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EMBEDDINGS OF ℓ_p INTO NON-COMMUTATIVE SPACES

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Abstract

Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with a faithful normal trace τ. We prove a Kadec-Pełczyński type dichotomy principle for subspaces of symmetric space of measurable operators of Rademacher type 2. We study subspace structures of non-commutative Lorentz spaces $L_{p,q}(\mathcal{M}, \tau)$, extending some results of Carothers and Dilworth to the non-commutative settings. In particular, we show that, under natural conditions on indices, ℓ_p cannot be embedded into $L_{p,q}(\mathcal{M}, \tau)$. As applications, we prove that for $0 < p < \infty$ with $p \neq 2$, ℓ_p cannot be strongly embedded into $L_p(\mathcal{M}, \tau)$. This provides a non-commutative extension of a result of Kalton for $0 < p < 1$ and a result of Rosenthal for $1 \leq p < 2$ on $L_p[0, 1]$.

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1. Introduction

The study of rearrangement invariant Banach spaces of measurable functions is a classical theme. Several studies have been devoted to characterizations of subspaces of rearrangement invariant spaces. Recently, the theory of rearrangement invariant Banach spaces of measurable operators affiliated with semi-finite von Neumann algebra have emerged as the natural non-commutative generalizations of Köthe functions spaces. This theory, which is based on the theory of non-commutative integration introduced by Segal [24], replaces the classical duality $(L_\infty(\mu), L_1(\mu))$ by the duality between a semi-finite von Neumann algebra and its predual. It provides a unified approach to the study of unitary ideals and rearrangement invariant spaces. Several authors have considered these non-commutative spaces of measurable operators (see for instance, [4, 6, 7, 8, 28]).

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The purpose of the present paper is to examine subspaces of symmetric spaces of measurable operators in which the norm topology and the measure topology coincide, and subspaces generated by disjointly supported basic sequences. Such subspaces are of particular importance as they represent in many cases the extreme structures. Our main method is to exploit the notion of uniform integrability of operators introduced in [21]. One of main results of this paper is a dichotomy type result for subspaces of symmetric spaces of measurable operators. More precisely, we prove that any given subspace of a symmetric space of measurable operators either is isomorphic to a Hilbert space or contains a basic sequence equivalent to a disjointly supported sequence.

The classical spaces $L_p(\mu)$ are of central importance and results in their structures go back to the work of Banach. Since their introduction by Lorentz in 1950, the Lorentz function spaces $L_{p,q}$ have been found to be of special interests in many aspects of analysis and probability theory. In [2] and [3], Carothers and Dilworth studied the spaces $L_{p,q}[0,1]$ and $L_{p,q}[0,\infty)$. They proved, among other things, that for some appropriate values of the indices p and q, $L_{p,q}[0,\infty)$ does not contain ℓ_p. Precisely, for $0 < p, q < \infty$, $p \neq q$ and $p \neq 2$, the sequence space ℓ_p does not embed into $L_{p,q}[0,\infty)$. Such result, not only is of interest in its own right, but also provides an alternative proof to some non-trivial results on L^p-spaces.

Motivated by such connections, we examine the subspace structures of non commutative Lorentz spaces $L_{p,q}(\mathcal{M},\tau)$, where $(\mathcal{M},\tau)$ is a semi-finite von Neumann algebra. Making use of our dichotomy result and some other results of general nature, we show that some of the results of [2] and [3] extend to the non-commutative settings. Our approach relies on a disjointification techniques based on the non-commutative Khintchine's inequalities ([17, 18]). As noted above, the initial basic question, that led to the consideration of these Lorentz spaces, is the question of embeddings of ℓ_p into $L_p(\mathcal{M},\tau)$. Clearly, any disjointly supported basic sequence in $L_p(\mathcal{M},\tau)$ is isomorphic to ℓ_p. For the commutative case, Rosenthal proved in [23] that if (Ω,Σ,μ) is a σ-finite measure space, $1 \leq p < 2$, and X is a subspace of $L_p(\Omega,\Sigma,\mu)$ containing ℓ_p, then the norm topology and the measure topology do not coincide on X. For $0 < p < 1$, the same result is implicit in a paper of Kalton [14]. This implies that for $0 < p < 2$, any basic sequence in $L_p(\Omega,\Sigma,\mu)$ that is equivalent to ℓ_p is essentially a perturbation of a disjointly supported basic sequence. We establish, as applications of our results on Lorentz spaces, that Kalton and Rosenthal's results extend to $L_p(\mathcal{M},\tau)$.

The paper is organized as follows. In Section 2 below, we gather some necessary definitions and present some basic facts concerning symmetric spaces of measurable operators that will be needed throughout. In Section 3, we present the Kadec-Pełczyński type dichotomy for subspaces of symmetric spaces of measurable operators. The final section is devoted entirely to the study of subspaces of Lorentz spaces and its applications.

2. Definitions and preliminaries

We begin by recalling some definitions and facts about function spaces. Let E be a complex quasi-Banach lattice. If $0 < p < \infty$, then E is said to be *p-convex* (respectively *p-concave*) if there exists a constant $C > 0$ such that for every finite sequence $\{x_n\}$ in E,

$$\left\| \left(\sum |x_n|^p \right)^{1/p} \right\|_E \leq C \left(\sum \|x_n\|^p \right)^{1/p}$$
$$\left(\text{respectively } \left\| \left(\sum |x_n|^p \right)^{1/p} \right\|_E \geq C^{-1} \left(\sum \|x_n\|^p \right)^{1/p} \right).$$

The least constant C is called the *p-convexity* (respectively *p-concavity*) *constant* of E and is denoted by $M^{(p)}(E)$ (respectively $M_{(p)}(E)$).

For $0 < p < \infty$, $E^{(p)}$ will denote the quasi-Banach lattice defined by

$$E^{(p)} := \{x : |x|^p \in E\}$$

equipped with $\|x\|_{E^{(p)}} = \||x|^p\|_E^{1/p}$. It is easy to verify that if E is α-convex and q-concave then $E^{(p)}$ is αp-convex and qp-concave with $M^{(\alpha p)}(E^{(p)}) \leq M^{(\alpha)}(E)^{1/p}$ and $M_{(qp)}(E^{(p)}) \leq M_{(q)}(E)^{1/p}$. Consequently, if E is α-convex then $E^{(1/\alpha)}$ is 1-convex and therefore can be equivalently renormed to be a Banach lattice [16].

The quasi-Banach lattice E is said to satisfy *a lower q-estimate* (respectively *upper p-estimate*) if there exists a positive constant $C > 0$ such that for all finite sequences of mutually disjoint elements of E

$$\left(\sum \|x_n\|_E^q \right)^{1/q} \leq C \left\| \sum x_n \right\|_E$$
$$\left(\text{respectively } \left(\sum \|x_n\|_E^p \right)^{1/p} \geq C^{-1} \left\| \sum x_n \right\|_E \right).$$

We denote by $\mathcal{M}$ a semi-finite von Neumann algebra on a Hilbert space $\mathcal{H}$, with a fixed faithful and normal semi-finite trace τ. The identity in $\mathcal{M}$ is denoted by $\mathbf{1}$, and we denote by $\mathcal{M}^p$ the set of all projections in $\mathcal{M}$. A linear operator $x : \mathrm{dom}(x) \to \mathcal{H}$, with domain $\mathrm{dom}(x) \subseteq \mathcal{H}$, is called *affiliated with $\mathcal{M}$* if $ux = xu$ for all unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. The closed and densely defined operator x, affiliated with $\mathcal{M}$, is called *τ-measurable* if for every $\epsilon > 0$ there exists $p \in \mathcal{M}^p$ such the $p(\mathcal{H}) \subseteq \mathrm{dom}(x)$ and $\tau(\mathbf{1} - p) < \epsilon$. With the sum and product defined as the respective closures of the algebraic sum and product, $\widetilde{\mathcal{M}}$ is a $*$-algebra. For standard facts concerning von Neumann algebras, we refer to [13, 26].

We recall the notion of generalized singular value function [10]. Given a self-adjoint operator x in $\mathcal{H}$ we denote by $e^x(\cdot)$ the spectral measure of x. Now assume that $x \in \widetilde{\mathcal{M}}$. Then $\chi_B(|x|) \in \mathcal{M}$ for all Borel sets $B \subseteq \mathbb{R}$, and there exists $s > 0$ such that $\tau(\chi_{(s,\infty)}(|x|)) < \infty$. For $x \in \widetilde{\mathcal{M}}$ and $t \geq 0$, we define

$$\mu_t(x) = \inf\big\{s \geq 0 : \tau(\chi_{(s,\infty)}(|x|)) \leq t\big\}.$$

The function $\mu(x) : [0, \infty) \to [0, \infty]$ is called the *generalized singular value function* (or decreasing rearrangement) of x; note that $\mu_t(x) < \infty$ for all $t > 0$. Suppose that $a > 0$. If we consider $\mathcal{M} = L_\infty([0, a), m)$, where m denotes Lebesgue measure on the interval $[0, a)$, as an abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L_2([0, a), m)$, with the trace given by integration with respect to m, it is easy to see that $\widetilde{\mathcal{M}}$ consists of all measurable functions on $[0, a)$ which are bounded except on a set of finite measure. Further, if $f \in \widetilde{\mathcal{M}}$, then the generalized singular value function $\mu(f)$ is precisely the classical non-increasing rearrangement of the function $|f|$. On the other hand, if $(\mathcal{M}, \tau)$ is the space of all bounded linear operators in some Hilbert space equipped with the canonical trace tr, then $\widetilde{\mathcal{M}} = \mathcal{M}$ and, if $x \in \mathcal{M}$ is compact, then the generalized singular value function $\mu(x)$ may be identified in a natural manner with the sequence $\{\mu_n(x)\}_{n=0}^{\infty}$ of singular values of $|x| = \sqrt{x^*x}$, repeated according to multiplicity and arranged in non-increasing order. By $L_0([0, a), m)$, we denote the space of all $\mathbb{C}$-valued Lebesgue measurable functions on the interval $[0, a)$ (with identification m-a.e.). A quasi-Banach space $(E, \|\cdot\|_E)$, where $E \subseteq L_0([0, a), m)$ is called a *rearrangement-invariant Banach function space* on the interval $[0, a)$, if it follows from $f \in E$, $g \in L_0([0, a), m)$ and $\mu(g) \leq \mu(f)$ that $g \in E$ and $\|g\|_E \leq \|f\|_E$. If $(E, \|\cdot\|_E)$ is a rearrangement-invariant quasi-Banach function space on $[0, a)$, then E is said to be *symmetric* if $f, g \in E$ and $g \prec\prec f$ imply that $\|g\|_E \leq \|f\|_E$. Here $g \prec\prec f$ denotes submajorization in the sense of Hardy-Littlewood-Polya

$$\int_0^t \mu_s(g)\, ds \leq \int_0^t \mu_s(f)\, ds, \quad \text{for all } t > 0.$$

The general theory of rearrangement-invariant spaces may be found in [1] and [16].

Given a semi-finite von Neumann algebra $(\mathcal{M}, \tau)$ and a symmetric quasi-Banach function space $(E, \|\cdot\|_E)$ on the measure space $([0, \tau(\mathbf{1})), m)$, we define the noncommutative space $E(\mathcal{M}, \tau)$ by setting

$$E(\mathcal{M}, \tau) := \{x \in \widetilde{\mathcal{M}} : \mu(x) \in E\} \quad \text{with}$$
$$\|x\|_{E(\mathcal{M},\tau)} := \|\mu(x)\|_E \quad \text{for } x \in E(\mathcal{M}, \tau).$$

It is known that if E is α-convex for some $0 < \alpha < \infty$ with $M^{(\alpha)}(E) = 1$, then $\|\cdot\|_{E(\mathcal{M},\tau)}$ is a norm for $\alpha \geq 1$ and an α-norm if $0 < \alpha < 1$. In this case, the space $(E(\mathcal{M}, \tau), \|\cdot\|_{E(\mathcal{M},\tau)})$ is a α-Banach space. Moreover, the inclusions

$$L_\alpha(\mathcal{M}, \tau) \cap \mathcal{M} \subseteq E(\mathcal{M}, \tau) \subseteq L_\alpha(\mathcal{M}, \tau) + \mathcal{M}.$$

hold with continuous embeddings. We remark that if $0 < p < \infty$ and $E = L_p[0, \tau(\mathbf{1}))$ then $E(\mathcal{M}, \tau)$ coincides with the definition of $L_p(\mathcal{M}, \tau)$ as in [19, 26]. In particular, if $\mathcal{M} = \mathcal{L}(\mathcal{H})$ with the standard trace then these L_p-spaces are precisely the *Schatten classes* $\mathcal{C}_p$.

We recall that the topology defined by the metric on $\widetilde{\mathcal{M}}$ obtained by setting

$$d(x, y) = \inf\{t \geq 0 : \mu_t(x - y) \leq t\}, \quad \text{for } x, y \in \widetilde{\mathcal{M}},$$

is called the *measure topology*. It is well known that a net $(x_\alpha)_{\alpha\in I}$ in $\widetilde{\mathcal{M}}$ converge to $x \in \widetilde{\mathcal{M}}$ in measure topology if and only if for every $\epsilon > 0$, $\delta > 0$, there exists $\alpha_0 \in I$ such that whenever $\alpha \geq \alpha_0$, there exists a projection $p \in \mathcal{M}^p$ such that

$$\|(x_\alpha - x)p\|_{\mathcal{M}} < \epsilon \quad \text{and} \quad \tau(\mathbf{1} - p) < \delta.$$

It was shown in [19] that $(\widetilde{\mathcal{M}}, d)$ is a complete metric, Hausdorff, topological $*$-algebra.

For $x \in \widetilde{\mathcal{M}}$, the right and left support projections of x are denoted by $r(x)$ and $l(x)$ respectively. Operators $x, y \in \widetilde{\mathcal{M}}$ are said to be right (respectively, left) disjointly supported if $r(x)r(y) = 0$ (respectively, $l(x)l(y) = 0$).

The following definition isolates the topic of this paper.

DEFINITION 2.1. Let E be a symmetric quasi-Banach function space on $[0, \tau(\mathbf{1}))$. We say that a subspace X of $E(\mathcal{M}, \tau)$ is *strongly embedded* into $E(\mathcal{M}, \tau)$ if the $\|\cdot\|_{E(\mathcal{M},\tau)}$-topology and the measure topology on X coincide.

The next definition was introduced in [21] as an analogue of the uniform integrability of families of functions.

DEFINITION 2.2. Let E be a symmetric quasi-Banach function space on $[0, \tau(\mathbf{1}))$. A bounded subset K of $E(\mathcal{M}, \tau)$ is said to be E*-uniformly-integrable* if

$$\lim_{n\to\infty} \sup_{x\in K} \|e_n x e_n\|_{E(\mathcal{M},\tau)} = 0$$

for every decreasing sequence $\{e_n\}_{n=1}^{\infty}$ of projections with $e_n \downarrow_n 0$.

A non-commutative extension of the Kadec-Pełczyński subsequence splitting lemma relative to the above notion of uniform integrability was considered in [21] (see [21, Theorem 3.1, Theorem 3.9, Corollary 3.10]) and will be used repeatedly throughout this paper. For convenience of the reader, we include the version that we need.

THEOREM 2.3 ([21]). *Let E be an order continuous symmetric quasi-Banach function space in* $[0, \tau(\mathbf{1}))$. *Assume that E is a Banach function space with the Fatou* *property or E is α-convex with constant 1 for some $0 < \alpha < 1$ which satisfies a lower q-estimate with constant 1 for some $q \geq \alpha$.*

Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence in $E(\mathcal{M}, \tau)$. Then there exist a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$, bounded sequences $\{\varphi_k\}_{k=1}^\infty$ and $\{\zeta_k\}_{k=1}^\infty$ in $E(\mathcal{M}, \tau)$ and mutually disjoint sequence of projections $\{e_k\}_{k=1}^\infty$ such that

(i) $x_{n_k} = \varphi_k + \zeta_k$ *for all* $k \geq 1$;
(ii) $\{\varphi_k : k \geq 1\}$ *is E-uniformly-integrable and* $e_k \varphi_k e_k = 0$ *for all* $k \geq 1$;
(iii) $\{\zeta_k\}_{k=1}^\infty$ *is such that* $e_k \zeta_k e_k = \zeta_k$ *for all* $k \geq 1$.

The following proposition is due to Sukochev [25] in the case where $\tau(\mathbf{1}) < \infty$ and will be used in the sequel.

PROPOSITION 2.4. *Let E be α-convex with constant 1 and assume that E is order continuous. Let $\{x_n\}_{n=1}^\infty$ be a basic sequence in $E(\mathcal{M}, \tau)$ such that $\{x_n\}_{n=1}^\infty$ is both right and left disjointly supported. Then $\{x_n\}_{n=1}^\infty$ is equivalent to a disjointly supported basic sequence in E.*

PROOF. For each $n \geq 1$, let $q_n := l(x_n)$ and $p_n := r(x_n)$ be the left and right support projection of x_n respectively. Both sequences $\{q_n\}_{n=1}^\infty$ and $\{p_n\}_{n=1}^\infty$ are mutually disjoint and for every $n \geq 1$, $x_n = q_n x_n p_n$. For any finite sequence of scalars $\{a_i\}_{i=1}^n$,

$$\left|\sum_{i=1}^n a_i x_i\right|^2 = \left(\sum_{i=1}^n \bar{a}_i p_i x_i^* q_i\right)\left(\sum_{i=1}^n a_i q_i x_i p_i\right)$$
$$= \sum_{i=1}^n |a_i|^2 p_i x_i^* q_i x_i p_i = \left|\sum_{i=1}^n a_i |x_i|\right|^2 .$$

Note that $\{|x_i|\}_{i=1}^\infty$ is disjointly supported by the projections $\{p_i\}_{i=1}^\infty$. For each $i \geq 1$, the semi-finiteness of p_i implies that the family $\{e_\beta\}_\beta$ of all projections in $p_i \mathcal{M} p_i$ of finite trace satisfies $0 \leq e_\beta \uparrow_\beta p_i$. Since E is order-continuous, it follows that

$$\left\| e_\beta |x_i| e_\beta - |x_i| \right\| \to \beta_0.$$

For each $i \geq 1$, choose a projection $\tilde{p}_i \leq p_i$ such that $\tau(\tilde{p}_i) < \infty$ and

$$\| \tilde{p}_i |x_i| \tilde{p}_i - |x_i| \|^\alpha \leq 2^{-i}.$$

CLAIM. *The sequence $\{\tilde{p}_i |x_i| \tilde{p}_i\}_{i=1}^\infty$ is equivalent to $\{|x_i|\}_{i=1}^\infty$.*

Let $p = \bigvee_{i=1}^\infty \tilde{p}_i$. For any $x = \sum_{i=1}^\infty a_i |x_i| \in \overline{\operatorname{span}}\{|x_i|, i \geq 1\}$, we have $\sum_{i=1}^\infty a_i \tilde{p}_i |x_i| \tilde{p}_i = p\big(\sum_{i=1}^\infty a_i |x_i|\big) p$ so the series $\sum_{i=1}^\infty a_i \tilde{p}_i |x_i| \tilde{p}_i$ is convergent whenever $\sum_{i=1}^\infty a_i |x_i|$ converges. Conversely, if $\{a_n\}_{n=1}^\infty$ is a bounded sequence of scalars such that $\sum_{i=1}^{\infty} a_i \tilde{p}_i |x_i| \tilde{p}_i$ is convergent, then for any subset S of $\mathbb{N}$,

$$\Big\| \sum_{i \in S} a_i |x_i| \Big\|_{E(\mathcal{M},\tau)}^{\alpha} \le \Big\| \sum_{i \in S} a_i \tilde{p}_i |x_i| \tilde{p}_i \Big\|_{E(\mathcal{M},\tau)}^{\alpha} + \Big\| \sum_{i \in S} a_i (\tilde{p}_i |x_i| \tilde{p}_i - |x_i|) \Big\|_{E(\mathcal{M},\tau)}^{\alpha}$$
$$\le \sup_{i \in S} |a_i|^{\alpha} \cdot \sum_{i \in S} 2^{-i} + \Big\| \sum_{i \in S} a_i \tilde{p}_i |x_i| \tilde{p}_i \Big\|_{E(\mathcal{M},\tau)}^{\alpha}.$$

This shows that the series $\sum_{i=1}^{\infty} a_i |x_i|$ is convergent. Let C_1 and C_2 be positive constants so that for any finite sequence of scalars $\{a_i\}_{i=1}^{n}$,

$$C_1 \Big\| \sum_{i=1}^{n} a_i |x_i| \Big\|_{E(\mathcal{M},\tau)} \le \Big\| \sum_{i=1}^{n} a_i \tilde{p}_i |x_i| \tilde{p}_i \Big\|_{E(\mathcal{M},\tau)} \le C_2 \Big\| \sum_{i=1}^{n} a_i |x_i| \Big\|_{E(\mathcal{M},\tau)}.$$

If $\alpha_1 = 0$ and $\alpha_n = \sum_{i=1}^{n} \tau(\tilde{p}_i) < \infty$, set $f_n := \mu_{(\cdot)-\alpha_{n-1}}(\tilde{p}_n |x_n| \tilde{p}_n)$ for $n \ge 1$. The sequence $\{f_n\}_{n=1}^{\infty}$ is disjointly supported in $E(0, \tau(\mathbf{1}))$ and $\{f_n\}_{n=1}^{\infty}$ is isometrically isomorphic to $\{\tilde{p}_n |x_n| \tilde{p}_n\}_{n=1}^{\infty}$. For any finite sequence of scalars $\{a_i\}_{i=1}^{n}$,

$$C_1 \Big\| \sum_{i=1}^{n} a_i x_i \Big\|_{E(\mathcal{M},\tau)} = C_1 \Big\| \sum_{i=1}^{n} a_i |x_i| \Big\|_{E(\mathcal{M},\tau)} \le \Big\| \sum_{i=1}^{n} a_i \tilde{p}_i |x_i| \tilde{p}_i \Big\|_{E(\mathcal{M},\tau)}$$
$$= \Big\| \sum_{i=1}^{n} a_i f_i \Big\|_{E(0,\tau(\mathbf{1}))} \le C_2 \Big\| \sum_{i=1}^{n} a_i x_i \Big\|_{E(\mathcal{M},\tau)}.$$

The proof is complete. □

3. Kadec-Pełczyński dichotomy

The main result of this section is the following theorem.

THEOREM 3.1. *Let E be an order continuous rearrangement invariant Banach function space on $[0, \tau(\mathbf{1}))$ with the Fatou property and assume that $E(\mathcal{M}, \tau)$ is of type 2. Then every subspace of $E(\mathcal{M}, \tau)$ either contains a basic sequence equivalent to a disjointly supported sequence in E or is isomorphic to a Hilbert space.*

REMARK 3.2. For the case of L^p with $p > 2$, the commutative case is a classical result of Kadec and Pełczyński [12]; the finite case is a result of Sukochev [25]. Recently, Raynaud and Xu [22] also obtained such dichotomy for the case of Haagerup L^p-spaces.

For the proof of Theorem 3.1, we need several results on $E(\mathcal{M},\tau)$, some of which could be of independent interest.

PROPOSITION 3.3. *Let E be a symmetric quasi-Banach function space on $[0,\tau(\mathbf{1}))$ that is order continuous and is α-convex with constant 1 for some $0<\alpha\leq 1$. Suppose that E satisfies a lower q-estimate with constant 1 for some $q\geq\alpha$. If X is a subspace of $E(\mathcal{M},\tau)$, then either X is strongly embedded into $E(\mathcal{M},\tau)$ or there exist a normalized basic sequence $\{y_n\}_{n=1}^\infty$ in X and a mutually disjoint sequence of projections $\{e_n\}_{n=1}^\infty$ in $\mathcal{M}$ such that*

$$\lim_{n\to\infty}\|y_n-e_ny_ne_n\|_{E(\mathcal{M},\tau)}=0.$$

In particular, $\{y_n\}_{n=1}^\infty$ has a subsequence that is equivalent to a disjointly supported basic sequence in E. Moreover, if X has a basis then the sequence $\{y_n\}_{n=1}^\infty$ can be chosen to be a block basis of the basis of X.

PROOF. Assume that X is not strongly embedded into $E(\mathcal{M},\tau)$ and set $j: E(\mathcal{M},\tau)\to\widetilde{\mathcal{M}}$ the natural inclusion. Since X is not strongly embedded into $E(\mathcal{M},\tau)$, the restriction $j|_X$ is not an isomorphism. There exists a sequence $\{y_n\}_{n=1}^\infty$ in the unit sphere of X which converges to zero in measure. Note that the bounded set $\{y_n, n\geq 1\}$ cannot be E-uniformly integrable. By Theorem 2.3, there exist a subsequence of $\{y_n\}_{n=1}^\infty$ (which we will denote again by $\{y_n\}_{n=1}^\infty$ for simplicity) and a mutually disjoint sequence of projections $\{e_n\}_{n=1}^\infty$ in $\mathcal{M}$ such that the set $\{y_n-e_ny_ne_n, n\geq 1\}$ is E-uniformly integrable. Since $\{y_n-e_ny_ne_n\}_{n=1}^\infty$ converges to zero in measure, we get that $\lim_{n\to\infty}\|y_n-e_ny_ne_n\|_{E(\mathcal{M},\tau)}=0$.

Assume now that X has a basis $\{x_n\}_{n=1}^\infty$. We will show that the sequence $\{y_n\}_{n=1}^\infty$ above can be chosen to be a block basis of $\{x_n\}_{n=1}^\infty$. In fact since $j(B_X)$ cannot be a neighbourhood of zero for the (relative) measure topology on X, for every $\epsilon>0$, $B_{\widetilde{\mathcal{M}}}(0,\epsilon)\cap X\not\subset B_X$ (where $B_{\widetilde{\mathcal{M}}}(0,\epsilon)$ denotes the ball centered at zero and with radius ϵ relative to the metric of the measure topology). Denote by π_n the projection X onto $\overline{\text{span}}\{x_k, k\leq n\}$. Fix $z_1\in S_X\cap B_{\widetilde{\mathcal{M}}}(0,2^{-1})$ and choose $k_1\geq 1$ so that $\|z_1-\pi_{k_1}(z_1)\|<2^{-1}$. The restriction of j on $(Id-\pi_{k_1})(X)$ cannot be an isomorphism. As above, one can choose $z_2\in S_X\cap B_{\widetilde{\mathcal{M}}}(0,2^{-2})$ and $\pi_{k_1}(z_2)=0$. Inductively, one can construct a sequence $\{z_n\}_{n=1}^\infty$ in S_X and a strictly increasing sequence of integers $\{k_n\}_{n=1}^\infty$ such that

(i) $z_n\in B_{\widetilde{\mathcal{M}}}(0,2^{-n})$ for all $n\geq 1$;
(ii) $\|z_n-\pi_{k_n}(z_n)\|<2^{-n}$ for all $n\geq 1$;
(iii) $(Id-\pi_{k_n})(z_{n+1})=0$ for all $n\geq 1$.

Set $y_n:=\pi_{k_n}(z_n)$ for all $n\geq 1$. Clearly $\{y_n\}_{n=1}^\infty$ is a block basic sequence, $\|y_n\|^\alpha\geq 1-2^{-n\alpha}$ for all $n\geq 1$ and $\{y_n\}_{n=1}^\infty$ converges to zero in measure. The proof is complete. □

The next result can be viewed as a non-commutative analogue of [16, Proposition 1.c.10, page 39]. Below, $\{r_n(\cdot)\}_{n=1}^{\infty}$ denotes the sequence of the Rademacher functions on $[0, 1]$.

PROPOSITION 3.4. *Let E be a symmetric Banach function space on $[0, \tau(\mathbf{1}))$. Assume that E is order continuous and satisfies the Fatou property. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $E(\mathcal{M}, \tau)$ such that*

(i) $\|x_n\| = 1$ *for all* $n \geq 1$;

(ii) *there exists a projection $e \in \mathcal{M}$ with $\tau(e) < \infty$ and $ex_n = x_n$ for all $n \geq 1$.*

Then either there exists a constant $C > 0$ such that for every choice of scalars $\{a_n\}_{n=1}^{\infty}$, we have $\int_0^1 \left\|\sum_{i=1}^n r_i(t)a_i x_i\right\|_{E(\mathcal{M},\tau)} dt \geq C\left(\sum_{i=1}^n |a_i|^2\right)^{1/2}$ for every $n \geq 1$ or $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ which is a basic sequence equivalent to a disjoint element of E.

PROOF. For $x \in E(\mathcal{M}, \tau)$, we set as in [25], $\sigma(x, \epsilon) := \chi_{[\epsilon\|x\|_{E(\mathcal{M},\tau)},\infty)}(|x|)$ and

$$M_{E(\mathcal{M},\tau)}(\epsilon) := \{x \in E(\mathcal{M}, \tau),\ \tau(\sigma(x, \epsilon)) \geq \epsilon\}.$$

Assume first that for every $\epsilon > 0$, there exists n_ϵ such that $|x_{n_\epsilon}^*|$ does not belong to $M_{E(\mathcal{M},\tau)}(\epsilon)$. We remark that $|x_n^*|$ is supported by the finite projection e. There exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that $\{|x_{n_j}^*|\}_{j=1}^{\infty}$ converges to zero in measure. In particular, $\{x_{n_j}\}_{j=1}^{\infty}$ converge to zero in measure. By Theorem 2.3, there exist a further subsequence (which we will denote again by $\{x_{n_j}\}_{j=1}^{\infty}$) and a disjoint sequence of projections $\{e_j\}_{j=1}^{\infty}$ so that the set $\{x_{n_j} - e_j x_{n_j} e_j,\ j \geq 1\}$ is E-uniformly integrable so by [21, Proposition 2.8], $\lim_{j\to\infty} \|x_{n_j} - e_j x_{n_j} e_j\| = 0$. This shows that a subsequence of $\{x_{n_j}\}_{j=1}^{\infty}$ can be taken to be equivalent to a disjoint sequence of E.

On the other hand, if $\{|x_n^*|, n \geq 1\} \subset M_{E(\mathcal{M},\tau)}(\epsilon)$ for some $\epsilon > 0$ then

$$\begin{aligned}1 = \|x\| = \||x_n^*|\| &\geq \||x_n^*|\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} \geq (\max(1, \tau(e)))^{-1} \||x_n^*|\|_1 \\ &\geq \epsilon\,(\max(1, \tau(e)))^{-1}\,\tau\left(\sigma(|x_n^*|, \epsilon)\right) \geq \epsilon^2(\max(1, \tau(e)))^{-1}.\end{aligned}$$

So for every $n \geq 1$, $\|x_n\|_1 = \|x_n^*\|_1 = \||x_n^*|\|_1 \geq \epsilon^2(\max(1, \tau(e)))^{-1}$. Since $L_1(\mathcal{M}, \tau)$ is of cotype 2 ([27]), there exists $A_1 > 0$ such that

$$\begin{aligned}\int_0^1 \left\|\sum_{i=1}^n r_i(t)a_i x_i\right\|_{E(\mathcal{M},\tau)} dt &= \int_0^1 \left\|e\left(\sum_{i=1}^n r_i(t)a_i x_i\right)\right\|_{E(\mathcal{M},\tau)} dt \\ &\geq \int_0^1 \left\|e\left(\sum_{i=1}^n r_i(t)a_i x_i\right)\right\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} dt \\ &\geq (\max(1, \tau(e)))^{-1} \int_0^1 \left\|e\left(\sum_{i=1}^n r_i(t)a_i x_i\right)\right\|_1 dt\end{aligned}$$

$$\geq A_1 \left(\max(1, \tau(e))\right)^{-1} \left(\sum_{i=1}^{n} |a_i|^2 \|x_i\|_1^2 \right)^{1/2}$$
$$\geq A_1 \epsilon^2 \left(\max(1, \tau(e))\right)^{-2} \left(\sum_{i=1}^{n} |a_i|^2 \right)^{1/2}.$$

The proof is complete. ☐

REMARKS 3.5. We do not know if condition (ii) can be removed. The same conclusion holds if (ii) is replaced by: (ii)′ there exists a projection $e \in \mathcal{M}$ with $\tau(e) < \infty$ and $x_n e = x_n$ for all $n \geq 1$.

PROPOSITION 3.6. *Let E be as in Theorem 3.1. Then every basic sequence $\{x_n\}_{n=1}^\infty$ in $E(\mathcal{M}, \tau)$ either contains a block basic sequence equivalent to a disjointly supported sequence in E or $\{r_n(\cdot) \otimes x_n\}_{n=1}^\infty$ is equivalent to ℓ_2.*

If $\tau(\mathbf{1}) < \infty$, then Proposition 3.6 is a simple corollary of Proposition 3.4 with the word 'block basic sequence' replaced by 'subsequence'.

For the semi-finite case, choose a mutually orthogonal family $\{f_i\}_{i \in I}$ of projections in $\mathcal{M}$ with $\sum_{i \in I} f_i = \mathbf{1}$ for the strong operator topology and $\tau(f_i) < \infty$ for all $i \in I$. Let $\{x_n\}_{n=1}^\infty$ be a basic sequence in $E(\mathcal{M}, \tau)$. Using a similar argument as in [28], one can deduce that there exists a countable subset $\{f_k\}_{k=1}^\infty$ of $\{f_i\}_{i \in I}$ such that for each f_i outside of $\{f_k\}_{k=1}^\infty$ and $n \geq 1$, $f_i x_n = x_n f_i = 0$. Let $f = \sum_{k=1}^\infty f_k$. For every $n \geq 1$, we have $f x_n = x_n f = x_n$. Replacing $\mathcal{M}$ by $f \mathcal{M} f$ and τ by its restriction on $f \mathcal{M} f$, we may assume that $f = \mathbf{1}$. For every $n \geq 1$, set $e_n := \sum_{k=1}^n f_k$. The sequence $\{e_n\}_{n=1}^\infty$ is such that $e_n \uparrow_n \mathbf{1}$ and $\tau(e_n) < \infty$ for all $n \geq 1$. Let $X := \overline{\operatorname{span}}\{x_n, n \geq 1\}$ and for $a \in \mathcal{M}$, let $aX := \{ax, x \in X\}$ and $Xa := \{xa, x \in X\}$.

LEMMA 3.7. *If for every $n \geq 1$, X is not isomorphic to $e_n X$, then there exist a normalized block basic sequence $\{y_k\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ and a strictly increasing sequence of integers $\{n_k\}_{k=1}^\infty$ so that $\|y_k - (e_{n_k} - e_{n_{k-1}}) y_k\| < 2^{-k}$, for $k \geq 1$. Similarly, if for every $n \geq 1$, X is not isomorphic to $X e_n$, then there exist a normalized block basic sequence $\{y_k\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ and a strictly increasing sequence of integers $\{n_k\}_{k=1}^\infty$ so that $\|y_k - y_k(e_{n_k} - e_{n_{k-1}})\| < 2^{-k}$, for $k \geq 1$.*

PROOF. Inductively, we will construct a sequence $\{y_k\}_{k=1}^\infty$ in the unit sphere of X, strictly increasing sequences of integers $\{m_k\}_{k=1}^\infty$ and $\{n_k\}_{k=1}^\infty$ such that

(i) $y_k \in \operatorname{span}\{x_n, m_{k-1} < n \leq m_k\}$ for all $k \geq 1$;
(ii) $\|e_{n_k} y_k\| < 2^{-(k+1)}$ for all $k \geq 1$;
(iii) $\|y_k - e_{n_k} y_k\| < 2^{-(k+1)}$ for all $k \geq 1$.

Fix y_1 a finitely supported vector in S_X and let $m_1 \geq 1$ so that $y_1 \in \mathrm{span}\{x_n, n \leq m_1\}$. Since $(\mathbf{1} - e_n) \downarrow_n 0$, there exists n_1 such that $\|y_1 - e_{n_1} y_1\| < 2^{-1}$.

Assume that the construction is done for $1, 2, \ldots, (j-1)$. Let $X_j = \overline{\mathrm{span}}\{x_n, n \geq m_{j-1}\}$. Since X_j is not isomorphic to $e_{n_{j-1}} X_j$, there exists $y_j \in S_{X_j}$ such that $\|e_{n_{j-1}} y_j\| < 2^{-(j+1)}$. By perturbation, we can assume that y_j is finitely supported. If we fix $n_j > n_{j-1}$ so that $\|y_j - e_{n_j} y_j\| < 2^{-(j+1)}$ then $\|y_j - (e_{n_j} - e_{n_{j-1}}) y_j\| < 2^{-j}$ and the lemma follows. □

Proof of Proposition 3.6. Assume first that there exists $n_0 \geq 1$ such that X is isomorphic to $e_{n_0} X$. Since $\tau(e_{n_0}) < \infty$, the sequence $\{e_{n_0} x_n\}_{n=1}^\infty$ satisfies the assumptions of Proposition 3.4. Since $E(\mathcal{M}, \tau)$ has type 2, either $\{r_n(\cdot) \otimes e_{n_0} x_n\}_{n=1}^\infty$ is equivalent to ℓ_2 or there exists a subsequence $\{e_{n_0} x_{n_j}\}_{j=1}^\infty$ which is equivalent to a sequence of disjoint elements of E and by isomorphism, the proposition follows.

Assume now that for every $n \geq 1$, X is not isomorphic to $e_n X$. By the above lemma, there exist a normalized block basic sequence $\{y_k\}_{k=1}^\infty$ and a strictly increasing sequence of integers $\{n_k\}_{k=1}^\infty$ so that for every $k \geq 1$,

$$\|y_k - (e_{n_k} - e_{n_{k-1}}) y_k\| < 2^{-k}. \tag{3.1}$$

Let $Y := \overline{\mathrm{span}}\{(e_{n_k} - e_{n_{k-1}}) y_k, k \geq 1\}$. As above, if there exists m_0 such that Y is isomorphic to $Y e_{m_0}$, then the conclusion follows. Otherwise, there exist a block basic sequence $\{z_k\}_{k=1}^\infty$ of $\{(e_{n_k} - e_{n_{k-1}}) y_k\}_{k=1}^\infty$ and a strictly increasing sequence of integers $\{m_k\}_{k=1}^\infty$ such that for every $k \geq 1$,

$$\|z_k - z_k(e_{m_k} - e_{m_{k-1}})\| < 2^{-k}. \tag{3.2}$$

We remark that since the sequence $\{z_k\}_{k=1}^\infty$ is a block basic sequence of $\{(e_{n_k} - e_{n_{k-1}}) y_k\}_{k=2}^\infty$, there exists a sequence $\{q_k\}_{k=1}^\infty$ of mutually disjoint projections such that for every $k \geq 1$, $z_k = q_k z_k$. Therefore, the sequence $\{z_k(e_{m_k} - e_{m_{k-1}})\}_{k=2}^\infty$ is both right and left disjointly supported and hence is equivalent to a disjointly supported sequence in E. By (3.2), we conclude that $\{z_k\}_{k=1}^\infty$ has a subsequence that is equivalent to a disjointly supported sequence in E (see for instance, [5, Theorem 9, page 46]). Since $\{z_k\}_{k=1}^\infty$ is a block basic sequence of $\{(e_{n_k} - e_{n_{k-1}}) y_k\}_{k=2}^\infty$, inequality (3.1) shows that the corresponding block of $\{y_k\}_{k=1}^\infty$ is equivalent to $\{z_k\}_{k=1}^\infty$. The proof of Proposition 3.6 is complete. □

Proof of Theorem 3.1. Let X be a subspace of $E(\mathcal{M}, \tau)$ and assume that X does not contain any basic sequence equivalent to a disjointly supported sequence in E. Let Y be a subspace of X with a basis. From the proof of Proposition 3.6 above, there exists an $n_0 \in \mathbb{N}$ such that Y is isomorphic to either $e_{n_0} Y$ or $Y e_{n_0}$. By Proposition 3.4, Y is of cotype 2 and therefore is isomorphic to a Hilbert space. □

4. Subspaces of Lorentz spaces and applications

In this section, we will specialize to the concrete case of Lorentz spaces. We begin by recalling some definitions and basic facts about Lorentz spaces.

For $0 < p < \infty$, $0 < q \leq \infty$, and $I = [0, 1]$ or $[0, \infty)$, the Lorentz function space $L_{p,q}(I)$ is the space of all (classes of) Lebesgue measurable functions f on I for which $\|f\|_{p,q} < \infty$, where

$$\|f\|_{p,q} = \begin{cases} \left(\int_I \mu_t^q(f)\, d(t^{q/p})\right)^{1/q}, & q < \infty; \\ \sup_{t\in I} t^{1/p}\mu_t(f), & q = \infty. \end{cases} \tag{4.1}$$

Clearly, $L_{p,p}(I) = L_p(I)$ for any $p > 0$. It is well known that for $1 \leq q \leq p < \infty$, (4.1) defines a norm under which $L_{p,q}(I)$ is a separable rearrangement invariant Banach function space; otherwise, (4.1) defines a quasi-norm on $L_{p,q}(I)$ (which is known to be equivalent to a norm if $1 < p < q < \infty$).

The following lemma was observed in [2]. It contains the technical ingredients for the construction of the non-commutative counterparts.

LEMMA 4.1. *Let* $0 < p < \infty$, $0 < q < \infty$.

(i) *If* $q < p$, *then* $L_{p,q}(I)$ *is* q*-convex with constant* 1 *and satisfies a lower* p*-estimate with constant* 1.

(ii) $L_{p,q}(I)$ *satisfies an upper* r*-estimate and lower* s*-estimate* (*with some constant* C), *where* $r = \min(p, q)$ *and* $s = \max(p, q)$.

For $0 < p < q < \infty$, $L_{p,q}(I)$ can be equivalently renormed to be a quasi-Banach lattice, that is, γ-convex (for $\gamma < p$) with constant 1 and satisfies a lower q-estimate of constant 1. Hence for any $0 < p, q < \infty$, we can define the non-commutative space $L_{p,q}(\mathcal{M}, \tau)$ as in Section 2. Since we are only interested in isomorphic properties, we will use the quasi-norm defined in (4.1). All results from Section 2 and Section 3 apply to $L_{p,q}(\mathcal{M}, \tau)$ with appropriate values of p and q.

The main result of this section extends a result of Carothers and Dilworth [3] to the non-commutative settings.

THEOREM 4.2. *Let* $0 < p < \infty$, $0 < q < \infty$, $p \neq q$ *and* $p \neq 2$. *Then* ℓ_p *does not embed into* $L_{p,q}(\mathcal{M}, \tau)$. *In particular, the Lorentz-Schatten ideal* $S_{p,q}$ *does not contain* ℓ_p.

The following application follows easily from Theorem 4.2. It characterizes strongly embedded subspaces in $L_p(\mathcal{M}, \tau)$ and generalizes results of Rosenthal and Kalton on $L_p[0, 1]$ to the non-commutative settings.

THEOREM 4.3. *Let* $0 < p < \infty$, $p \neq 2$ *and* X *be a subspace of* $L_p(\mathcal{M}, \tau)$. *Then the following are equivalent*:

(1) X *contains* ℓ_p.
(2) X *is not strongly embedded into* $L_p(\mathcal{M}, \tau)$.

PROOF. Let X be a subspace of $L_p(\mathcal{M}, \tau)$ and assume that X contains ℓ_p. Since for $p < q$, $\|\cdot\|_{p,q} \leq C\|\cdot\|_p$, for some constant C (see [1, Proposition 4.2, page 217]). There exists an inclusion map from $L_p(\mathcal{M}, \tau)$ into $L_{p,q}(\mathcal{M}, \tau)$. If X is strongly embedded into $L_p(\mathcal{M}, \tau)$, then X is isomorphic to a subspace of $L_{p,q}(\mathcal{M}, \tau)$. In particular, ℓ_p embeds into $L_{p,q}(\mathcal{M}, \tau)$. This contradicts Theorem 4.2.

The converse is a direct consequence of Theorem 4.7. □

REMARK 4.4. For $1 \leq p < 2$ and $\mathcal{M}$ being finite, Theorem 4.3 also appeared in recent work of Haagerup, Rosenthal and Sukochev [11, Theorem 5.4]. Their approach is completely different from the one taken in this paper.

For the proof of Theorem 4.2, we need some preparation. First, we recall that for any given $0 < p < \infty$ and $0 < q \leq \infty$, the space $L_{p,q}(I)$ is equal (up to an equivalent quasi-norm) to the spaces $(L_{p_1}(I), L_{p_2}(I))_{\theta,q}$ constructed using the real interpolation method where $0 < p_1 < p_2 < \infty$, $0 < \theta < 1$ and $1/p = (1-\theta)/p_1 + \theta/p_2$. From general theory of lifting of interpolations to non commutative settings, the same result remains valid for $L_{p,q}(\mathcal{M}, \tau)$ (see for instance [20]).

LEMMA 4.5. *If* $0 < p_1, p_2, q < \infty$ *and* $0 < \theta < 1$, *then*

$$(L_{p_1}(\mathcal{M}, \tau), L_{p_2}(\mathcal{M}, \tau))_{\theta,q} = L_{p,q}(\mathcal{M}, \tau)$$

(*with equivalent quasi-norms*), *where* $1/p = (1-\theta)/p_1 + \theta/p_2$.

Combining [3, Lemma 2.4] with Proposition 2.4, we can also state:

LEMMA 4.6. *Let* $0 < p < \infty$ *and* $0 < q < \infty$. *Let* $\{x_n\}_{n=1}^\infty$ *be a normalized basic sequence in* $L_{p,q}(\mathcal{M}, \tau)$. *If* $\{x_n\}_{n=1}^\infty$ *is both right and left disjointly supported, then* $\overline{\text{span}}\{x_n, n \geq 1\}$ *contains a copy of* ℓ_q.

The next result can be viewed as a particular case of a result of Levy on real-interpolation [15]. It can also be deduced directly from Proposition 3.3 and Lemma 4.6

PROPOSITION 4.7. *Let* $0 < p < \infty$, $0 < q < \infty$, *and let* X *be a subspace of* $L_{p,q}(\mathcal{M}, \tau)$. *Then either* X *is strongly embedded into* $L_{p,q}(\mathcal{M}, \tau)$ *or* X *contains a copy of* ℓ_q.

For the next result, we need to fix some notation. Let $\mathcal{N}$ be a von Neumann algebra on a given Hilbert space H with semi-finite trace φ. Define

$$[\mathcal{N}] := \left\{(a_{ij})_{ij};\ \forall\, i,\, j,\ a_{ij} \in \mathcal{N},\ \|(a_{ij})_{ij}\|_{B(\ell_2(H))} < \infty\right\}.$$

Clearly, $[\mathcal{N}]$ is a von Neumann algebra over the Hilbert space $\ell_2(H)$ and the functional $[\varphi]((a_{ij})_{ij}) = \sum_{i=1}^{\infty} \varphi(a_{ii})$ defines a normal semi-finite trace on $[\mathcal{N}]$. The von Neumann algebra $[\mathcal{N}]$ is formally $\mathcal{N} \bar{\otimes} B(\ell_2)$ and $[\varphi] = \varphi \otimes tr$, where tr is the usual trace on $B(\ell_2)$.

Let $\{y_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{N}$. For each $k \geq 1$, we define $[y_k] = ([y_k]_{ij})_{ij}$ by setting: $[y_k]_{1,k} = y_k$ and $[y_k]_{ij} = 0$ for $(i,\, j) \neq (1, k)$, that is, for $k \geq 1$,

$$[y_k] := \begin{pmatrix} 0 & \cdots & 0 & y_k & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

This amounts to placing the sequence $\{y_k\}_{k=1}^{\infty}$ in the first row of an infinite matrix i.e for every $k \geq 1$, $[y_k] = y_k \otimes e_{1,k}$.

LEMMA 4.8. *Let $0 < p < 2$ and $\{y_k\}_{k=1}^{\infty}$ be a sequence in $L_{p,q}(\mathcal{N}, \varphi)$. There exists an absolute constant C such that for every choice of scalars $\{a_k\}_{k=1}^{\infty}$ and every $n \geq 1$,*

$$\int_0^1 \left\| \sum_{k=1}^{n} r_k(t) a_k y_k \right\|_{L_{p,q}(\mathcal{N},\varphi)}^2 dt \leq C \min \left\{ \left\| \sum_{k=1}^{n} a_k [y_k] \right\|_{L_{p,q}([\mathcal{N}],[\varphi])}^2 , \left\| \sum_{k=1}^{n} \bar{a}_k [y_k^*] \right\|_{L_{p,q}([\mathcal{N}],[\varphi])}^2 \right\}. \tag{4.2}$$

PROOF. We first remark from non-commutative Kintchine's inequalities ([18] for $1 \leq p < 2$ and [20, Remark 6.3] for $0 < p < 1$) that

$$\left(\int_0^1 \left\| \sum_{k=1}^{n} r_k(t) a_k y_k \right\|_{L_p(\mathcal{N},\varphi)}^2 dt \right)^{1/2} \leq \left\| \left(\sum_{k=1}^{n} |a_k|^2 y_k y_k^* \right)^{1/2} \right\|_{L_p(\mathcal{N},\varphi)}. \tag{4.3}$$

Note that

$$\left| \sum_{k=1}^{n} \bar{a}_k [y_k]^* \right|^p = \begin{pmatrix} \left(\sum_{k=1}^{n} |a_k|^2 y_k y_k^*\right)^{p/2} & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \cdots \end{pmatrix}$$

therefore

$$\left\| \sum_{k=1}^{n} \bar{a}_k [y_k]^* \right\|_{L_p([\mathcal{N}],[\varphi])} = \left\| \left(\sum_{k=1}^{n} |a_k|^2 y_k y_k^* \right)^{1/2} \right\|_{L_p(\mathcal{N},\varphi)} .$$

Hence

$$\left\| \sum_{k=1}^{n} a_k [y_k] \right\|_{L_p([\mathcal{N}],[\varphi])} = \left\| \left(\sum_{k=1}^{n} |a_k|^2 y_k y_k^* \right)^{1/2} \right\|_{L_p(\mathcal{N},\varphi)}$$

and by (4.3),

$$\int_0^1 \left\| \sum_{k=1}^{n} r_k(t) a_k y_k \right\|^2_{L_p(\mathcal{N},\varphi)} dt \le \left\| \sum_{k=1}^{n} a_k [y_k] \right\|^2_{L_p([\mathcal{N}],[\varphi])} .$$

SUBLEMMA 4.9. *For every $0 < p < 1$, the map $(a_{ij})_{ij} \to \sum_k r_k(\cdot) a_{1k}$ is bounded as a linear map from $L_p([\mathcal{N}],[\varphi])$ into $L_2([0,1], L_p(\mathcal{N},\varphi))$.*

Let $a = (a_{ij})_{ij}$ be an element of $L_p([\mathcal{N}],[\varphi])$ and consider $|a^*|^2 = (b_{ij})_{ij}$. Clearly, $b_{11} = \sum_{k=1}^{\infty} a_{1k} a_{1k}^*$. Set e to be the projection in $[\mathcal{N}]$ defined by $e = \mathbf{1} \otimes e_{1,1}$, that is, $e = (\alpha_{ij})_{ij}$ with $\alpha_{11} = \mathbf{1}$ and $\alpha_{ij} = 0$ for $(i,j) \neq (1,1)$. We have

$$e|a^*|^2 e = \begin{pmatrix} \sum_{k=1}^{\infty} a_{1k} a_{1k}^* & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \cdots \end{pmatrix}$$

so $\| e|a^*|^2 e \|_{L_{p/2}([\mathcal{N}],[\varphi])} = \left\| \left(\sum_{k=1}^{\infty} a_{1k} a_{1k}^* \right)^{1/2} \right\|^2_{L_p(\mathcal{N},\varphi)}$ and as above,

$$\int_0^1 \left\| \sum_{k=1}^{\infty} r_k(t) a_{1k} \right\|^2_{L_p(\mathcal{N},\varphi)} dt \le \left\| \sum_{k=1}^{\infty} [a_{1k}] \right\|^2_{L_p([\mathcal{N}],[\varphi])} = \left\| \left(\sum_{k=1}^{\infty} a_{1k} a_{1k}^* \right)^{1/2} \right\|^2_{L_p(\mathcal{N},\varphi)}$$
$$= \left\| e|a^*|^2 e \right\|_{L_{p/2}([\mathcal{N}],[\varphi])} \le \|a\|^2_{L_p([\mathcal{N}],[\varphi])} .$$

The sublemma follows.

By interpolation, the map $(a_{ij})_{ij} \to \sum_k r_k(\cdot) a_{1k}$ is also a bounded map from $L_{p,q}([\mathcal{N}],[\varphi])$ into $L_2([0,1], L_{p,q}(\mathcal{N},\varphi))$. In particular, there exists an absolute constant C such that

$$\int_0^1 \left\| \sum_{k=1}^{n} r_k(t) a_k y_k \right\|^2_{L_{p,q}(\mathcal{N},\varphi)} dt \le C \left\| \sum_{k=1}^{n} a_k [y_k] \right\|^2_{L_{p,q}([\mathcal{N}],[\varphi])} .$$

By taking adjoints, the other inequality follows. The proof of Lemma 4.8 is complete. ☐

Our next result is a disjointification of sequences in $L_{p,q}(\mathcal{M}, \tau)$ and could be of independent interest.

PROPOSITION 4.10. *Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra. There exists a semi-finite von Neumann algebra $\mathcal{S}$ equipped with a faithful normal semi-finite trace ω with the following properties*:

(i) *$\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{S}$;*

(ii) *τ is the restriction of ω on $\mathcal{M}$;*

(iii) *for $0 < p < 2$ and $0 < q < \infty$, there exists a constant K such that for any given basic sequence $\{x_n\}_{n=1}^{\infty}$ in $L_{p,q}(\mathcal{M}, \tau)$, there exists a left and right disjointly supported sequence $\{s_n\}_{n=1}^{\infty}$ in $L_{p,q}(\mathcal{S}, \omega)$ such that for any choice of scalars $\{a_k\}_{k=1}^{\infty}$ and $n \geq 1$,*

$$\int_0^1 \left\| \sum_{k=1}^n a_k r_k(t) x_k \right\|_{L_{p,q}(\mathcal{M},\tau)}^2 dt \leq K \left\| \sum_{k=1}^n a_k s_k \right\|_{L_{p,q}(\mathcal{S},\omega)}^2 .$$

PROOF. Using the above notation, let $\mathcal{N} = [\mathcal{M}]$, $\varphi = [\tau]$. Clearly, $(\mathcal{N}, \varphi)$ is a semi-finite von Neumann algebra on the Hilbert space $H = \ell_2(\mathcal{H})$. Set $\mathcal{S} = [\mathcal{N}]$ and $\omega = [\varphi]$. As above, $\mathcal{M}$ can be identified as a von Neumann subalgebra of $\mathcal{S}$ with τ being the restriction of ω on $\mathcal{M}$.

Let $\{x_n\}_{n=1}^{\infty}$ be a basic sequence in $L_{p,q}(\mathcal{M}, \tau)$. Consider the sequence $\{[x_n]\}_{n=1}^{\infty}$ in $\mathcal{N} = [\mathcal{M}]$.

CLAIM. *The sequence $\{[x_n]\}_{n=1}^{\infty}$ is right disjointly supported.*

To verify this claim, recall that elements of $\mathcal{N}$ are infinite matrices with entries in $\mathcal{M}$. For $n \geq 1$, let $\pi_n = (a_{ij})_{ij}$ with $a_{n,n} = \mathbf{1}$ and $a_{i,j} = 0$ for $(i, j) \neq (n, n)$. Clearly, $\{\pi_n\}_{n=1}^{\infty}$ is a mutually disjoint sequence of projection in $\mathcal{N}$ and for each $n \geq 1$, $[x_n]\pi_n = [x_n]$.

For each $n \geq 1$, let $z_n = [x_n] \in L_{p,q}(\mathcal{N}, \varphi)$ and consider the sequence $\{s_n\}_{n=1}^{\infty}$ in $L_{p,q}(\mathcal{S}, \omega)$ defined by $s_n := [z_n^*]^*$.

CLAIM. *The sequence $\{s_n\}_{n=1}^{\infty}$ is left and right disjointly supported.*

First note that, as above, the sequence $\{[z_n^*]\}_{n=1}^{\infty}$ is right disjointly supported so its adjoints $\{s_n\}_{n=1}^{\infty}$ is left disjointly supported. To prove that it is right disjointly supported, consider the following sequence $\{e_n\}_{n=1}^{\infty}$ in $\mathcal{S}$: $e_n = (a_{ij}^{(n)})_{ij}$, where $a_{11}^{(n)} = \pi_n$ and $a_{ij}^{(n)} = 0$ for $(i, j) \neq (1, 1)$.

It is clear that the e_n's are projections in $\mathcal{S}$ and since $\{\pi_n\}_{n=1}^{\infty}$ is mutually disjoint in $\mathcal{N}$, $\{e_n\}_{n=1}^{\infty}$ is mutually disjoint and one can see that for every $n \geq 1$, $s_n e_n = s_n$.

To complete the proof, we use Lemma 4.8,

$$\begin{aligned}\int_0^1 \left\|\sum_{k=1}^n r_k(t) a_k x_k\right\|^2_{L_{p,q}(\mathcal{M},\tau)} dt &\le C \left\|\sum_{k=1}^n a_k [x_k]\right\|^2_{L_{p,q}(\mathcal{N},\varphi)} \\ &= C \int_0^1 \left\|\sum_{k=1}^n r_k(t) a_k [x_k]\right\|^2_{L_{p,q}(\mathcal{N},\varphi)} dt \\ &= C \int_0^1 \left\|\sum_{k=1}^n r_k(t) a_k z_k\right\|^2_{L_{p,q}(\mathcal{N},\varphi)} dt.\end{aligned}$$

Applying Lemma 4.8 on the von Neumann algebra $\mathcal{N}$,

$$\begin{aligned}\int_0^1 \left\|\sum_{k=1}^n r_k(t) a_k x_k\right\|^2_{L_{p,q}(\mathcal{M},\tau)} dt &\le C^2 \left\|\sum_{k=1}^n \bar{a}_k [z_k^*]\right\|^2_{L_{p,q}(\mathcal{S},\omega)} \\ &= C^2 \left\|\sum_{k=1}^n a_k [z_k^*]^*\right\|^2_{L_{p,q}(\mathcal{S},\omega)} = C^2 \left\|\sum_{k=1}^n a_k s_k\right\|^2_{L_{p,q}(\mathcal{S},\omega)}.\end{aligned}$$

The proof is complete □

PROOF OF THEOREM 4.2. The proof will be divided into several cases. First, notice that since $p \neq q$, Proposition 4.7 shows that every subspace of $L_{p,q}(\mathcal{M},\tau)$ equivalent to ℓ_p (and therefore not containing any copy of ℓ_q) is strongly embedded into $L_{p,q}(\mathcal{M},\tau)$. Fix $r > q$, then $\|\cdot\|_{p,r} \le C\|\cdot\|_{p,q}$, where C is a constant depending only on p, q and r (see for instance [1, Proposition 4.2, page 217]). In particular, there exists a continuous inclusion from $L_{p,q}(\mathcal{M},\tau)$ into $L_{p,r}(\mathcal{M},\tau)$ and if X is a strongly embedded subspace of $L_{p,q}(\mathcal{M},\tau)$ then X is isomorphic to a subspace of $L_{p,r}(\mathcal{M},\tau)$ so without loss of generality, we can assume that $p < q$ and $1 < q$.

Case $0 < p < q < \infty$ *and* $p < 2$.

Assume that there exists a sequence $\{x_n\}_{n=1}^\infty$ that is M-equivalent to ℓ_p in $L_{p,q}(\mathcal{M},\tau)$ and consider the disjoint sequence $\{y_n\}_{n=1}^\infty$ in $L_{p,q}(\mathcal{S},\omega)$ as in Proposition 4.10. For every finite sequence of scalars $\{a_n\}$, we have:

$$\begin{aligned}\left(\sum_n |a_n|^p\right)^{1/p} &\le M\left(\int_0^1 \left\|\sum_n r_n(t) a_n x_n\right\|^2_{L_{p,q}(\mathcal{M},\tau)} dt\right)^{1/2} \\ &\le M.\sqrt{K}\left\|\sum_n a_n y_n\right\|_{L_{p,q}(\mathcal{S},\omega)} \le N.M.\sqrt{K}\left\|\sum_n a_n \varphi_n\right\|_{L_{p,q}(0,\infty)}\end{aligned}$$

where $\{\varphi_k\}_{k=1}^\infty$ is a disjoint sequence in $L_{p,q}[0,\infty)$ and $N > 0$. Since $p < q$, the space $L_{p,q}[0,\infty)$ satisfies an upper p-estimate hence there exists constants C_1 and C_2 such that

$$C_1\left(\sum_{k=1}^n |a_k|^p\right)^{1/p} \leq \left\|\sum_{k=1}^n a_k\varphi_k\right\|_{L_{p,q}[0,\tau(\mathbf{1}))} \leq C_2\left(\sum_{k=1}^n |a_k|^p\right)^{1/p}.$$

But this is a contradiction since $\overline{\text{span}}\{\varphi_k, k \geq 1\}$ contains a copy of ℓ_q.

Case $2 < p < q < \infty$.

We remark that combining [9] with [16, Proposition 2g.22, page 230], $L_{p,q}(\mathcal{M},\tau)$ is of type 2 and therefore Theorem 3.1 applies to $L_{p,q}(\mathcal{M},\tau)$. Assume that there exists a sequence $\{x_n\}_{n=1}^\infty$ in $L_{p,q}(\mathcal{M},\tau)$ that is equivalent to ℓ_p. Since $p \neq 2$, Theorem 3.1 implies that $\{x_n\}_{n=1}^\infty$ contains a block basic sequence $\{y_n\}_{n=1}^\infty$ that is equivalent to a disjointly supported normalized sequence in $L_{p,q}[0,\tau(\mathbf{1}))$ so $\overline{\text{span}}\{y_n, n \geq 1\}$ does not contain ℓ_p. This is a contradiction since $\{y_n\}_{n=1}^\infty$ is equivalent to ℓ_p. □

We conclude the paper with an observation on copies of ℓ_p in $L_p(\mathcal{M},\tau)$. It extends a well known results for copies of ℓ_1 in preduals of von Neumann algebras.

COROLLARY 4.11. *Let* $1 \leq p < \infty$, $p \neq 2$. *If* $\{x_n\}_{n=1}^\infty$ *is a sequence in* $L_p(\mathcal{M},\tau)$ *that is equivalent to* ℓ_p *and* $\{\varepsilon_n\}_{n=1}^\infty$ *is a sequence in the interval* $(0,1)$ *with* $\varepsilon_n \downarrow_n 0$, *then there exists a block basis* $\{y_n\}_{n=1}^\infty$ *of* $\{x_n\}_{n=1}^\infty$ *such that*

$$\left(\sum_n |a_n|^p\right)^{1/p} - \left(\sum_n |a_n|^p {\varepsilon_n}^p\right)^{1/p} \leq \left\|\sum_n a_n y_n\right\|$$
$$\leq \left(\sum_n |a_n|^p\right)^{1/p} + \left(\sum_n |a_n|^p {\varepsilon_n}^p\right)^{1/p}$$

for all finite sequence $(a_n)_n$ *of scalars. In particular, for every* $k \geq 1$, *the sequence* $\{y_n\}_{n=k}^\infty$ *is* $(1+\varepsilon_k)$*-equivalent to* ℓ_p.

PROOF. Since ℓ_p is not strongly embedded into $L_p(\mathcal{M},\tau)$, Proposition 3.3 implies the existence of a block basic sequence $\{z_n\}_{n=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ and a sequence $\{p_n\}_{n=1}^\infty$ of mutually disjoint projections in $\mathcal{M}$ such that

$$\lim_{n\to\infty} \|z_n - p_n z_n p_n\| = 0.$$

Note that $\liminf_{n\to\infty} \|p_n z_n p_n\| > 0$. By taking a subsequence (if necessary), we will assume that for every $n \geq 1$,

$$\frac{\|z_n - p_n z_n p_n\|}{\|p_n z_n p_n\|} \leq \varepsilon_n 2^{-n}.$$

For $n \geq 1$, set $y_n := z_n / \| p_n z_n p_n \|$. If $(a_n)_n$ is a finite sequence of scalars then

$$\left\| \sum_n a_n y_n \right\| \leq \sum_n |a_n| \, \| y_n - p_n z_n p_n \| + \left(\sum_n |a_n|^p \right)^{1/p}$$
$$\leq \left(\sum_n |a_n|^p \varepsilon_n^p \right)^{1/p} \left(\sum_n 2^{-nq} \right)^{1/q} + \left(\sum_n |a_n|^p \right)^{1/p}$$

where $1/p + 1/q = 1$. This shows that

$$\left\| \sum_n a_n y_n \right\| \leq \left(\sum_n |a_n|^p \right)^{1/p} + \left(\sum_n |a_n|^p \varepsilon_n^p \right)^{1/p}.$$

The other inequality can be obtained with similar estimates. □

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