

## ON PERMUTATION GROUPS WITH CONSTANT MOVEMENT

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### Abstract

Let  $G$  be a permutation group on a set  $\Omega$  with no fixed point in  $\Omega$ . If for each subset  $\Gamma$  of  $\Omega$  the size  $|\Gamma^g - \Gamma|$  is bounded, for  $g \in G$ , we define the movement of  $g$  as the  $\max|\Gamma^g - \Gamma|$  over all subsets  $\Gamma$  of  $\Omega$ . In particular, if all non-identity elements of  $G$  have the same movement, then we say that  $G$  has constant movement. In this paper we will first give some families of groups with constant movement. We then classify all transitive permutation groups with a given constant movement  $m$  on a set of maximum size.

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### 1. Introduction

Let  $G$  be a permutation group on a set  $\Omega$  with no fixed points in  $\Omega$  and let  $m$  be a positive integer. If for each subset  $\Gamma$  of  $\Omega$  and each element  $g \in G$ , the size  $|\Gamma^g - \Gamma|$  is bounded, we define the *movement* of  $\Gamma$  as  $\text{move}(\Gamma) = \max_{g \in G} |\Gamma^g - \Gamma|$ . If  $\text{move}(\Gamma) \leq m$  for all  $\Gamma \subseteq \Omega$ , then  $G$  is said to have *bounded movement* and the *movement* of  $G$  is defined as the maximum of  $\text{move}(\Gamma)$  over all subsets  $\Gamma$ . This notion was introduced in [6]. Similarly, for each  $1 \neq g \in G$ , we define the movement of  $g$  as the  $\max|\Gamma^g - \Gamma|$  over all subsets  $\Gamma$  of  $\Omega$ . If all non-identity elements of  $G$  have the same movement, then we say that  $G$  has *constant movement*.

Clearly every permutation group with constant movement has bounded movement. By [6, Theorem 1], if  $G$  has bounded movement equal to  $m$ , then  $\Omega$  is finite, and its size is bounded by a function of  $m$ .

For transitive groups of movement  $m$ , the following bounds on  $\Omega$  were obtained in [6].

**LEMMA 1.1.** *Let  $G$  be a transitive permutation group on a set  $\Omega$  such that  $G$  has movement  $m$ .*

(a) *If  $G$  is a 2-group then  $|\Omega| \leq 2m$ .*

(b) *If  $G$  is not a 2-group and  $p$  is the least odd prime dividing  $|G|$ , Then  $|\Omega| \leq \lfloor 2mp/(p - 1) \rfloor$ . (For  $x \in R$ ,  $\lfloor x \rfloor$  denotes the integer part of  $x$ .)*

There are various types of permutation groups with constant movement for which the bounds in Lemma 1.1 may be attained. For example, let  $G$  be either a  $p$ -group of exponent  $p$  or a 2-group. If we consider  $G$  as a permutation group in its regular representation, then we see that all non-identity elements have the same movement.

The purpose of this paper is to classify all transitive permutation groups  $G$  of maximum degree  $n$  with constant movement  $m$ , (where  $n = 2m$  if  $G$  is a 2-group and otherwise  $n = \lfloor 2mp/(p - 1) \rfloor$  if  $p$  is the least odd prime dividing  $|G|$  and by [6] these are the maximum sizes of  $n$ ).

**THEOREM 1.2.** *Let  $m$  be a positive integer, and let  $G$  be a transitive permutation group on a set  $\Omega$  of maximum size  $n$  with constant movement  $m$ . Then either  $G$  is a 2-group in its regular representation, or for an odd prime  $p$  one of the following holds:*

(1)  $|\Omega| = p$ ,  $m = (p - 1)/2$  and  $G$  is the semi-directed product of  $Z_p Z_{2^a}$ , where  $2^a | (p - 1)$  for some  $a \geq 1$ ;

(2)  $G := A_4, A_5$ ,  $|\Omega| = 6$  and  $m = 2$ ;

(3)  $G$  is a  $p$ -group of exponent  $p$  in its regular representation.

*Moreover, all permutation groups listed above have constant movement.*

All the groups in Theorem 1.2 are examples (see Section 2). In Section 3, we prove the above theorem, which is a classification theorem for the transitive permutation groups of maximal degree with constant movement.

## 2. Attaining the bounds: examples

Let  $G$  be a transitive permutation group on a finite set  $\Omega$ . Then by [9, Theorem 3.26], which we shall refer to as Burnside’s Lemma, the average number of fixed points in  $\Omega$  of elements of  $G$  is equal to the number of  $G$ -orbits in  $\Omega$ , namely 1, and since  $1_G$  fixes  $|\Omega|$  points and  $|\Omega| > 1$ , it follows that there is some element of  $G$  which has no fixed points in  $\Omega$ . We shall say that such elements are fixed point free on  $\Omega$ .

Let  $1 \neq g \in G$  and suppose that  $g$  in its disjoint cycle representation has  $t$  nontrivial cycles of lengths  $l_1, \dots, l_t$ , say. We might represent  $g$  as

$$g = (a_1 a_2 \cdots a_{l_1})(b_1 b_2 \cdots b_{l_2}) \cdots (z_1 z_2 \cdots z_{l_t}).$$

Let  $\Gamma(g)$  denote a subset of  $\Omega$  consisting of  $\lfloor l_i/2 \rfloor$  points from  $i^{th}$  cycle, for each  $i$ , chosen in such way that  $\Gamma(g)^g \cap \Gamma(g) = \emptyset$ .

For example we could choose  $\Gamma(g) = \{a_2, a_4, \dots, b_2, b_4, \dots, z_2, z_4, \dots\}$ . Note that  $\Gamma(g)$  is not uniquely determined as it depends on the way each cycle is written down. For any set  $\Gamma(g)$  of this kind, we say that  $\Gamma(g)$  consists of every second point of every cycle of  $g$ . From the definition of  $\Gamma(g)$  we see that

$$|\Gamma(g)^g - \Gamma(g)| = |\Gamma(g)| = \sum_{i=1}^t \lfloor l_i/2 \rfloor.$$

In [3] we have shown that this quantity is an upper bound for  $|\Gamma^g - \Gamma|$  for an arbitrary subset  $\Gamma$ . Thus the movement of  $g$  is  $|\Gamma(g)|$ .

Now we will show that there certainly are some families of examples of transitive groups with constant movement for which the bound of Lemma 1.1 holds, for any prime  $p$ . First we look at groups of exponent  $p$ .

**LEMMA 2.1.** (a) *Let  $m := p^{a-1}(p - 1)/2$  for some  $a \geq 1$ , where  $p$  is an odd prime and suppose that  $G$  is a regular permutation group of exponent  $p$  on a set  $\Omega$  of size  $p^a = 2mp/(p - 1)$ . Then  $G$  has constant movement  $m$ .*

(b) *Let  $m$  be a power of 2, and suppose that  $G$  is a 2-group of order  $2m$ . Then the regular representation of  $G$  is a permutation group of constant movement  $m$ .*

**PROOF.** Let  $1 \neq g \in G$  and let  $\Gamma \subseteq \Omega$ . By [3, Lemma 2.1],  $|\Gamma^g - \Gamma| \leq m$ . Since  $G$  is regular,  $g$  is fixed point free on  $\Omega$ . Suppose that  $\Gamma(g)$  consists of every second point of every cycle of  $g$ . Then by definition  $\Gamma(g)^g \cap \Gamma(g) = \emptyset$ . If  $p$  is an odd prime, then  $|\Gamma(g)^g - \Gamma(g)| = |\Gamma(g)| = (|\Omega|/p)(p - 1)/2 = p^{a-1}(p - 1)/2 = m$ .

Thus  $G$  has constant movement  $m$ . Also with the same argument it can be shown that every 2-group of degree  $2m$  in its regular representation has constant movement  $m$ .  $\square$

In what follows we will see that the regularity condition for each transitive  $p$ -group is a necessary condition. Let  $H$  be a core-free subgroup of a  $p$ -group  $G$  and consider the permutation representation by right multiplication on the right cosets of  $H$ . If  $H \neq 1$ , then  $G$  is not regular in this action and does not have constant movement. An example of such a core-free subgroup  $H$  in a  $p$ -group  $G$  of exponent  $p$  is the cyclic group generated by any non-central element. Such elements exist provided that  $G$  is non-abelian.

Let  $H = \langle h \rangle \cong Z_n$ , and let  $K = \langle k \rangle \cong Z_m$  be such that  $K$  is a subgroup of  $\text{Aut}(H)$ . Then  $h^k = h^r$  for some positive integer  $r$  such that  $r^m \equiv 1 \pmod{n}$ . Let  $G = HK$  be the natural semi-direct product of  $H$  by  $K$ . Then  $G$  is given by the defining relations:  $h^n = 1, k^m = 1, k^{-1}hk = h^r$ , with  $r^m \equiv 1 \pmod{n}$ .

Here every element of  $G$  is uniquely expressible as  $h^i k^j$ , where  $0 \leq i \leq n - 1, 0 \leq j \leq m - 1$ . Certain semi-direct products of this type also provide examples of

groups with constant movement where the bound in Lemma 1.1 holds. (We note that, if  $n = p$ , a prime, then this group  $G$  is a subgroup of the group  $AGL(1, p) = Z_p Z_{p-1}$ .)

**LEMMA 2.2.** *Let  $G := Z_p Z_{2^a}$  denote a group defined as above of order  $p \cdot 2^a$ , where  $2^a | (p - 1)$  for some  $a \geq 1$ . Then  $G$  acts transitively on a set  $\Omega$  of size  $p$  and in this action  $G$  has constant movement  $(p - 1)/2$ .*

**PROOF.** The group  $G$  is a Frobenius group and has up to permutational isomorphism a unique transitive representation of degree  $p$ , on a set  $\Omega$ , say. Let  $g \in G$  be such that  $o(g) = p$ . Then by [3, Lemma 2.1],  $|\Gamma^g - \Gamma| \leq m = (p - 1)/2$  for all subsets  $\Gamma$ , and if  $\Gamma(g)$  consists of every second point of the unique cycle of  $g$ , then  $|\Gamma(g)^g - \Gamma(g)|$  has size equal to  $m$ . Suppose now that  $g \in G$  has order  $o(g)$  a power of 2. Then  $g$  has one fixed point and  $(p - 1)/o(g)$  cycles of length  $o(g)$  in  $\Omega$ . For each  $\Gamma \subseteq \Omega$ ,  $|\Gamma^g - \Gamma|$  consists of at most  $o(g)/2$  points from each cycle of  $g$  of length  $o(g)$ , and hence has size at most  $m$ . Since each non-identity element of  $G$  is either a 2-element or has order  $p$ , it follows that  $G$  has constant movement equal to  $m$ . □

**LEMMA 2.3.** *The groups  $A_4$ , and  $A_5$  acting transitively on a set of size 6 have constant movement equal to 2.*

**PROOF.** By [2, 4, 5] the groups  $A_4$  and  $A_5$  have bounded movement equal to 2. Using similar argument as in [3, Lemma 3.3], we will show that they also have constant movement 2. Let  $1 \neq g \in A_4$ . Then  $g$  has order 2 or 3. If  $g$  has order 2 then  $g$  has two cycles of length 2 and hence  $|\Gamma(g)^g - \Gamma(g)| = 2$ . Similarly, if  $g$  has order 3 then  $g$  has two cycles of length 3 and again  $|\Gamma(g)^g - \Gamma(g)| = 2$ . As for  $A_5$ , since every non-identity element of  $A_5$  has order 2, 3 or 5, as above it is easy to see that every element of  $A_5$  has movement equal to 2. Hence both of them have constant movement 2. □

### 3. Proof of Theorem 1.2

Let  $m$  be a positive integer. Suppose that  $G$  is a transitive permutation group on a set  $\Omega$  of size  $n$  with constant movement  $m$ , which have maximal degree. (Where  $n = 2m$  if  $G$  is a 2-group and otherwise  $n = \lfloor 2mp/(p - 1) \rfloor$  where  $p$  is the least odd prime dividing  $|G|$ .) By [1, Theorem 1], for some prime  $q$  dividing  $|G|$ , there exists a  $q$ -element  $g$  of order  $q^a$  (for some positive integer  $a$ ) in  $G$  which is fixed point free on  $\Omega$ . Then  $g$  has  $b_i$  cycles of length  $q^i$  for  $i = 1, \dots, a$ , where  $\sum_{i=1}^a b_i q^i = n$  and  $b_a > 0$ . Now we consider two cases:

Case 1: Suppose that  $q$  is odd. Then by the definition of  $\Gamma(g)$  we have,

$$m = |\Gamma(g)| = \sum_{i=1}^a b_i \frac{q^i - 1}{2}.$$

Suppose that  $a \geq 2$ , and consider  $h = g^{q^{a-1}}$ , say. Then  $h$  has  $b_a q^{a-1}$  cycles of length  $q$ , so by the definition of every second point of every cycles of  $h$  we have,

$$\begin{aligned} m = |\Gamma(h)| &= b_a q^{a-1} \frac{q-1}{2} = \sum_{i=1}^{a-1} b_i \frac{q^i - 1}{2} + b_a \frac{q^a - 1}{2} \\ &\geq b_a \frac{q^a - 1}{2} = b_a \frac{q-1}{2} (q^{a-1} + \dots + q + 1) \\ &= b_a q^{a-1} \frac{q-1}{2} + b_a \frac{q-1}{2} (q^{a-2} + \dots + q + 1) \\ &= b_a q^{a-1} \frac{q-1}{2} + b_a \frac{q^{a-1} - 1}{2} > b_a q^{a-1} \frac{q-1}{2}, \end{aligned}$$

which is a contradiction. Hence  $a = 1$  and therefore  $b_a = b_1 = n/q$ , and  $m = (n/q)(q-1)/2$ . Suppose there exists an odd prime  $r$  dividing  $|G|$  such that  $r \leq q$ , and let  $x \in G$ ,  $o(x) = r$ . Then

$$m = |\Gamma(x)| \leq \frac{n}{r} \frac{r-1}{2} = \frac{2mq}{q-1} \frac{r-1}{2r}.$$

So  $(q-1)r \leq q(r-1)$  and hence  $q \leq r$  which is a contradiction. Hence  $q$  is the least odd prime dividing  $|G|$ , that is, we have proved that  $q = p$ .

Case 2: Now we suppose that  $q = 2$ , so as above we can assume that  $o(g) = 2^a$  for some positive integer  $a$ , and  $g$  is a fixed point free element on  $\Omega$ . Then  $g$  has  $b_i$  cycles of length  $2^i$  for  $i = 1, \dots, a$ , where  $n = \sum_{i \leq a} b_i 2^i$ ,  $b_a > 0$ , and

$$m = |\Gamma(g)| = \sum_{i=1}^a b_i 2^{i-1}.$$

Suppose that  $a \geq 2$ , and consider  $g^{2^{a-1}} = h$ , say. Then  $h$  has  $b_a 2^{a-1}$  cycles of length 2, so

$$b_a 2^{a-1} = |\Gamma(h)| = m = \sum_{i=1}^{a-1} b_i 2^{i-1} + b_a 2^{a-1}.$$

The above equality is true if  $b_i = 0$  for each  $i < a$ . So all  $g$ -cycles have length  $2^a$ , and hence  $2^a |n$ .

We first suppose that  $G$  is a transitive permutation group on a set of size  $n = 2m$  and  $G$  is a 2-group. As each  $1 \neq g \in G$  has constant movement  $m$ ,  $|\text{supp}(g)| = 2m$ , where  $\text{supp}(g) = \{\alpha \in \Omega \mid \alpha^g \neq \alpha\}$ . Thus  $g$  is a fixed point free element on  $\Omega$ , that is,  $G_\alpha = 1$  for each  $\alpha \in \Omega$ . Hence  $G$  is a regular 2-group.

Now suppose that  $p$  is an odd prime. Then  $G$  is not a 2-group. Since  $G$  is a transitive permutation group with maximal degree, by [7, Theorem 6.4]

$$|\Omega| = \left\lfloor \frac{2mp}{p-1} \right\rfloor = \frac{2mp}{p-1},$$

where  $p$  is the least odd prime dividing  $|G|$ . (Since  $2m < 2mp/(p-1)$ , so if  $G$  is not a 2-group with maximal degree then  $|\Omega| \neq 2m$ .) Then by [2, 3, 4, 5], one of the following holds:

- (1)  $|\Omega| = p, m = (p-1)/2$  and  $G$  is the semi-direct product  $Z_p Z_{2^a}$  where  $2^a \mid (p-1)$  for some  $a \geq 1$ .
- (2)  $G$  is the semi-direct product  $KP$  with  $K$  a 2-group and  $P = Z_p$  is fixed point free on  $\Omega$ ;  $|\Omega| = 2^s p, m = 2^{s-1}(p-1)$ , and  $2^s < p$ , where  $K$  has  $p$ -orbits of length  $2^s$ , and each element of  $K$  moves at most  $2^s(p-1)$  points of  $\Omega$ . (We note that  $A_4 \cong (Z_2)^2 Z_3$  is a transitive permutation group of degree 6 which has constant movement 2, this occur in this case where  $p = 3$  and  $m = 2$ .)
- (3)  $G$  is a  $p$ -group.
- (4)  $p = 3, m = 2$ , and  $G = A_5$ .

All groups in part (1) are examples for Theorem 1.2. In parts (2) and (4), except for the groups  $A_4$  and  $A_5$  acting on a set of size 6, the other groups have some elements whose movements are less than  $m$ , which contradicts the fact that  $G$  has constant movement, (since  $G = KP$  has constant movement  $m$ , each non-identity element  $k \in K$  has  $(p-1)$  cycles of length  $2^s$ . We consider the element  $kk^g$  of  $K$ . This element is fixed point free on  $\Omega$  and so has movement  $p2^{s-1}$ , which is a contradiction). In part (3), by Burnside’s lemma,  $G$  has a fixed point free element, say  $g$ , on a set of size  $p^a$  for some positive integer  $a$ . Since every fixed point free element has order  $p$  with movement  $p^a(p-1)/2$  (see [3, Proposition 4]),  $o(g) = p$  and hence  $\text{move}(g) = p^{a-1}(p-1)/2$ . But, by our assumption,  $G$  has constant movement  $m$  and so  $m = p^{a-1}(p-1)/2$ . Therefore, each non-identity element  $g$  of  $G$  is a fixed point free element, so that  $G$  is a regular  $p$ -group of exponent  $p$ . This completes the proof of Theorem 1.2. □

### 4. Intransitive examples

In this section we show that there certainly are families of examples of intransitive permutation groups with constant movement, for any prime  $p$ . First for  $p = 2$ , we

have the following example.

**EXAMPLE 4.1.** Let  $m = 2^{r-1} \geq 1$  and let  $G := Z_2^r$ . Then  $G$  has  $2^r - 1 = 2m - 1$  subgroups of index 2, say  $H_1, \dots, H_{2m-1}$ . For  $i = 1, \dots, 2m - 1$ , let  $\Omega_i$  denote the set of two cosets of  $H_i$  in  $G$ , and set

$$\Omega := \bigcup_{i=1}^{2m-1} \Omega_i.$$

Then  $G$  acts faithfully on  $\Omega$  by right multiplication with  $2m - 1$  orbits  $\Omega_1, \dots, \Omega_{2m-1}$ , each of length 2. Each nontrivial element  $g \in G$  lies in exactly  $2^{r-1} - 1 = m - 1$  of the subgroups  $H_i$  and permutes nontrivially the remaining  $m = 2^{r-1}$  points of  $\Omega_i$ . Thus each nontrivial element of  $G$  has  $m = 2^{r-1}$  cycles of length 2 in  $\Omega$ . For any subset  $\Gamma \subseteq \Omega$  and any  $1 \neq g \in G$ , the set  $(\Gamma^g - \Gamma)$  consists of at most 1 point from each of the  $G$ -orbits on which  $g$  acts nontrivially, and hence  $\max|\Gamma^g - \Gamma| = m$ . It follows that  $G$  has constant movement  $m$ .

The following example shows that intransitive  $p$ -groups,  $p$  odd, with constant movement do exist.

**EXAMPLE 4.2.** Let  $d$  be a positive integer, let  $G := Z_p^d$ , let  $t := (p^d - 1)/(p - 1)$ , and let  $H_1, \dots, H_t$  be an enumeration of the subgroups of index  $p$  in  $G$ . Define  $\Omega_i$  to be the coset space of  $H_i$  in  $G$  and  $\Omega = \Omega_1 \cup \dots \cup \Omega_t$ . If  $g \in G - \{1\}$ , then  $g$  lies in  $(p^{d-1} - 1)/(p - 1)$  of the groups  $H_i$  and therefore acts on  $\Omega$  as a permutation with  $p(p^{d-1} - 1)/(p - 1)$  fixed points and  $p^{d-1}$  orbits of length  $p$ . Taking every second point from each of these  $p$ -cycles to form a set  $\Gamma$  we see that  $\text{move}(g) = m \geq p^{d-1}(p - 1)/2$ , and it is not hard to prove that in fact  $\text{move}(g) = m = p^{d-1}(p - 1)/2$ . Since  $g$  is non-trivial,  $G$  has constant movement  $p^{d-1}(p - 1)/2$ .

The last example for  $p = 3$ , inclined to the following example not only are examples of permutation groups with constant movement equal to  $3^{d-1}$  and 2 respectively, but also gives some positive answer to the Question 1.5 in [8].

**EXAMPLE 4.3.** Let  $\Omega = \Omega_1 \cup \Omega_2$  be a set of size 7, such that  $\Omega_1 = \{1, 2, 3\}$  and  $\Omega_2 = \{1', 2', 3', 4'\}$ . Moreover, suppose that  $Z_2^2 \cong \langle (1'2')(3'4'), (1'3')(2'4') \rangle$  and  $Z_3 \cong \langle (123)(1'2'3') \rangle$

Then the semi-direct product  $G := Z_2^2 Z_3$  with normal subgroup  $Z_2^2$  is a permutation group on a set  $\Omega$  with 2-orbits which has constant movement 2, since each non-identity element of  $G$  has two cycle of length 2 or two cycle of length 3.

Finally, one may ask whether there exist further examples of intransitive groups, which have constant movement.

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