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ENGEL SERIES EXPANSIONS OF LAURENT SERIES AND HAUSDORFF DIMENSIONS

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Abstract

For any positive integer $q \geq 2$, let $\mathbb{F}_q$ be a finite field with q elements, $\mathbb{F}_q((z^{-1}))$ be the field of all formal Laurent series $x = \sum_{n=v}^{\infty} c_n z^{-n}$ in an indeterminate z, I denote the valuation ideal $z^{-1}\mathbb{F}_q[[z^{-1}]]$ in the ring of formal power series $\mathbb{F}_q[[z^{-1}]]$ and $\mathbf{P}$ denote probability measure with respect to the Haar measure on $\mathbb{F}_q((z^{-1}))$ normalized by $\mathbf{P}(I) = 1$. For any $x \in I$, let the series $\sum_{n=1}^{\infty} 1/(a_1(x)a_2(x)\cdots a_n(x))$ be the Engel expansion of Laurent series of x. Grabner and Knopfmacher have shown that the $\mathbf{P}$-measure of the set $A(\alpha) = \{x \in I : \lim_{n\to\infty} \deg a_n(x)/n = \alpha\}$ is 1 when $\alpha = q/(q-1)$, where $\deg a_n(x)$ is the degree of polynomial $a_n(x)$. In this paper, we prove that for any $\alpha \geq 1$, $A(\alpha)$ has Hausdorff dimension 1. Among other things we also show that for any positive integer m, the following set $B(m) = \{x \in I : \deg a_{n+1}(x) - \deg a_n(x) = m \text{ for any } n \geq 1\}$ has Hausdorff dimension 1.

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1. Introduction

The most frequently applied operation of mathematics is series representation of 'numbers'. As a matter of fact, in all practical applications we replace arbitrary 'numbers' by their decimal expansions after a certain number of 'digits'. Recently Knopfmacher and Knopfmacher [8, 9] introduced and studied some properties of various unique expansions of formal Laurent series over a field $\mathbb{F}$, as the sums of reciprocals of polynomials, involving 'digits' $a_1, a_2, \ldots$ lying in a polynomial ring $\mathbb{F}[z]$ over $\mathbb{F}$. In particular, one of these expansions was constructed to be analogous to the so-called Engel expansion of a real number, discussed in Galambos [5]. A number

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of famous expansions including those of Euler and the Rogers-Ramanujan identities are, in fact, special cases of Engel expansions of formal Laurent series. Andrews, Knopfmacher and Knopfmacher [1] and Andrews, Knopfmacher and Paule [2] have shown how so called Engel expansions of formal Laurent series can be used to give new and exciting proofs of the Rogers-Ramanujan and related identities.

Erdös, Rényi and Szüsz [3] (see also Rényi [11] or Galambos [5]) have studied the metric properties for real Engel expansions, and similar metric results for Engel expansions of Laurent series have been derived by Grabner and Knopfmacher [6]. The aim of this paper is to discuss the fractal properties of sets related to Engel expansions of Laurent series. The corresponding results for real Engel expansions have been obtained by Liu and the author [10].

2. Engel expansions of Laurent series

In order to explain the conclusions, we first fix some notations and describe Engel expansions to be considered.

Let $\mathscr{L} = \mathbb{F}((z^{-1}))$ denote the field of all formal Laurent series $A = \sum_{n=v}^{\infty} c_n z^{-n}$ in an indeterminate z, with coefficients c_n all lying in a given field $\mathbb{F}$. (We consider $\mathbb{F}((z^{-1}))$ rather than $\mathbb{F}((z))$ as in [8] and [9] since it turns out to be more convenient for stating our results.)

We also consider the ring $\mathbb{F}[z]$ of polynomials in z with coefficients in $\mathbb{F}$.

If $c_v \neq 0$, we call $v = v(A)$ the order of A above, and define the *norm* (or *valuation*) of A to be $\|A\| = q^{-v(A)}$, where initially $q > 1$ may be an arbitrary constant, but later will be chosen as $q = \sharp(\mathbb{F})$, the cardinality of $\mathbb{F}$, if $\mathbb{F}$ is finite. Letting $v(0) = +\infty$, $\|0\| = 0$, one then has (see for example, Jones and Thron [7, Chapter 5]).

$$\begin{cases} \|A\| \geq 0 \quad \text{with } \|A\| = 0 \text{ if and only if } A = 0, \\ \|AB\| = \|A\| \cdot \|B\| \quad \text{and} \\ \|\alpha A + \beta B\| \leq \max(\|A\|, \|B\|) \quad \text{for non-zero } \alpha, \beta \in \mathbb{F} \\ \qquad\qquad \text{with equality when } \|A\| \neq \|B\|. \end{cases}$$

From above, the norm $\|\cdot\|$ is non-Archimedean, and it is well known that $\mathscr{L}$ forms a complete metric space under the metric ρ such that $\rho(A, B) = \|A - B\|$.

REMARK 1. Since the metric ρ is non-Archimedean, it follows that each point of a disc may be considered its center and thus if two discs intersect, then one contains the other.

For $A = \sum_{n=v}^{\infty} c_n z^{-n} \in \mathscr{L}$, let $[A] = \sum_{v \leq n \leq 0} c_n z^{-n} \in \mathbb{F}[z]$, and refer to $[A]$ the integral part of $A \in \mathscr{L}$. Then $-v = -v(A)$ is the degree $\deg[A]$ of $[A]$ relative to z.

Given $A \in \mathscr{L}$, now note that $[A] = a_0 \in \mathbb{F}[z]$ if and only if $v(A_1) \geq 1$, where $A_1 = A - a_0$. As in [8, 9], if $A_n \neq 0$ ($n > 0$) is already defined, we then let $a_n = [1/A_n]$ and put $A_{n+1} = a_n A_n - 1$. If $A_n = 0$ or $a_n = 0$ for some n, this recursive process stops. It was shown in [8, 9] that this algorithm leads to a finite or convergent (relative to ρ) Engel series expansion of Laurent series.

THEOREM 1 ([8, 9]). *Every $x \in \mathscr{L}$ has a finite or convergent (relative to ρ) series expansion of the form*

$$x = a_0(x) + \frac{1}{a_1(x)} + \frac{1}{a_1(x)a_2(x)} + \cdots + \frac{1}{a_1(x)a_2(x)\cdots a_n(x)} + \cdots, \tag{1}$$

where $a_n(x) \in \mathbb{F}[z]$, $a_0(x) = [x]$, and

$$\deg a_n(x) \geq n \quad \text{and} \quad \deg(a_{n+1}(x)) \geq \deg(a_n(x)) + 1 \quad \text{for } n \geq 1. \tag{2}$$

The series (1) is unique for x subject to the preceding conditions on the 'digits' $a_n(x)$.

From now on we assume $q \geq 2$ is a positive integer and $\mathbb{F} = \mathbb{F}_q$ is a finite field with exactly q elements. Let I denote the valuation ideal $z^{-1}\mathbb{F}[[z^{-1}]]$ in the ring of formal power series $\mathbb{F}_q[[z^{-1}]]$, then I is compact under the metric ρ. Let $\mathbf{P}$ denote probability measure with respect to the Haar measure on $\mathscr{L}$ normalized by $\mathbf{P}(I) = 1$. The Haar measure on I is the product measure on $\prod_{n=1}^{\infty} \mathbb{F}_q$ defined by $\mathbf{P}(\{x\}) = q^{-1}$ for each factor and any element $x \in \mathbb{F}_q$.

Analogous to Engel series representation for real numbers, Grabner and Knopfmacher ([6]) have studied metric properties of Engel properties of Engel expansions of Laurent series and proved the following result.

THEOREM 2 ([6]). *For any $x \in I$, let*

$$x = \frac{1}{a_1(x)} + \frac{1}{a_1(x)a_2(x)} + \cdots + \frac{1}{a_1(x)a_2(x)\cdots a_n(x)} + \cdots$$

be the Engel expansion of Laurent series of x. Then

(i) *for almost all $x \in I$,*

$$\|a_n(x)\|^{1/n} \to q^{q/(q-1)} \quad \text{as } n \to \infty. \tag{3}$$

(ii) *For almost all $x \in I$,*

$$\limsup_{n\to\infty} \frac{\deg a_{n+1}(x) - \deg a_n(x)}{\log_q n} = 1, \tag{4}$$

and

$$\liminf_{n\to\infty} \deg a_{n+1}(x) - \deg a_n(x) = 1. \tag{5}$$

(iii) *For almost all* $x \in I$,

$$\|x - p_n/q_n\| = q^{-(qn^2/2(q-1))(1+o(1))} \quad \text{as } n \to \infty, \tag{6}$$

where $p_n/q_n = \sum_{k=1}^{n} 1/(a_1(x)a_2(x)\cdots a_k(x))$, $q_n = a_1(x)a_2(x)\cdots a_n(x)$.

The definition of Hausdorff measure on I is the same as on $\mathbb{R}^n$. Given $s > 0$ and a subset E of I, the Hausdorff s-measure is given by

$$\mathscr{H}^s(E) = \lim_{\delta \to 0} \left\{ \inf \sum_j (\operatorname{diam} D_j)^s \right\},$$

where the infimum is over all covers of E by discs D_j of diameter (in the metric ρ) at most δ and diam denotes the diameter of a set. The Hausdorff dimension of E is defined by $\dim E = \inf\{s : \mathscr{H}^s(E) = 0\}$.

REMARK 2. From the definition of Hausdorff dimension, it is easy to see that for any Borel subset E of I, if $\mathbf{P}(E) > 0$, then $\dim E = 1$.

Note that for any $x \in \mathbb{F}_q[z]$, $\|x\| = q^{\deg x}$, thus for almost all $x \in I$, the formula (3) is equivalent to

$$\frac{1}{n} \deg a_n(x) \to \frac{q}{q-1} \quad \text{as } n \to \infty. \tag{7}$$

Also note that since $\deg a_n(x) \geq n$, it is natural to consider the following set

$$A(\alpha) = \left\{ x \in I : \lim_{n \to \infty} \frac{1}{n} \deg a_n(x) = \alpha \right\} \tag{8}$$

for any $\alpha \geq 1$. In Section 3, we discuss the Hausdorff dimension of $A(\alpha)$ and obtain the following result.

THEOREM 3. *For any* $\alpha \geq 1$, $\dim A(\alpha) = 1$.

If α is an integer in Theorem 3, we can get the following quite strong result. For any positive integer m, let

$$B(m) = \{x \in I : \deg a_{n+1}(x) - \deg a_n(x) = m \text{ for any } n \geq 1\}. \tag{9}$$

THEOREM 4. *For any positive integer* m, $\dim B(m) = 1$.

As corollaries of Theorem 4, both the Hausdorff dimension of the set where (3), (4) and (5) fail and the Hausdorff dimension of the set where (6) fails are 1.

3. Proof of Theorem 3 and Theorem 4

The aim of this section is to prove the main results of this paper.

We first state the mass distribution principle [4, Proposition 4.2] that will be used later.

LEMMA 1. *Suppose $E \subset I$ and μ is a measure with $\mu(E) > 0$. If there exist constants $c > 0$ and $\delta > 0$ such that $\mu(D) \le c(\operatorname{diam} D)^s$ for all disc D with diameter* $\operatorname{diam} D \le \delta$. *Then* $\dim E \ge s$.

PROOF OF THEOREM 3. For any $n \ge 1$, let $\mathbb{F}_q^{(n)}[z]$ denote the polynomials in $\mathbb{F}_q[z]$ with degree n, that is,

$$\mathbb{F}_q^{(n)}[z] = \left\{ x \in \mathbb{F}_q[z] : x = \sum_{k=0}^{n} c_k z^k, c_i \in \mathbb{F}_q, (1 \le i \le n), \text{ and } c_n \ne 0 \right\}.$$

For any $n \ge 1$ and $b_k \in \mathbb{F}_q^{(\operatorname{int}(k\alpha))}[z]$, $k = 1, \dots, n$, where $\operatorname{int}(a)$ denotes the integer part for any real number a, define

$$J(b_1, \dots, b_n) = \{x \in I : a_1(x) = b_1, \dots, a_n(x) = b_n\}.$$

We call $J(b_1, \dots, b_n)$ an n-order disc. Note that $\operatorname{int}((n+1)\alpha) \ge \operatorname{int}(n\alpha) + 1$ for any $n \ge 1$, by Theorem 1, we have $J(b_1, \dots, b_n)$ is a disc with center at $\sum_{k=1}^{n} 1/(b_1 \cdots b_k)$ and diameter $q^{-\sum_{k=1}^{n} \operatorname{int}(k\alpha) - \operatorname{int}(n\alpha) - 1}$. Also by Theorem 1, we have

(i) If $(b_1, \dots, b_n) \ne (b_1', \dots, b_n')$, $J(b_1, \dots, b_n) \bigcap J(b_1', \dots, b_n') = \emptyset$.
(ii) $J(b_1, \dots, b_n, b_{n+1}) \subset J(b_1, \dots, b_n)$ for any $n \ge 1$.

Let $E_n = \bigcup J(b_1, \dots, b_n)$, where the union is over all $b_k \in \mathbb{F}_q^{(\operatorname{int}(k\alpha))}[z], k = 1, \dots, n$. Then

$$E_n = \{x \in I : \deg a_1(x) = \operatorname{int}(\alpha), \dots, \deg a_n(x) = \operatorname{int}(n\alpha)\}, \tag{10}$$

and E_n consists of $(q-1)^n q^{\sum_{k=1}^{n} \operatorname{int}(k\alpha)}$ disjoint discs with diameter $q^{-\sum_{k=1}^{n} \operatorname{int}(k\alpha) - \operatorname{int}(n\alpha) - 1}$. Define $E = \bigcap_{n=1}^{\infty} E_n$. It is obvious that

$$E = \{x \in I : \deg a_k(x) = \operatorname{int}(k\alpha) \text{ for any } k \ge 1\}. \tag{11}$$

Thus $E \subset A(\alpha)$. Now we estimate the Hausdorff dimension of E.

Let μ be a mass distribution supported on E such that for any $n \ge 1$ and $b_k \in \mathbb{F}_q^{(\operatorname{int}(k\alpha))}[z], k = 1, \dots, n$,

$$\mu(J(b_1, \dots, b_n)) = (q-1)^{-n} q^{-\sum_{k=1}^{n} \operatorname{int}(k\alpha)}. \tag{12}$$

For any $\varepsilon > 0$, choose n_0 large enough such that for any $n \geq n_0$,

$$n^2 - 3n + 2 \geq (n^2 + 3n + 2)(1 - \varepsilon). \tag{13}$$

For any $x \in I$ and $m \geq \mathrm{int}(\alpha) + \mathrm{int}(2\alpha) + \cdots + \mathrm{int}(n_0\alpha)$, choose $k \geq n_0$ such that

$$\mathrm{int}(\alpha) + \cdots + \mathrm{int}(k\alpha) \leq m < \mathrm{int}(\alpha) \cdots + \mathrm{int}((k+1)\alpha).$$

This implies

$$\begin{aligned} &-\mathrm{int}(\alpha) - \cdots - \mathrm{int}((k+1)\alpha) \\ &\quad < -m \leq -\mathrm{int}(\alpha) - \cdots - \mathrm{int}((k-1)\alpha) - \mathrm{int}((k-1)\alpha) - 1, \end{aligned} \tag{14}$$

thus $B(x, q^{-m}) := \{y \in I : \|y - x\| \leq q^{-m}\}$ can intersect at most one $(k-1)$-order disc. In fact, if there exist $b_i, b_i' \in \mathbb{F}_q^{(\mathrm{int}(i\alpha))}[z]$, $1 \leq i \leq k-1$ such that $(b_1, b_2, \ldots, b_{k-1}) \neq (b_1', b_2', \ldots, b_{k-1}')$, $B(x, q^{-m}) \bigcap J(b_1, b_2, \ldots, b_{k-1}) \neq \emptyset$ and $B(x, q^{-m}) \bigcap J(b_1', b_2', \ldots, b_{k-1}') \neq \emptyset$, then $B(x, q^{-m}) \subset J(b_1, b_2, \ldots, b_{k-1})$ and $B(x, q^{-m}) \subset J(b_1', b_2', \ldots, b_{k-1}')$. Thus by Remark 1,

$$\begin{aligned} &J(b_1', \ldots, b_{k-1}') \subset J(b_1, \ldots, b_{k-1}) \quad \text{or} \\ &J(b_1, \ldots, b_{k-1}) \subset J(b_1', \ldots, b_{k-1}'), \end{aligned}$$

and this contradicts $J(b_1, \ldots, b_{k-1}) \bigcap J(b_1', \ldots, b_{k-1}') = \emptyset$. Therefore, by (13) and (14),

$$\begin{aligned} \mu(B(x, q^{-m})) &\leq (q-1)^{-(k-1)} q^{-\mathrm{int}(\alpha) - \mathrm{int}(2\alpha) - \cdots - \mathrm{int}((k-1)\alpha)} \\ &\leq q^{-\mathrm{int}(\alpha) - \mathrm{int}(2\alpha) - \cdots - \mathrm{int}((k-1)\alpha)} \leq q^{-(k(k-1)/2)\alpha + k - 1} \leq q^{-(k(k-1)/2)\alpha + (k-1)\alpha} \\ &= q^{(-k^2 + 3k - 2/2)\alpha} \leq q^{-((k+1)(k+2)/2)(1-\varepsilon)\alpha} \leq \left(\mathrm{diam}\, B(x, q^{-m})\right)^{1-\varepsilon}. \end{aligned}$$

By Lemma 1, we have $\dim E \geq 1 - \varepsilon$. Since ε is arbitrary, we have $\dim E = 1$. Note that $E \subset A(\alpha)$, thus $\dim A(\alpha) = 1$. The proof of Theorem 3 is finished. □

Proof of Theorem 4. For any positive integer m, let $\alpha = m$ and E_n, E be constructed in the same way as in the proof of Theorem 3. Then $E \subset B(m)$ and $\dim E = 1$ by the proof of Theorem 3. Thus $\dim B(m) = 1$ and we complete the proof of Theorem 4. □

By Theorem 4, we can get the following corollaries immediately.

Corollary 1. *The Hausdorff dimension of the set where* (3), (4) *and*(5) *fail is* 1.

Corollary 2. *For any positive integer m, let*

$$C(m) = \left\{x \in I : \|x - p_n/q_n\| = q^{-(m(n+1)(n+2))/2} \text{ for any } n \geq 1\right\}.$$

Then $\dim C(m) = 1$. *In particular, the Hausdorff dimension of the set where* (6) *fails is* 1.

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References

[1] G. E. Andrews, A. Knopfmacher and J. Knopfmacher, 'Engel expansions and the Rogers-Ramanujan identities', *J. Number Theory (2)* **80** (2000), 273–290.
[2] G. E. Andrews, A. Knopfmacher and P. Paule, 'An infinite family of Engel expansions of Rogers-Ramanujan type', *Adv. Appl. Math. (1)* **25** (2000), 2–11.
[3] P. Erdös, A. Rényi and P. Szüsz, 'On Engel's and Sylvester's series', *Ann. Univ. Sci. Budapest, Sectio Math.* **1** (1958), 7–32.
[4] K. J. Falconer, *Fractal geometry, mathematical foundations and applications* (John Wiley, New York, 1990).
[5] J. Galambos, *Representations of real numbers by infinite series*, Lecture Notes in Math. 502 (Springer, Berlin, 1976).
[6] P. J. Grabner and A. Knopfmacher, 'Metric properties of Engel series expansions of Laurent series', *Math. Slovaca (3)* **48** (1998), 233–243.
[7] W. B. Jones and W. J. Thron, *Continued fractions* (Addison-Wesley, 1980).
[8] A. Knopfmacher and J. Knopfmacher, 'Inverse polynomial expansions of Laurent series', *Constr. Approx. (4)* **4** (1988), 379–389.
[9] ———, 'Inverse polynomial expansions of Laurent series, II', *J. Comput. Appl. Math.* **28** (1989), 249–257.
[10] Y. Y. Liu and J. Wu, 'Hausdorff dimensions in Engel expansions', *Acta Arith. (1)* **99** (2001), 79–83.
[11] A. Renyi, 'A new approach to the theory of Engel's series', *Ann. Univ. Sci. Budapest, Sectio Math.* **5** (1962), 25–32.

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