with more than one point.

Step 8: The only complete cycle of $g^{(7)}$ is (ξ_0), so for example if $\xi_1 \sim \xi_0$, then at step 8 we must extend $g^{(7)}$ by adjoining an incomplete 2-cycle of non-neighbours of ξ_0.

Step 9: Nothing need be done, as w_0, w_1 have length 1 and distinct words of length 1 cannot have a common next letter as required for a coincidence.

Finally, we specify that $w_1 = g$ has two compatibility types K_1^1 and K_2^1, corresponding to neighbours and non-neighbours of ξ_0 respectively, so $h(1) = 2$.

It follows immediately from Lemma 4.3 and the remark following its statement that Steps $5n$ and $5n + 1$ are possible. It is also straightforward to see that Step $5n + 3$ is possible, since we can construct new $w_n^{(5n+3)}$-chains of length l_n using new points.

Step $k = 5n + 2$. This is the most troublesome step. Suppose that $\Phi(n) = (q_1, q_2, q_3)$ with $q_1 < n$ and write $w := w_{q_1}$ and $l := \ell(w)$. Let $(\ldots, \epsilon_1, \ldots, \epsilon_c, \ldots)$ be the $w^{(k-1)}$-cycle containing ξ_{q_2}, with corresponding $w^{(k-1)}$-chain $(\delta_1, \ldots, \delta_r)$, and

$(\dots, \epsilon'_1, \dots, \epsilon'_d, \dots)$ be the $w^{(k-1)}$-cycle containing ξ_{q_2}, with corresponding $w^{(k-1)}$-chain $(\delta'_1, \dots, \delta'_s)$. We may suppose that these two cycles are distinct, as otherwise there is nothing to do. By extending these two cycles if necessary (using Lemma 4.3), we may assume that they are the two longest $w^{(k-1)}$-cycles, with $c > d$.

Let $l' := \max\{l_0, \dots, l_n\}$. We first apply Lemma 4.3 to add between $l'^2 + l'$ and $l'^2 + 2l'$ new points to each end of the $w^{(k-1)}$-chain $(\delta_1, \dots, \delta_r)$ to obtain a new $w^{(k*)}$-chain $(\alpha_1, \dots, \alpha_{r'})$ with w-cycle $(\dots, \epsilon_{-m}, \dots, \epsilon_{c+m+1}, \dots)$, where $\alpha_1 = \epsilon_{-m}$ and $\alpha_{r'} = \epsilon_{c+m+1}$ (so $m \geq l'$). Likewise, we can find a good extension of the $w^{(k-1)}$-chain $(\delta'_1, \dots, \delta'_s)$ to a chain $(\alpha'_1, \dots, \alpha'_{s'})$ with $w^{(k*)}$-cycle $(\dots, \epsilon'_{-m}, \dots, \epsilon'_{d+m+1}, \dots)$, where $\alpha'_1 = \epsilon'_{-m}$ and $\alpha'_{s'} = \epsilon'_{d+m+1}$. This is done so Lemma 4.4 (i) can be applied later.

We now adopt the notation of Definition 4.1 for the complete cycles $C_1, \dots, C_p$ of $w^{(k-1)}$. In particular, $D_i := \{x \in C_i : x \sim \epsilon_1\}$ and $D'_i = \{x \in C_i : x \sim \epsilon'_1\}$, for each $i = 1, \dots, p$. By compatibility, there is a such that $a \equiv s_i \pmod{r_i}$ for each $i = 1, \dots, p$. Put $b := a - (c + 2m + 1)$. The idea here is to ensure that $\epsilon_1 w^a = \epsilon'_1$. Now $\epsilon_1 w^{c+m} = \epsilon_{c+m+1}$, and $\epsilon'_{-m} w^{m+1} = \epsilon'_1$. Thus we need b such that $\epsilon_{c+m+1} w^b = \epsilon'_{-m}$. That gives $a = c + m + b + m + 1$. For later convenience, we choose a so that $b > c + 2m + 2$.

We shall find new $\gamma_1, \dots, \gamma_{lb-1}$ so that there is a good extension $f^{(k+)}, g^{(k+)}$ of $f^{(k*)}, g^{(k*)}$ such that there is a $w^{(k+)}$-chain $(\alpha_1, \dots, \alpha_{r'}, \gamma_1, \dots, \gamma_{lb-1}, \alpha'_1, \dots, \alpha'_{s'})$ with $\alpha_{r'}(w^{(k+)})^b = \alpha'_1$. To smooth out notation, we put

$$\gamma_{-(r'-1)} := \alpha_1, \dots, \gamma_0 := \alpha_{r'}, \gamma_{lb} := \alpha'_1, \dots, \gamma_{lb+s'-1} := \alpha'_{s'}.$$

The process is inductive. After a typical step $k*$ we will have found $\gamma_1, \dots, \gamma_{i-1}$, so that $(\gamma_{-(r'-1)}, \dots, \gamma_{i-1})$ is a $w^{(k*)}$-chain. At step $k+$ we must find γ_i so that the following conditions hold (they are assumed inductively to hold after step $k*$). Below, we say that a word z *potentially takes* α to β if, for any extension of $f^{(k+)}, g^{(k+)}$ to $f^{(k)}, g^{(k)}$ (partial permutations, not necessarily automorphisms) given by choosing $\gamma_{i+1}, \dots, \gamma_{lb-1}$ so that $(\gamma_{-(r'-1)}, \dots, \gamma_{lb+s'-1})$ is a $w^{(k)}$-chain, we have $\alpha z^{(k)} = \beta$. Thus, for example, before finding γ_1 the word w^b potentially takes γ_0 to γ_{lb}, and if f is the first letter of w, then $w^b f$ potentially takes γ_0 to γ_{lb+1}. The idea of (1)–(4) below is that we have an implicit commitment that a certain final subword of w^b must eventually take γ_i to γ_{lb}. We will also sometimes say that a word z will *eventually take* α to $\gamma_j \in \{\gamma_{i+1}, \dots, \gamma_{lb-1}\}$, or write that $\alpha z^{(k)} = \gamma_j$, meaning that for any extension $f^{(k)}, g^{(k)}$ as above, we have $\alpha z^{(k)} = \gamma_j$.

(1) Automorphism conditions: $f^{(k+)}, g^{(k+)}$ are partial automorphisms.

(2) Compatibility conditions: for $i < n$, if two partial cycles of $w_i^{(k*)}$ have the same compatibility type, so do their extensions after step $k+$.

(3) If $\alpha, \beta \in \Omega$ with $\alpha z^{(k+)} = \beta$, and $\mu, \lambda \in \Omega$ and $\mu z^{(k+)}$ is undefined but z potentially takes μ to λ, $\mu \sim \alpha \leftrightarrow \lambda \sim \beta$.

(4) If $\alpha, \beta, \alpha', \beta' \in \Omega$ and $\alpha z^{(k+)}$, $\beta z^{(k+)}$ are undefined but z potentially takes $\langle \alpha, \beta \rangle$ to $\langle \alpha', \beta' \rangle$, then $\alpha \sim \alpha' \leftrightarrow \beta \sim \beta'$.

Conditions (3) and (4) deal with commitments arising because of the intention later to add $\gamma_{i+1}, \ldots, \gamma_{lb-1}$. They become important when we choose γ_{lb-1}. Up until then, using Lemma 4.3, we could make choices preserving just (1) and (2). However, to ensure that we can choose γ_{lb-1} so that (1) and (2) still hold, we need to preserve (3) and (4) throughout the construction (and this will suffice). To see this, suppose after step $(k*)$ we have found $\gamma_1, \ldots, \gamma_{lb-2}$, and must find γ_{lb-1}, subject say to $\gamma_{lb-2} f^{(k+)} = \gamma_{lb-1}$ and $\gamma_{lb-1} g^{(k+)} = \gamma_{lb}$. Suppose say that it is impossible to find γ_{lb-1} (subject to (1) above). In this case there are λ, μ, ν such that $\lambda f^{(k*)} = \mu$ and $\lambda \not\sim \gamma_{lb-2}$, and $\mu g^{(k*)} = \nu$ and $\nu \sim \gamma_{lb}$ (or the same holds with $\sim$ and $\not\sim$ reversed). Then after step $(k*)$ we had that fg potentially takes γ_{lb-2} to γ_{lb}, and $\lambda f^{(k*)} g^{(k*)} = \nu$, but $\lambda \not\sim \gamma_{lb-2}$ but $\nu \sim \gamma_{lb}$, contrary to (3) or (4) at step $k*$.

Condition (2) above poses no problems, essentially by Lemma 4.4 (i) (and the fact that we extended the chain $(\delta_1, \ldots, \delta_r)$ sufficiently). For when we add $\gamma_1, \ldots, \gamma_{lb-1}$, no cycles for $w_0, \ldots, w_{n-1}$, other than the obvious one for w, are affected (we may create some new cycles for other words, but they will be compatible with previous cycles). Also, as in the proof of Lemma 4.3, we may at each stage choose γ_i such that (1) holds (with the argument in the last paragraph for $i = lb - 1$). Thus, the problem is to show that (3) and (4) hold before γ_1 is chosen, and that, assuming that $\gamma_1, \ldots, \gamma_{i-1}$ are chosen to satisfy (1)–(4) and that the choice of γ_i also satisfies (1), then it can be arranged that (3) and (4) also hold after the choice of γ_i.

We first simplify (3). Suppose that $u^{(k+)}$ is the word which will take γ_0 along the chain $(\gamma_0, \gamma_1, \ldots, \gamma_i)$ to γ_i, and that $w^b = uv$ (so that v potentially takes γ_i to γ_{lb}). Then, since we assume γ_i is chosen to satisfy (1), an easy induction argument on the length of z in (3) allows us to assume that $z = v$, with $\alpha = \gamma_i$ and $\beta = \gamma_{lb}$. We omit the other case, when $z = v^{-1}$ and $\alpha = \gamma_{lb}$, $\beta = \gamma_0$.

Starting the induction. First, note that condition (3) holds before γ_1 is chosen, since the $w^{(k*)}$-cycles

$$(\ldots, \gamma_{-(r'-1)}, \ldots, \gamma_0, \ldots) \quad \text{and} \quad (\ldots, \gamma_{lb}, \ldots, \gamma_{lb+s'-1}, \ldots)$$

satisfy condition (ii) of Definition 4.1, and since by the choice of b, there are no other w-cycles of length b.

We show now that (4) holds at the beginning, that is, when $(k+)$ is the step before γ_1 is chosen. Suppose not, and let the word z be a counterexample to (4) of least length. Then we can write z as $x_1 y_1 x_2 y_2 \ldots x_p y_p$, where the x_i, y_i are reduced words (y_p possibly empty, the other y_i non-empty), with $x_1, \ldots, x_r \in \{w^b, w^{-b}\}$. Furthermore, we may suppose $\alpha \in \{\gamma_0, \gamma_{lb}\}$, say $\alpha = \gamma_0$, in which case $x_1 = w^b$. Slightly abusing notation, we shall write that $\alpha x_1^{(k+)} = \gamma_{lb}$. Since we are reducing to the case $p = 1$, we first suppose $p > 1$. Each $y_i^{(k+)}$ is defined on $\alpha(x_1y_1\cdots x_{i-1})^{(k+)}$, and $\alpha(x_1y_1\cdots x_{i-1})^{(k+)}$ is equal to γ_{lb} if $x_{i-1} = w^b$, or γ_0 if $x_{i-1} = w^{-b}$. By minimality of z (using (1) and that (3) holds at the beginning), $\alpha' x_1^{(k+)}$, $\alpha'(x_1y_1)^{(k+)}$ are undefined, that is, x_1, x_1y_1 will eventually take α' to points in $\{\gamma_1, \ldots, \gamma_{lb-1}\}$. Since x_1 ends in a copy of w, and x_2 starts with a copy of w or w^{-1}, it follows by Lemma 4.4 (ii) that

$$\alpha' x_1^{(k)}, \alpha'(x_1y_1)^{(k)} \in \{\gamma_0 w^i : i = 1, \ldots, b-1\}.$$

In particular, y_1 is a power of w (clearly a positive power as it is defined on $\gamma_{lb} = \alpha x_1^{(k+)}$). However, in this case $\gamma_{lb}y_1 = \gamma_{l(b+j)}$ for some $j > 0$, contradicting that $\gamma_{lb}y_1 \in \{\gamma_0, \gamma_{lb}\}$

Thus, we have $p = 1$, so $z = w^b y_1$, with $\gamma_{lb}y_1$ defined after step $(k+)$. By minimality of z, we also have $z = y_1' w^b$ or $z = y_1' w^{-b}$, and we suppose the former (the latter is similar). Then $\beta' = \gamma_{lb}$. We wish to show y_1 or y_1' is a power of w, for then (4) holds by condition (i) of Definition 4.1.

Suppose first $l(y_1') \geq lb$. Then, since w^b is an initial segment of z, we have $z = w^b u w^b$ for some u. In this case, as w^b is defined (before the choice of γ_1) on $\gamma_{lb}u^{(k*)}$, the element $\gamma_{lb}u^{(k*)}$ lies on a finite complete w-cycle, as does $\alpha(u^{-1})^{(k*)}$. Now $\alpha \sim \alpha' \leftrightarrow \beta \sim \alpha'$ (as the two w-cycles being joined satisfy Definition 4.1 (ii)). Since u is defined on α', β before γ_1 is chosen, $\beta \sim \alpha' \leftrightarrow \beta' \sim \alpha$. By Definition 4.1 (ii) again, $\beta' \sim \alpha \leftrightarrow \beta' \sim \beta$, whence $\alpha \sim \alpha' \leftrightarrow \beta \sim \beta'$, as required. Thus, we may assume $\ell(y_1') < lb$, so y_1' is an initial subword of w^b (and likewise y_1 is a final subword). It now follows by Lemma 4.4 (ii) that y_1' is a power of w, since otherwise z would not potentially take α to β. This starts the induction.

The inductive step. We now suppose that (3) and (4) hold after step $k*$ (when γ_{i-1} was chosen), and verify that γ_i can be chosen so they hold after step $k+$. We may suppose that γ_i is to be chosen to equal $\gamma_{i-1}f^{(k+)}$. Recall the simplification of (3) before the inductive step, and the choice of v. In particular, after step $k*$, fv potentially takes γ_{i-1} to γ_{lb}.

First note that (3) does not conflict with a condition of type (1). Suppose $\mu, \lambda \in \Omega$ with $\mu v^{(k*)} = \lambda$, and that $\lambda \sim \gamma_{lb}$. We must choose γ_i so $\mu \sim \gamma_i$. If this clashes with (1), then $\mu(f^{-1})^{(k*)} \not\sim \gamma_{i-1}$. However, fv potentially takes (at step $k*$) γ_{i-1} to γ_{lb} and $(\mu(f^{-1})^{(k*)})(fv)^{(k*)} = \lambda$. Thus, $\lambda \sim \gamma_{lb}$ and $\mu(f^{-1})^{(k*)} \not\sim \gamma_{i-1}$, contrary to (3) at the previous substep. We should also consider here the case $\lambda = \gamma_i$, in which case we must ensure $\mu \sim \gamma_i \leftrightarrow \gamma_i \sim \gamma_{lb}$. Again, this is consistent.

Thus, it remains to show that (4) is preserved, under the assumption that (1)–(3) hold after step $k+$. So suppose γ_i is chosen so that (1)–(3) hold but (4) does not hold, and that the word z is a counterexample to (4) of minimal length. By this minimality we may suppose that in (4), $\alpha = \gamma_i$, and that the word z has form $vy_1x_2y_2\cdots x_ry_r$, where $x_2, \ldots, x_r \in \{v, v^{-1}\}$, and the y_i are arbitrary (reduced) words. Here, after step $k*$, each y_i is defined on the potential image of γ_i under $vx_2y_2\cdots x_i$, and the decomposition of z is chosen so that $\alpha vy_1\dots y_j$ is potentially α if $x_{j+1}=v$, or γ_{lb} if $x_{j+1}=v^{-1}$.

We suppose for a contradiction that $r>1$. By minimality of $\ell(z)$ and the assumption that (1)-(3) hold, $v^{(k+)}$ and $(vy_1)^{(k+)}$ are not potentially defined on α' (so eventually, i.e., after step k, they will take α' to points in $\{\gamma_{i+1},\dots,\gamma_{lb-1}\}$). Also, as $\gamma_{lb}y_1^{(k*)}\in\{\gamma_i,\gamma_{lb}\}$ and any word at step $k*$ of length at most l^2 defined on γ_{lb} is an initial subword of w^l, $\ell(y_1)\geq l^2$, and y_1 has an initial subword w. It follows that after step k, w will be defined on $\alpha'v^{(k)}$. In particular, as eventually we will have $\alpha'vw\in\{\gamma_{i+1},\dots,\gamma_{lb}\}$ we have $\ell(v)>\ell(w)$, and we can write $v=v'w^q$ where $\ell(v')<\ell(w)$ and $q>0$. By Lemma 4.4 (ii), after step k $\alpha'v$ will lie on the w-cycle of γ_0. Assume first $\ell(v')>0$. Then as v' is a proper final subword of w (and the maximal $w^{(k*)}$-chain of γ_0 begins with a point on the w-cycle of γ_0), it follows that $\alpha'':=\alpha'f^{-1}$ is defined. Hence, after step $k*$ fz potentially takes $\langle\gamma_{i-1},\alpha''\rangle$ to $\langle\beta,\beta'\rangle$. Hence $\gamma_{i-1}\sim\alpha''\leftrightarrow\beta\sim\beta'$, so as f is an automorphism, $\gamma_i\sim\alpha'\leftrightarrow\beta\sim\beta'$, as required. If v' is empty, so $v=w^q$, then, by considering Lemma 4.4 (ii) applied to $\alpha'v,\alpha'vy_1,\alpha'vy_1x_2$, we get that y_1 is a positive power of w. This contradicts that

$$\gamma_{lb}y_1^{(k+)}\in\{\gamma_i,\gamma_{lb}\}.$$

Thus, we reduce to the case $r=1$, that is, $z=vy$. Again, by minimality of $\ell(z)$ we may suppose that after step k, each point on the $y^{(k)}$-chain from $\alpha'v^{(k)}$ to $\alpha'(vy)^{(k)}$ will lie in $\{\gamma_{i+1},\dots,\gamma_{lb-1}\}$, except for $\alpha'(vy)^{(k)}$ which is one of $\gamma_i,\dots,\gamma_{lb}$. In particular, $\ell(y)<\ell(v)$. Write $v=v'w^q$ where $\ell(v')<l$.

Suppose first $\ell(y)\geq\ell(w)$. Then as $\gamma_{lb}y^{(k*)}$ is defined, y has w as an initial segment. Hence, by Lemma 4.4 (ii), as $y^{(k)}$ will be defined on $\alpha'v^{(k)}$, $\alpha'v^{(k)}$ will be on the w-cycle of γ_0. In particular, y is a power of w, and $\beta'=\gamma_{lb}$. In this case, vy is a final segment of a power of w, and it potentially takes α to β and α' to β' along a subset of the w-chain from $\gamma_{-r'-1}$ to $\gamma_{lb+s'-1}$. We may suppose v' is non-empty, as otherwise z is a power of w and hence satisfies (4) by Definition 4.1 (i). In particular, as above $\alpha'':=\alpha'f^{-1}$ is defined at step $k*$, and fvy potentially takes $\langle\gamma_{i-1},\alpha''\rangle$ to $\langle\beta,\beta'\rangle$. Since (4) held at step $k*$ and f is a partial automorphism, (4) holds after step $k+$.

Alternatively, $\ell(y)<\ell(w)$. There are two cases, according to whether β' is γ_{lb} or γ_i. In the first case, we have to *choose* γ_i to ensure $\gamma_i\sim\alpha'\leftrightarrow\beta\sim\gamma_{lb}$, and in the second case we choose γ_i so that $\gamma_i\sim\alpha'\leftrightarrow\beta\sim\gamma_i$. It can be checked that this does not conflict with other constraints.

The above argument shows that the two cycles of $w^{(k-1)}$ can be joined, without creating new incompatible cycles for $w_0,\dots,w_{n-1}$. It remains to check that after Step $5n+2$, there are no coincidences. By the proof of Lemma 4.3, any such coincidence must involve one of the δ_i and one of the δ_i' and for some $e\leq n-1$ must involve a w_e-chain from δ_i to δ'_j along $\gamma_0, \ldots, \gamma_{lb}$ (we do not mean that the δ_i, δ'_i are *endpoints* of the chains in the coincidence). Furthermore, by Lemma 4.4 (i), $w_e = w$.

So suppose we have a $(w, w_{e'}, k, \alpha, \beta)$-coincidence. Since the 'next' letter of the w-chain from α to β is undefined, we must have $\beta = \alpha'_{s'}$. However, in this case, some chain of length at least $l^2 + l$ is both a w-chain and a $w_{e'}$-chain, so by Lemma 4.4 (i), $w = w_{e'}$, and the two chains between α and β are equal, contrary to the definition of coincidence.

Step $k = 5n + 4$. Put $l := \max\{l_0, \ldots, l_n\}$. Consider all pairs $\langle \epsilon, w \rangle$, where ϵ is an old point and w is a reduced word of length $l^2 + l$, whose first letter h does not have $\epsilon \in \mathrm{dom}(h^{(k-1)})$. For each such pair, use Lemma 4.3 to add a $w^{(k)}$-chain of length $l^2 + l$, so that $\epsilon w^{(k)}$ is defined. We do this by a good extension, in such a way that there are no overlaps between the added points for $\langle \epsilon, w \rangle$ and for any other $\langle \epsilon', w' \rangle$, except those forced because $\epsilon = \epsilon'$ and w, w' have a common initial subword. It follows from Lemma 4.4 that after this step there is no $(w_a, w_b, k, \alpha, \beta)$ coincidence for any $a, b \leq n$. $\square$

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