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ON PRODUCTS IN LATTICE-ORDERED ALGEBRAS

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Abstract

Let A be a uniformly complete vector sublattice of an Archimedean semiprime f-algebra B and $p \in \{1, 2, \dots\}$. It is shown that the set $\Pi_p^B(A) = \{f_1 \cdots f_p : f_k \in A, k = 1, \dots, p\}$ is a uniformly complete vector sublattice of B. Moreover, if A is provided with an almost f-algebra multiplication $*$ then there exists a positive operator T_p from $\Pi_p^B(A)$ into A such that $f_1 * \cdots * f_p = T_p(f_1 \cdots f_p)$ for all $f_1, \dots, f_p \in A$.

As application, being given a uniformly complete almost f-algebra $(A, *)$ and a natural number $p \geq 3$, the set $\Pi_p^*(A) = \{f_1 * \cdots * f_p : f_k \in A, k = 1, \dots, p\}$ is a uniformly complete semiprime f-algebra under the ordering and the multiplication inherited from A.

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1. Introduction

The only lattice-ordered algebras under consideration are almost f-algebras, f-algebras and commutative d-algebras. The definition of an almost f-algebra was first given in 1967 by Birkhoff [4]. Contrary to the f-algebras, introduced in 1956 by Birkhoff and Pierce [5], the almost f-algebras provoked little interest for a long period. In fact, it is only since 1981, year of the publication of Scheffold's fundamental paper [16], that this scope has attracted more attention and works in this field have been more prolific. We cite particularly the important paper of Bernau and Huijsmans [2], in which the authors give some almost f-algebras properties and especially an elegant proof of commutativity of Archimedean almost f-algebras. In their work, Bernau and Huijsmans present also a study on d-algebras, whose definition seems to go back to Kudláček [13]. We also mention the work of Buskes and van Rooij [10] which includes, among others, a survey on products of two (and three) elements in almost f-algebras. Our works [7, 8, 6] on products in lattice-ordered algebras and more precisely in almost f-algebras have been undoubtedly motivated by the Buskes and van Rooij paper; It is in this context that, in [6], we proved the following results: if A is a uniformly complete vector sublattice of an Archimedean semiprime f-algebra B then $\Pi_2^B(A) = \{fg : f, g \in A\}$ is a uniformly complete vector sublattice of B with $\Sigma_2^B(A) = \{f^2 : f \in A^+\}$ as positive cone. Moreover, if A is, in addition, an almost f-algebra with respect to a multiplication $*$ then there exists a positive operator T_2 from $\Pi_2^B(A)$ into A such that $f * g = T_2(fg)$ for all $f, g \in A$.

In this paper, and more precisely in the third section, we shall generalize these results in the following sense: given a natural number $p \in \{1, 2, \ldots\}$ and a uniformly complete vector sublattice A of an Archimedean semiprime f-algebra B, the set

$$\Pi_p^B(A) = \{f_1 \cdots f_p : f_k \in A,\ k = 1, \ldots, p\}$$

is a uniformly complete vector sublattice of B with $\Sigma_p^B(A) = \{f^p : f \in A^+\}$ as positive cone. Furthermore, if A is furnished with an almost f-algebra multiplication $*$ then there exists a positive operator T_p from $\Pi_p^B(A)$ into A such that

$$f_1 * \cdots * f_p = T_p(f_1 \cdots f_p) \quad \text{for all} \quad f_1, \ldots, f_p \in A.$$

The main topic of Section 4 of this work is to investigate $\ker T_p$, the kernel of T_p. To be more precise, we prove that if $p \geq 3$ then $\ker T_p$ is an order ideal of $\Pi_p^B(A)$. An example is produced showing that this result fails in the case $p = 2$.

The last section of this paper deals mainly with $T_p(\Pi_p^B(A))$, the range of T_p. In particular, we show that if $p \geq 3$ then $T_p(\Pi_p^B(A))$ is a uniformly complete vector lattice with respect to the ordering inherited from A, and we give an example proving that this result does not hold if $p = 2$. As application, we re-prove (see [8]) that if $(A, *)$ is a given uniformly complete almost f-algebra then for every natural number $p \geq 3$, the set

$$\Pi_p^*(A) = \{f_1 * \cdots * f_p : f_k \in A,\ k = 1, \ldots, p\}$$

is a uniformly complete semiprime f-algebra under the ordering and the multiplication inherited from A with

$$\Sigma_p^*(A) = \{f * \cdots * f\ (p\text{-times}) : f \in A^+\}$$

as positive cone. We note that the case where $(A, *)$ is a commutative d-algebra (respectively, f-algebra) is also treated.

Finally, we point out that the second section is devoted to some preliminaries that will be useful throughout this paper.

2. Preliminaries

For terminology, notations and elementary properties of vector lattices not explained or proved below we refer the reader to the standard works [1, 14, 17].

Let E be a vector lattice with positive cone E^+ and $e \in E^+$. The sequence $\{f_n\}_{n=1}^{n=\infty}$ in E is said to *converge e-uniformly* to $f \in E$ whenever, for every $\varepsilon > 0$, there exists a natural number N_ε such that $|f - f_n| \leq \varepsilon e$ for all $n \geq N_\varepsilon$. The sequence $\{f_n\}_{n=1}^{n=\infty}$ is said to *converge* (*relatively*) *uniformly* to $f \in E$ if it converges e-uniformly to f for some $0 \leq e \in E$. In like manner the notion of (*relatively*) *uniform Cauchy* sequence is defined. Uniform limits are unique if and only if E is Archimedean [14, Theorem 63.2]. For this reason all vector lattices and lattice-ordered algebras under consideration are supposed to be Archimedean. The (Archimedean) vector lattice E is called *uniformly complete* whenever every uniform Cauchy sequence in E has a (unique) limit. More about the (relatively) uniform convergence can be found in [14].

Let E be a vector lattice. A vector subspace I of E is said to be an *order ideal* of E whenever $|f| \leq |g|$ in E and $g \in I$ imply $f \in I$. Every order ideal of E is a vector sublattice of E. The order ideal *generated* by an element $e \in E$ is denoted by E_e and it is the smallest (with respect to the inclusion) order ideal that contains e. Every order ideal of the form E_e is referred to as a *principal order ideal*. Moreover, if E is uniformly complete then so is E_e. An element $0 \leq e \in E$ is called a *strong order unit* whenever $E = E_e$. In particular, e is a strong order unit in E_e. Let G be a non-empty subset of E. The collection G^d of all elements $f \in E$ such that $|f| \wedge |g| = 0$ for every $g \in G$ is an order ideal of E.

A vector lattice A is said to be a *lattice-ordered algebra* (*or ℓ-algebra*) if there exists an associative multiplication in A with the usual algebra properties such that $fg \geq 0$ for all $0 \leq f, g \in A$. The ℓ-algebra A is called an *f-algebra* if A has the property that $f \wedge g = 0$ in A implies $(fh) \wedge g = (hf) \wedge g = 0$ for all $0 \leq h \in A$. An *almost f-algebra* is an ℓ-algebra A such that $f \wedge g = 0$ in A implies $fg = 0$ (equivalently, $f^2 = |f|^2$ for all $f \in A$). The ℓ-algebra A is said to be a *d-algebra* whenever $f \wedge g = 0$ in A implies $(fh) \wedge (gh) = (hf) \wedge (hg) = 0$ for all $0 \leq h \in A$ (equivalently, $|fg| = |f||g|$ for all $f, g \in A$).

We recall that all ℓ-algebras considered in this paper are supposed to be Archimedean. Any f-algebra is an almost f-algebra and a d-algebra but not conversely. Almost f-algebras need not be d-algebras and vice versa. An (almost) f-algebra is automatically commutative. In general, d-algebras are not commutative. A d-algebra which is commutative is an almost f-algebra. We summarize as follows:

$$f\text{-algebra} \Rightarrow \text{commutative } d\text{-algebra} \Rightarrow \text{almost } f\text{-algebra}.$$

For any ℓ-algebra A, we denote by $N(A)$ the set of all nilpotent elements of A. The ℓ-algebra A is said to be *semiprime* if $N(A) = \{0\}$. Any (almost) f-algebra (or

d-algebra) with positive multiplication unit is semiprime and any semiprime almost f-algebra (or d-algebra) is an f-algebra. The subset $\{f \in A : f^2 = 0\}$ of an ℓ-algebra A is denoted by $N_2(A)$ and the subset $\{f \in A : f^3 = 0\}$ is denoted by $N_3(A)$. If A is an almost f-algebra then

$$N_2(A) = \{f \in A : fg = 0 \text{ for all } g \in A\}$$

and

$$N(A) = N_3(A) = \{f \in A : fgh = 0 \text{ for all } g, h \in A\}.$$

If A is an f-algebra then $N(A) = N_2(A) = \{f \in A : fg = 0 \text{ for all } g \in A\}$ and $fg \in N(A)^d$ for all $f, g \in A$. For more informations about f-algebras (respectively, almost f-algebras and d-algebras) refer to [11, 17] (respectively, [2, 7]).

Let E and F be vector lattices. An operator τ from E into F is said to be *order-bounded* if the image under τ of an order-bounded subset of E is an order-bounded subset of F. The collection of all order-bounded operators from E into F is denoted by $\mathscr{L}_b(E, F)$ and by $\mathscr{L}_b(E)$ if $E = F$. An operator τ from E into F is called *positive* if $\tau(f) \geq 0$ in F for all $f \geq 0$ in E. The set of all positive operators from E into F is denoted by $\mathscr{L}_b(E, F)^+$. This notation is justified by the fact that $\mathscr{L}_b(E, F)$ is an ordered vector space with $\mathscr{L}_b(E, F)^+$ as positive cone [1]. The operator τ from E into F is called a *lattice homomorphism* whenever $f \wedge g = 0$ in E implies $\tau(f) \wedge \tau(g) = 0$ in F (equivalently, $|\tau(f)| = \tau(|f|)$ in F for all $f \in E$). Obviously, every lattice homomorphism is positive. An order-bounded operator τ of E is said to be an *orthomorphism* if $|f| \wedge |g| = 0$ implies $|\tau(f)| \wedge |g| = 0$. A positive operator τ of E is an orthomorphism if and only if $f \wedge g = 0$ implies $\tau(f) \wedge g = 0$. The collection $\mathrm{Orth}(E)$ of all orthomorphisms of E is, under the usual vector space operations, the ordering inherited from $\mathscr{L}_b(E)$ and the composition as multiplication, an Archimedean f-algebra with the identity mapping I_E on E as unit element. If E is a uniformly complete vector lattice then $\mathrm{Orth}(E)$ is as well and if, in addition, $0 \leq e$ is a strong order unit of E then the map

$$\begin{aligned} \mathrm{Orth}(E) &\to E \\ \tau &\mapsto \tau(e) \end{aligned}$$

is a lattice isomorphism. In particular, if $0 \leq f \in E$ then there exists a unique $0 \leq \tau_f \in \mathrm{Orth}(E)$ such that $f = \tau_f(e)$ [11, Theorem 12.1 and Remark 19.5]. More about orthomorphisms can be found in [11, 17].

Let E and F be vector lattices and $p \in \{2, 3, \ldots\}$. The p-linear map ψ from $E^p = E \times \cdots \times E$ (p-times) into F is called *positive* if $\psi(f_1, \ldots, f_p) \geq 0$ in F for all $f_1, \ldots, f_p \geq 0$ in E. The positive p-linear map ψ is said to *have the property* (AF) if $f_i \wedge f_j = 0$ for some $i, j \in \{1, \ldots, p\}$ implies $\psi(f_1, \ldots, f_p) = 0$. In the proof of commutativity of Archimedean almost f-algebras [2, Theorem 2.15], Bernau and

Huijsmans do not make use of associativity. This shows that every positive bilinear map with the property (AF) is automatically symmetrical.

The following two results are important in the context of this work; they are already proven in our paper [8] but for the sake of completeness we reproduce the proofs.

PROPOSITION 2.1. *Let E and F be Archimedean vector lattices, $p \in \{2, 3, \dots\}$, Ψ be a positive p-linear map from E^p into F having the property (AF) and σ be a permutation of the set $\{1, \dots, p\}$. Then $\Psi(f_1, \dots, f_p) = \Psi(f_{\sigma(1)}, \dots, f_{\sigma(p)})$ for all $f_1, \dots, f_p \in E$.*

PROOF. By multilinearity, we can establish the desired equality only for positive elements $f_1, \dots, f_p \in E$. Besides, as groups of permutations are generated by transpositions, it suffices to prove that if $i \neq j \in \{1, \dots, p\}$ then $\Psi(\dots, f_i, \dots, f_j, \dots) = \Psi(\dots, f_j, \dots, f_i, \dots)$. Let Φ be the map defined from E^2 into F by

$$\Phi(u, v) = \Psi(f_1, \dots, \overset{i}{u}, \dots, \overset{j}{v}, \dots, f_p) \quad \text{for all } u, v \in E^2.$$

It is easy to see that Φ is a positive bilinear map with the property (AF) and therefore symmetrical. Consequently,

$$\Psi(\dots, f_i, \dots, f_j, \dots) = \Phi(f_i, f_j) = \Phi(f_j, f_i) = \Psi(\dots, f_j, \dots, f_i, \dots)$$

as required. □

As consequence, we get the following theorem which will turn out to be useful later.

THEOREM 2.2. *Let E and F be Archimedean vector lattices, $p \in \{2, 3, \dots\}$, Ψ be a positive p-linear map from E^p into F having the property (AF) and $\tau \in \mathrm{Orth}(E)$. Then, for every $i \neq j \in \{1, \dots, p\}$*

$$\Psi(f_1, \dots, \tau(f_i), \dots, f_j, \dots, f_p) = \Psi(f_1, \dots, f_i, \dots, \tau(f_j), \dots, f_p)$$

for all $f_1, \dots, f_p \in E$.

PROOF. It is clear that it suffices to prove this result for a positive orthomorphism $\tau \in \mathrm{Orth}(E)$. Let $i \neq j \in \{1, \dots, p\}$ and define the map $\Phi : E^p \to F$ by

$$\Phi(f_1, \dots, f_p) = \Psi(f_1, \dots, \tau(f_i), \dots, f_p)$$

for all $f_1, \dots, f_p \in E$. The fact that Φ is a positive p-linear map having the property (AF) is derived immediately from the definition of orthomorphisms. Proposition 2.1 applied to Φ and to the transposition $\sigma = (i, j)$ yields that

$$\Psi(f_1, \dots, \tau(f_i), \dots, f_j, \dots, f_p) = \Psi(f_1, \dots, \tau(f_j), \dots, f_i, \dots, f_p). \tag{2.1}$$

Again by Proposition 2.1, applied to Ψ and σ, we get

$$\Psi(f_1, \dots, \tau(f_j), \dots, f_i, \dots, f_p) = \Psi(f_1, \dots, f_i, \dots, \tau(f_j), \dots, f_p). \tag{2.2}$$

The desired result is gotten combining (2.1) with (2.2). ☐

Throughout this paper, we will keep the following notations: if $(A, *)$ is an ℓ-algebra, $p \in \{2, 3, \dots\}$ and $f \in A$ then

(1) $f^{*p} = f * \cdots * f$ (p-times).
(2) $N^*(A)$ is the set of all nilpotent elements of A.
(3) $N_2^*(A) = \{f \in A : f^{*2} = 0\}$.
(4) $N_3^*(A) = \{f \in A : f^{*3} = 0\}$.

3. Almost f-algebras as vector sublattices of f-algebras

Throughout this section, B stands for an Archimedean semiprime f-algebra and A stands for a uniformly complete vector sublattice of B.

In [6] we have shown that $\Pi_2^B(A) = \{fg : f, g \in A\}$ is a vector sublattice of B with $\Sigma_2^B(A) = \{f^2 : f \in A^+\}$ as positive cone and if A is, in addition, equipped with an almost f-algebra multiplication $*$ then there exists a positive operator T_2 from $\Pi_2^B(A)$ into B such that $f * g = T_2(fg)$ for all $f, g \in A$. Our aim in this section is to generalize these results to an arbitrary natural number $p \geq 2$ (note that the case $p = 1$ is obvious).

Let's fix a natural number $p \geq 2$. Choose $0 \leq f \in B$ and assume that there exists $0 \leq g \in B$ such that $g^p = f$. As B is semiprime, g is the unique positive element of B satisfying the equation $g^p = f$ [3, Proposition 2]. We say that g is the pth *root* of f in B and we denote $g = f^{1/p}$.

We start this section by the following lemma.

LEMMA 3.1. *Let $2 \leq p$ be a natural number, A be a uniformly complete vector sublattice of an Archimedean semiprime f-algebra B and $0 \leq f, g, f_1, \dots, f_p \in A$. Then $(f^p + g^p)^{1/p}$ and $(f_1 \cdots f_p)^{1/p}$ exist and belong to A.*

PROOF. Let $0 \leq f, g \in A$ and put $e = f + g$. As A is uniformly complete, the principal order ideal A_e generated by e is a uniformly complete vector sublattice of A with e as a strong order unit. Therefore, there exist τ_f and τ_g, positive orthomorphisms on A_e, such that $\tau_f(e) = f$ and $\tau_g(e) = g$. Define the map ψ from $(A_e)^p$ into B by

$$\psi(u_1, \dots, u_p) = u_1 \cdots u_p \quad \text{for all } u_1, \dots, u_p \in A_e.$$

Since B is an f-algebra, ψ is a positive p-linear map having the property (AF). Hence, using Theorem 2.2

$$\begin{aligned} f^p + g^p &= \psi(f, \dots, f) + \psi(g, \dots, g) \\ &= \psi(\tau_f(e), \dots, \tau_f(e)) + \psi(\tau_g(e), \dots, \tau_g(e)) \\ &= \psi(e, \dots, e, \tau_f^p(e)) + \psi(e, \dots, e, \tau_g^p(e)) \\ &= \psi(e, \dots, e, (\tau_f^p + \tau_g^p)(e)). \end{aligned}$$

Now, A_e is uniformly complete and so is the unital f-algebra $\mathrm{Orth}(A_e)$. So the pth root $(\tau_f^p + \tau_g^p)^{1/p}$ of $\tau_f^p + \tau_g^p$ exists in $\mathrm{Orth}(A_e)$ [3, Corollary 6]. Consequently,

$$\begin{aligned} f^p + g^p &= \psi(e, \dots, e, ((\tau_f^p + \tau_g^p)^{1/p})^p(e)) \\ &= \psi(((\tau_f^p + \tau_g^p)^{1/p})(e), \dots, ((\tau_f^p + \tau_g^p)^{1/p})(e)) \\ &= [((\tau_f^p + \tau_g^p)^{1/p})(e)]^p \end{aligned}$$

(where we use Theorem 2.2). We infer that $(f^p + g^p)^{1/p} = ((\tau_f^p + \tau_g^p)^{1/p})(e)$ exists and belongs to A_e and thus to A.

The result concerning $(f_1 \cdots f_p)^{1/p}$ is obtained by an analogous method. $\square$

At this point, we put

$$\Pi_p^B(A) = \{f_1 \cdots f_p : f_k \in A,\ k = 1, \dots, p\}$$

and

$$\Sigma_p^B(A) = \{f^p : f \in A^+\}.$$

The previous lemma implies that the set $\Sigma_p^B(A)$ is closed under addition and therefore it is a positive cone in B. Furthermore, if $0 \le f, g \in A$, then

$$(f \vee g)^p = f^p \vee g^p$$

[3, Proposition 1]. As consequence, $\Sigma_p^B(A)$ is closed under (finite) supremum. It follows from [15, Proposition 1.1.4] that

$$\Sigma_p^B(A) - \Sigma_p^B(A) = \{f^p - g^p : f, g \in A^+\}$$

is a vector sublattice of B with $\Sigma_p^B(A)$ as positive cone.

Now, we are in position to prove the first main result of this section.

THEOREM 3.2. *Let $2 \le p$ be a natural number and A be a uniformly complete vector sublattice of an Archimedean semiprime f-algebra B. Then $\Pi_p^B(A)$ is a uniformly complete vector sublattice of B with $\Sigma_p^B(A)$ as positive cone.*

PROOF. We begin by showing that $\Pi_p^B(A) = \Sigma_p^B(A) - \Sigma_p^B(A)$. Take $0 \leq f, g \in A$ and observe that

$$f^p - g^p = (f - g)\left(\sum_{k=0}^{p-1} f^k g^{p-1-k}\right)$$

(by agreement, $x^0 y = y x^0 = y$ for all $x, y \in B^+$). According to Lemma 3.1, $f^k g^{p-1-k} \in \Sigma_{p-1}^B(A)$ for every $k \in \{0, \ldots, p-1\}$. But then

$$\sum_{k=0}^{p-1} f^k g^{p-1-k} \in \Sigma_{p-1}^B(A)$$

since $\Sigma_{p-1}^B(A)$ is closed under addition. Therefore $f^p - g^p \in \Pi_p^B(A)$ and thus

$$\Sigma_p^B(A) - \Sigma_p^B(A) \subset \Pi_p^B(A).$$

Conversely, let $f_1, \ldots, f_p \in A$. It is easy to see that $f_1 \cdots f_p = F - G$, where F (and G) is a sum of products of p positive elements of A (where we use an argument of multilinearity). Using Lemma 3.1, we infer that $F, G \in \Sigma_p^B(A)$ and the second inclusion follows. Summarizing, $\Pi_p^B(A)$ is a vector sublattice of B with $\Sigma_p^B(A)$ as positive cone.

In the remainder of this proof, we shall show that $\Pi_p^B(A)$ is uniformly complete. Let $\{f_n\}_{n=1}^{n=\infty}$ be a sequence in A^+ and $g \in A^+$ such that $\{f_n^p\}_{n=1}^{n=\infty}$ is a g^p-uniform Cauchy sequence in $\Pi_p^B(A) \subset B$. It is shown in [3, Corollary 3] that $\{f_n\}_{n=1}^{n=\infty}$ is a g-uniform Cauchy sequence in B and therefore in A, which is uniformly complete. Hence there exists $f \in A^+$ such that $\{f_n\}_{n=1}^{n=\infty}$ converges g-uniformly to f. Finally $\{f_n^p\}_{n=1}^{n=\infty}$ converges g^p-uniformly to f^p [3, Corollary 3]. The proof is complete. $\square$

Now, let's equip A with a multiplication $*$ in such a manner that $(A, *)$ becomes an almost f-algebra. We recall that for every $f, g, f_1, \ldots, f_p \in A^+$, both $(f^p + g^p)^{1/p}$ (the pth root of $f^p + g^p$ in B) and $(f_1 \cdots f_p)^{1/p}$ (the pth root of $f_1 \cdots f_p$ in B) exist and belong to A (Lemma 3.1). In the next lemma, we shall calculate the two powers $((f^p + g^p)^{1/p})^{*p}$ and $((f_1 \cdots f_p)^{1/p})^{*p}$.

LEMMA 3.3. *Let B be an Archimedean semiprime f-algebra, $(A, *)$ be a uniformly complete almost f-algebra such that A is a vector sublattice of B, $2 \leq p$ be a natural number and $0 \leq f, g, f_1, \ldots, f_p \in A$. Then*

$$\left((f^p + g^p)^{1/p}\right)^{*p} = f^{*p} + g^{*p} \quad \textit{and} \quad \left((f_1 \cdots f_p)^{1/p}\right)^{*p} = f_1 * \cdots * f_p.$$

PROOF. We opt for the gait followed in the proof of Lemma 3.1.

Let $0 \leq f, g \in A$ and put $e = f + g$. We consider τ_f (respectively, τ_g) the positive orthomorphism of A_e such that $\tau_f(e) = f$ (respectively, $\tau_g(e) = g$). Define the map ψ^* from $(A_e)^p$ into A by $\psi^*(u_1, \ldots, u_p) = u_1 * \cdots * u_p$ for all $u_1, \ldots, u_p \in A_e$. It is shown in the proof of Lemma 3.1 that $(f^p + g^p)^{1/p} = ((\tau_f^p + \tau_g^p)^{1/p})(e)$. Therefore,

$$\begin{aligned}\left((f^p + g^p)^{1/p}\right)^{*p} &= \psi^*\left((f^p + g^p)^{1/p}, \ldots, (f^p + g^p)^{1/p}\right)\\ &= \psi^*\left(((\tau_f^p + \tau_g^p)^{1/p})(e), \ldots, ((\tau_f^p + \tau_g^p)^{1/p})(e)\right).\end{aligned}$$

Since $(A, *)$ is an almost f-algebra, ψ^* is a positive p-linear map having the property (AF). In view of Theorem 2.2, we obtain

$$\begin{aligned}((f^p + g^p)^{1/p})^{*p} &= \psi^*(e, \ldots, e, ((\tau_f^p + \tau_g^p)^{1/p})^p(e))\\ &= \psi^*(e, \ldots, e, (\tau_f^p + \tau_g^p)(e))\\ &= \psi^*(e, \ldots, e, \tau_f^p(e)) + \psi^*(e, \ldots, e, \tau_g^p(e))\\ &= \psi^*(f, \ldots, f) + \psi^*(g, \ldots, g) = f^{*p} + g^{*p}.\end{aligned}$$

The second assertion can be obtained in the same way. □

At this point, we are able to prove the second principal result of this section, namely a generalization of [6, Theorem 1] to a natural number $p \geq 2$.

THEOREM 3.4. *Let B be an Archimedean semiprime f-algebra, $(A, *)$ be a uniformly complete almost f-algebra such that A is a vector sublattice of B and $2 \leq p$ be a natural number. There exists a positive operator T_p from $\Pi_p^B(A)$ into A such that*

$$T_p(f_1 \cdots f_p) = f_1 * \cdots * f_p \quad \textit{for all} \quad f_1, \ldots, f_p \in A.$$

PROOF. Let $0 \leq f, g \in A$ such that $f^p = g^p$. Since B is semiprime, $f = g$ [3, Proposition 2] and thus $f^{*p} = g^{*p}$. Therefore a map $T_p : \Sigma_p^B(A) \to A^+$ can be defined by putting $T_p(f^p) = f^{*p}$ for all $0 \leq f \in A$. Set now $h = (f^p + g^p)^{1/p}$. Lemma 3.3 implies that

$$T_p(f^p + g^p) = T_p\left[\left((f^p + g^p)^{1/p}\right)^p\right] = \left((f^p + g^p)^{1/p}\right)^{*p} = f^{*p} + g^{*p}$$

for all $0 \leq f, g \in A$. Hence, T_p is additive on $\Sigma_p^B(A)$. As $\Sigma_p^B(A)$ is the positive cone of the vector lattice $\Pi_p^B(A)$ (Theorem 3.2), T_p extends uniquely to a positive operator from $\Pi_p^B(A)$ into A [1, Theorem 1.7]. This extension is also denoted by T_p. We intend to show that $T_p(f_1 \cdots f_p) = f_1 * \cdots * f_p$ for all $f_1, \ldots, f_p \in A$. It suffices to prove this formula for $f_1, \ldots, f_p \geq 0$ (the general case follows straightforwardly from multilinearity). To do this, put $f = (f_1 \cdots f_p)^{1/p}$. Using again Lemma 3.3, we get

$$T(f_1 \cdots f_p) = T(f^p) = f^{*p} = f_1 * \cdots * f_p$$

and we are done. □

4. The kernel of the operator T_p

We already mentioned in the preliminaries that every f-algebra is an almost f-algebra but not conversely. It is a very natural question to ask under which supplementary condition any (Archimedean) almost f-algebra will be an f-algebra? The answer is given in the next proposition.

PROPOSITION 4.1. *Let A be an Archimedean almost f-algebra. The following statements are equivalent*:

(i) *A is an f-algebra.*

(ii) *A has the property that $0 \leq f, g, h$ and $fgh = 0$ imply $(fg) \wedge h = 0$.*

PROOF. (i) implies (ii). Assume that A is an f-algebra and let $0 \leq f, g, h \in A$ such that $fgh = 0$. So, $0 \leq ((fg) \wedge h)^2 \leq fgh = 0$ whence,

$$(fg) \wedge h \in N(A). \tag{4.1}$$

Moreover, $0 \leq (fg) \wedge h \leq fg \in N(A)^d$. Since $N(A)^d$ is an order ideal in A, we get

$$(fg) \wedge h \in N(A)^d. \tag{4.2}$$

Combining (4.1) with (4.2), we obtain $(fg) \wedge h = 0$.

(ii) implies (i). Let $0 \leq f, g, h \in A$ such that $g \wedge h = 0$. Since A is an almost f-algebra, $gh = 0$ and thus $fgh = 0$. It results from the hypothesis that $(fg) \wedge h = 0$ which implies that A is an f-algebra, as desired. $\square$

Let $(A, *)$ be an Archimedean almost f-algebra. With every $f \in A$, we associate π_f the element of $\mathcal{L}_b(A)$ defined by $\pi_f(g) = f * g$ for all $g \in A$. The set of all π_f is denoted by $\sigma(A)$. Putting $T = I_A$ in [7, Theorem 4.4], we deduce that $\sigma(A)$ is, with respect to the composition and the ordering inherited from $\mathcal{L}_b(A)$, an Archimedean f-algebra in its own right with the following supremum and infimum:

$$\pi_f \vee \pi_g = \pi_{f \vee g}, \quad \pi_f \wedge \pi_g = \pi_{f \wedge g}$$

for all $f, g \in A$. In other words, the map $\sigma : A \to \sigma(A)$, defined by $\sigma(f) = \pi_f$ for all $f \in A$, is a lattice homomorphism. Furthermore,

$$\begin{aligned}(\sigma(f)\sigma(g))(h) &= \sigma(f)(\sigma(g)(h)) = \sigma(f)(g * h) \\ &= f * g * h = \sigma(f * g)(h)\end{aligned}$$

for all $f, g, h \in A$ (the multiplication of $\sigma(A)$ is denoted by juxtaposition). Thus σ is also an algebra homomorphism.

In order to prove the following corollary, we recall that $f * g = (f \vee g) * (f \wedge g)$ for all $f, g \in A$ [2, Proposition 1.13].

COROLLARY 4.2. *Let* $(A, *)$ *be an Archimedean almost* f*-algebra,* $3 \leq p$ *be a natural number and* $f_1, \ldots, f_p \in A$*. If* $f_1 * \cdots * f_p = 0$ *then* $|f_1| * \cdots * |f_k| = 0$.

PROOF. The method of the proof is by induction on $p \geq 3$.

Let $f, g, h \in A$ satisfying $f * g * h = 0$. Since σ is a lattice and algebra homomorphism and $\sigma(A)$ is an f-algebra, we get

$$\begin{aligned}\pi_{|f|}\pi_{|g|}\pi_{|h|} &= \sigma(|f|)\sigma(|g|)\sigma(|h|) = |\sigma(f)|\,|\sigma(g)|\,|\sigma(h)| \\ &= |\sigma(f)\sigma(g)\sigma(h)| = |\sigma(f * g * h)| = 0 \text{ in } \sigma(A).\end{aligned}$$

Applying Proposition 4.1 to the f-algebra $\sigma(A)$, we obtain

$$\pi_{(|f|*|g|)\wedge|h|} = \pi_{(|f|*|g|)} \wedge \pi_{|h|} = (\pi_{|f|}\pi_{|g|}) \wedge \pi_{|h|} = 0.$$

Finally,

$$\begin{aligned}|f| * |g| * |h| &= [(|f| * |g|) \wedge |h|] * [(|f| * |g|) \vee |h|] \\ &= \pi_{(|f|*|g|)\wedge|h|}((|f| * |g|) \vee |h|) = 0.\end{aligned}$$

The case $p = 3$ being treated, assume that the desired result is true for a natural number $p \geq 3$ and let $f_1, \ldots, f_{p+1} \in A$ such that $f_1 * \cdots * f_{p+1} = 0$. By induction hypothesis, we obtain $(|f_1| * \cdots * |f_{p-1}|) * |f_p * f_{p+1}| = 0$. Consequently,

$$0 \leq |(|f_1| * \cdots * |f_{p-1}|) * f_p * f_{p+1}| \leq |f_1| * \cdots * |f_{p-1}| * |f_p * f_{p+1}| = 0.$$

Therefore,

$$(|f_1| * |f_2|) * (|f_1| * \cdots * |f_{p-1}|) * f_p * f_{p+1} = (|f_1| * \cdots * |f_{p-1}|) * f_p * f_{p+1} = 0.$$

Hence, again by induction hypothesis

$$|f_1| * \cdots * |f_{p+1}| = \big||f_1| * |f_2|\big| * |f_3| * \cdots * |f_{p+1}| = 0$$

which finishes the induction step. □

The next example shows that Corollary 4.2 fails in the case $p = 2$.

EXAMPLE 1. Take $A = C[-1, 1]$ with the usual operations and order. For every $f, g \in A$, we put

$$(f * g)(x) = \begin{cases} |x + 1/3|\, f(x)g(x) & \text{if } -1 \leq x \leq -1/3; \\ 0 & \text{if } -1/3 \leq x \leq 2/3; \\ \displaystyle\int_{2/3-x}^{x-2/3} f(t)g(t)\,dt & \text{if } 2/3 \leq x \leq 1. \end{cases}$$

It is easy to show that $(A, *)$ is an Archimedean almost f-algebra. Consider now $\alpha, \beta \in A$, defined by

$$\alpha(x) = \begin{cases} -1 & \text{if } -1 \leq x \leq -1/3; \\ 3x & \text{if } -1/3 \leq x \leq 1/3; \\ 1 & \text{if } 1/3 \leq x \leq 1 \end{cases}$$

and

$$\beta(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq -1/3; \\ 3x+1 & \text{if } -1/3 \leq x \leq 0; \\ -3x+1 & \text{if } 0 \leq x \leq 1/3; \\ 0 & \text{if } 1/3 \leq x \leq 1. \end{cases}$$

A straightforward computation shows that $\alpha * \beta = 0$. However,

$$(|\alpha| * |\beta|)(1) = 6 \int_0^{1/3} t(-3t+1)\, dt > 0.$$

At this point, let $(A, *)$ be a uniformly complete almost f-algebra, B be a semiprime f-algebra such that A is a vector sublattice of B and $3 \leq p$ be a natural number. According to Theorem 3.4, there exists a positive operator T_p from $\Pi_p^B(A)$ into A defined by $T_p(f_1 \cdots f_p) = f_1 * \cdots * f_p$ for all $f_1, \ldots, f_p \in A$. The topic of the following proposition is the kernel, $\ker(T_p)$, of T_p.

PROPOSITION 4.3. *Let B be an Archimedean semiprime f-algebra, $(A, *)$ be a uniformly complete almost f-algebra such that A is a vector sublattice of B and $3 \leq p$ be a natural number. Then $\ker(T_p)$ is an order ideal in $\Pi_p^B(A)$.*

PROOF. Let $f_1, \ldots, f_p \in A$ such that $T_p(f_1 \cdots f_p) = 0$. Then

$$f_1 * \cdots * f_p = T_p(f_1 \cdots f_p) = 0.$$

It follows from Corollary 4.2 that

$$T_p(|f_1 \cdots f_p|) = T_p(|f_1| \cdots |f_p|) = |f_1| * \cdots * |f_p| = 0.$$

The rest is obvious. □

The following example shows that Proposition 4.3 fails in the case $p = 2$.

EXAMPLE 2. It is known that $B = C[-1, 1]$, equipped by the usual algebra operations and order is a uniformly complete unital (and therefore semiprime) f-algebra.

Consider $(A, *)$ the almost f-algebra defined in Example 1. The operator T_2 is, in this case, defined by

$$T_2(f)(x) = \begin{cases} |x+1/3|\, f(x) & \text{if } \ -1 \leq x \leq -1/3; \\ 0 & \text{if } \ -1/3 \leq x \leq 2/3; \\ \displaystyle\int_{2/3-x}^{x-2/3} f(t)\, dt & \text{if } \ 2/3 \leq x \leq 1 \end{cases}$$

for all $f \in A$. It is shown in Example 1 that $T_2(\alpha\beta) = \alpha * \beta = 0$ and $T_2(|\alpha\beta|) = |\alpha| * |\beta| \neq 0$. Therefore, $\ker(T_2)$ is not an order ideal.

5. The range of the operator T_p

We start this section by mentioning some well-known facts concerning ordered vector spaces. Consider an operator T from a vector lattice E into an arbitrary vector space F such that $\ker(T)$ is an order ideal of E. The range $T(E)$ of T is a vector lattice with $T(E^+)$ as positive cone. The lattice operations in $T(E)$ are given by $\sup\{T(f), T(g)\} = T(f \vee g)$ and $\inf\{T(f), T(g)\} = T(f \wedge g)$. In other words, T is a lattice homomorphism from E into $T(E)$. Furthermore, if F is, in addition, an ordered vector space then the ordering on $T(E)$ inherited from F coincides with the initial ordering on $T(E)$ if and only if $F^+ \cap T(E) = T(E^+)$ (more details can be found in [14, Section 18]).

Throughout this section, $(A, *)$ designates a uniformly complete almost f-algebra, B an Archimedean semiprime f-algebra such that A is a vector sublattice of B and $3 \leq p$ a natural number. Recall that $\Pi_p^B(A) = \{f_1 \cdots f_p : f_k \in A, k = 1, \ldots, p\}$ is a vector sublattice of B with $\Sigma_p^B(A) = \{f^p : f \in A^+\}$ as positive cone (Theorem 3.2) and there exists a positive operator T_p from the $\Pi_p^B(A)$ into A defined by

$$T_p(f_1 \cdots f_p) = f_1 * \cdots * f_p \quad \text{for all} \quad f_1, \ldots, f_p \in A$$

(Theorem 3.4). Observe that

$$T_p(\Pi_p^B(A)) = \{f_1 * \cdots * f_p : f_k \in A, k = 1, \ldots, p\}$$

and

$$T_p(\Sigma_p^B(A)) = \{f^{*p} : f \in A^+\}.$$

Since $\ker(T_p)$ is an order ideal of $\Pi_p^B(A)$ (Proposition 4.3), $T_p(\Pi_p^B(A))$ is a vector lattice with $T_p(\Sigma_p^B(A))$ as positive cone and T_p is a lattice homomorphism from $\Pi_p^B(A)$ into $T_p(\Pi_p^B(A))$. We obtain the following theorem.

THEOREM 5.1. *Let B be an Archimedean semiprime f-algebra, $(A, *)$ be a uniformly complete almost f-algebra such that A is a vector sublattice of B and $3 \leq p$ be a natural number. Then*

(i) *$T_p(\Pi_p^B(A))$ is a vector lattice under the ordering inherited from A with $T_p(\Sigma_p^B(A))$ as positive cone.*
(ii) *T_p is a lattice homomorphism from $\Pi_p^B(A)$ to $T_p(\Pi_p^B(A))$. In particular, the absolute value of $f_1 * \cdots * f_p$ in $T_p(\Pi_p^B(A))$ is $|f_1| * \cdots * |f_p|$ for all $f_1, \ldots, f_p$.*
(iii) *$T_p(\Pi_p^B(A))$ is uniformly complete.*

PROOF. (i) The only point that needs some details is the fact that the initial ordering on $T_p(\Pi_p^B(A))$ coincides with the ordering inherited from A. To this end, we need to show that $T_p(\Sigma_p^B(A)) = A^+ \cap T_p(\Pi_p^B(A))$.

The inclusion $T_p(\Sigma_p^B(A)) \subset A^+ \cap T_p(\Pi_p^B(A))$ being obvious, prove the second one. Let $f_1, \ldots, f_p \in A$ such that $0 \leq f_1 * \cdots * f_p$ (in A) and consider the lattice and algebra homomorphism σ from A into the f-algebra $\sigma(A) = \{\pi_f : f \in A\}$ defined by $\sigma(f)(g) = \pi_f(g) = f * g$ for all $f, g \in A$ (see Section 4). Hence $0 \leq \sigma(f_1) \cdots \sigma(f_p)$ in $\sigma(A)$. As $\sigma(A)$ is an f-algebra, we get

$$\begin{aligned}\sigma(f_1 * \cdots * f_p) &= \sigma(f_1) \cdots \sigma(f_p) = |\sigma(f_1) \cdots \sigma(f_p)| = |\sigma(f_1)| \cdots |\sigma(f_p)| \\ &= \sigma(|f_1|) \cdots \sigma(|f_p|) = \sigma(|f_1| * \cdots * |f_p|)\end{aligned}$$

and thus $|f_1| * \cdots * |f_p| - f_1 * \cdots * f_p \in \ker(\sigma)$.

Using multilinearity, $|f_1| * \cdots * |f_p| - f_1 * \cdots * f_p$ is a sum of products (under $*$) of p positive elements of A. It follows from Lemma 3.3 that there exists $0 \leq g \in A$ such that $g^{*p} = |f_1| * \cdots * |f_p| - f_1 * \cdots * f_p$. Observe now that

$$\begin{aligned}\ker(\sigma) &= \{f \in A : \sigma(f) = 0\} = \{f \in A : \sigma(f)(g) = 0 \text{ for all } g \in A\} \\ &= \{f \in A : f * g = 0 \text{ for all } g \in A\} = N_2^*(A) \subset N^*(A).\end{aligned}$$

Therefore $g \in N^*(A) = N_3^*(A)$ and thus $g^{*3} = 0$. Since $p \geq 3$, $g^{*p} = 0$. This gives, *via* Lemma 3.3, that

$$f_1 * \cdots * f_p = |f_1| * \cdots * |f_p| = ((|f_1| \cdots |f_p|)^{1/p})^{*p} \in T_p(\Sigma_p^B(A))$$

which is the desired result.

(ii) This follows immediately from the introduction made in the beginning of this section.

(iii) This is a straightforward inference from the fact that the range of a lattice homomorphism defined on a uniformly complete vector lattice is also a uniformly complete vector lattice [13, Theorem 59.3]. ☐

In the following example, we show that the previous result is not true in the case $p = 2$.

EXAMPLE 3. The set $B = C[-1, 1]$ is a uniformly complete unital (and then semiprime) f-algebra under the usual algebra operations and order. Take $A = B$ and put

$$(f * g)(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0; \\ \displaystyle\int_{-1}^{x-1} f(t)g(t)\,dt & \text{if } 0 \leq x \leq 1 \end{cases}$$

for all $f, g \in A$. A simple calculation shows that $(A, *)$ is an almost f-algebra under the multiplication $*$. In this case, T_2 is defined by

$$T_2(f)(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0; \\ \displaystyle\int_{-1}^{x-1} f(t)\,dt & \text{if } 0 \leq x \leq 1 \end{cases}$$

for all $f \in A$. An element $f \in T_2(\Pi_2^B(A))$ if and only if $f(x) = 0$ for all $x \in [-1, 0]$ and the restriction of f to $[0, 1]$ belongs to $C^1[0, 1]$. Therefore $T_2(\Pi_2^B(A))$ can not be a vector lattice under the ordering inherited from A.

We said in Theorem 5.1 (i) that if $p \geq 3$ then $T_p(\Pi_p^B(A))$ is a vector lattice under the ordering inherited from A. However, in general, $T_p(\Pi_p^B(A))$ is not a vector sublattice of A as it shown in the next example.

EXAMPLE 4. Consider the uniformly complete unital f-algebra $B = C([-1, 1])$. Take $A = B$ and define $\alpha \in A$ by

$$\alpha(x) = \begin{cases} -x & \text{if } -1 \leq x \leq 0; \\ 0 & \text{if } 0 \leq x \leq 1. \end{cases}$$

For $f, g \in A$, we put

$$(f * g)(x) = \begin{cases} \alpha(x) f(x) g(x) & \text{if } -1 \leq x \leq 0; \\ \displaystyle\int_{-x}^{0} f(t)g(t)\,dt & \text{if } 0 \leq x \leq 1. \end{cases}$$

It is not hard to show that A is a uniformly complete almost f-algebra under the multiplication $*$. In this case, the operator T_3 is defined by

$$T_3(f)(x) = \begin{cases} \alpha(x)^2 f(x) & \text{if } -1 \leq x \leq 0; \\ \displaystyle\int_{-x}^{0} \alpha(t) f(t)\,dt & \text{if } 0 \leq x \leq 1 \end{cases}$$

for all $f \in A$. Define now $\varphi \in A$ by $\varphi(x) = 2x + 1$ for all $x \in [-1, 1]$. By a simple calculation, we get

$$|T_3(\varphi)|(1) = \frac{1}{10} \neq (|\varphi| * |\varphi| * |\varphi|)(1) = \frac{1}{8}.$$

This translates that the absolute value in $T_3(\Pi_3^B(A))$ doesn't coincide with the absolute value in A.

Assume now that $(A, *)$ is, in addition, a d-algebra (in other words, $(A, *)$ is an uniformly complete commutative d-algebra) or, in particular, an f-algebra. Then the situation improves, in this case the property $|f * g| = |f| * |g|$ holds in A. We obtain straightforwardly the following corollary.

COROLLARY 5.2. *Let B be an Archimedean semiprime f-algebra, $(A, *)$ be a uniformly complete commutative d-algebra (respectively, f-algebra) such that A is a vector sublattice of B and $2 \leq p$ be a natural number. Then*

(i) *$T_p(\Pi_p^B(A))$ is a vector sublattice of A with $T_p(\Sigma_p^B(A))$ as positive cone.*
(ii) *T_p is a lattice homomorphism from $\Pi_p^B(A)$ into A.*
(iii) *$T_p(\Pi_p^B(A))$ is uniformly complete.*

Note that the case $p = 2$ in the previous corollary follows immediately from [6, Corollary 2].

In the last paragraph of this section, $(A, *)$ is a given uniformly complete ℓ-algebra. For every natural number $p \geq 3$, we put

$$\Pi_p^*(A) = \{f_1 * \cdots * f_p : f_k \in A, k = 1, \ldots, p\}$$

and $\Sigma_p^*(A) = \{f^{*p} : f \in A^+\}$.

A classical result of vector lattices theory is that there exists an Archimedean unital (and then semiprime) f-algebra having A as a vector sublattice (we refer to [9] for a Zorn Lemma-free proof of this existence or [1] for an approach using orthomorphisms). Let B be such an f-algebra, whose multiplication is denoted by juxtaposition. According to Theorem 5.1 and Corollary 5.2, we obtain the following result.

COROLLARY 5.3. *Let $(A, *)$ be a uniformly complete ℓ-algebra and p be a natural number.*

(i) *If $(A, *)$ is an almost f-algebra and $p \geq 3$ then $\Pi_p^*(A)$ is a uniformly complete semiprime f-algebra under the ordering and the multiplication inherited from A with $\Sigma_p^*(A)$ as positive cone and with the following supremum and infimum.*

$$f^{*p} \wedge_p g^{*p} = (f \wedge g)^{*p} \quad \textit{and} \quad f^{*p} \vee_p g^{*p} = (f \vee g)^{*p} \quad \textit{for all } 0 \leq f, g \in A.$$

The absolute value in $\Pi_p^*(A)$ *is given by* $|f_1 * \cdots * f_p|_p = |f_1| * \cdots * |f_p|$, *for all* $f_1, \ldots, f_p \in A$.

(ii) *If* $(A, *)$ *is a commutative* d*-algebra and* $p \geq 3$ *then* $\Pi_p^*(A)$ *is a uniformly complete semiprime* f*-subalgebra of* A.

(iii) *If* $(A, *)$ *is a commutative* d*-algebra then* $\Pi_2^*(A)$ *is a uniformly complete* f*-subalgebra of* A.

(iv) *If* $(A, *)$ *is an* f*-algebra and* $p \geq 2$ *then* $\Pi_p^*(A)$ *is a uniformly complete semiprime* f*-subalgebra of* A.

PROOF. (i) We prove only that $\Pi_p^*(A)$ is a semiprime f-algebra, the rest of the proof is a simple inference from Theorem 5.1. Obviously, $\Pi_p^*(A)$ is an ℓ-algebra under the multiplication inherited from A. Let $0 \leq f, g \in A$ such that $f^{*p} \wedge_p g^{*p} = 0$. Hence $(f \wedge g)^{*p} = 0$ and $f \wedge g \in N^*(A) = N_3^*(A)$. Consequently,

$$\begin{aligned} f^{*p} * g^{*p} &= (f * g) * (f * g)^{*(p-1)} \\ &= (f \wedge g) * (f \vee g) * (f * g)^{*(p-1)} = 0. \end{aligned}$$

We deduce that $(\Pi_p^*(A), *)$ is an almost f-algebra. It suffices therefore to show that it is semiprime. To do this, choose $0 \leq f \in A$ such that $(f^{*p})^n = 0$ for some $n \in \{1, 2, \ldots\}$. This implies that $f^{*pn} = 0$ and thus $f \in N^*(A) = N_3^*(A)$. We infer that $f^{*3} = 0$ and, as $p \geq 3$, $f^{*p} = 0$.

(ii) We obtain this assertion by combining (i) with Corollary 5.2.

(iii) The fact that $\Pi_2^*(A)$ is a uniformly complete vector lattice follows immediately from Corollary 5.2. For the remainder, let $f, g, h \in A^+$ such that $f^{2*} \wedge g^{2*} = 0$. Therefore,

$$\begin{aligned} 0 &\leq (h^{2*} f^{2*}) \wedge g^{2*} = ((hf) \wedge g)^{2*} \\ &\leq h * f * g = h * (f \vee g) * (f \wedge g). \end{aligned}$$

Observe now that $(f \wedge g)^{2*} = f^{2*} \wedge g^{2*} = 0$. Hence $f \wedge g \in N^*(A)$ and thus $h * (f \vee g) * (f \wedge g) = 0$. Finally $(h^{2*} f^{2*}) \wedge g^{2*} = 0$. We deduce that $\Pi_2^*(A)$ is an f-algebra.

(iv) It only remains that $\Pi_2^*(A)$ is semiprime. For this, take $0 \leq f \in A$ such that $(f^{*2})^n = 0$ for some $n \in \{1, 2, \ldots\}$. Hence $f \in N^*(A) = N_2^*(A)$. Finally, $f^{*2} = 0$ and we may conclude. $\square$

Remark that Example 4 shows that, in general, $\Pi_p^*(A)$ is not a vector sublattice of A when $(A, *)$ is a uniformly complete almost f-algebra.

In the end of this paper, we present an example of a uniformly complete commutative d-algebra $(A, *)$ such that, contrary to the f-algebras case, the f-subalgebra $\Pi_2^*(A)$ is not semiprime

EXAMPLE 5. Let A be the set of all real sequences with the usual addition, scalar multiplication, partial ordering and the multiplication $*$ defined by

$$(u_n)_{n\geq 0} * (v_n)_{n\geq 0} = (w_n)_{n\geq 0}$$

with $w_0 = u_2 v_2$, $w_1 = u_1 v_1$ and $w_n = 0$ for all $n \geq 2$. It is easily verified that $(A, *)$ is a uniformly complete commutative d-algebra and $\Pi_2^*(A) = \{(u_n)_{n\geq 0} : u_n = 0 \text{ for all } n \geq 2\}$. Now that $(1, 0, \dots) \in \Pi_2^*(A)$ and $(1, 0, \dots)^{*2} = 0$. Therefore $\Pi_2^*(A)$ is not semiprime.

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