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MULTIPLICITIES OF HIGHER LIE CHARACTERS

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Abstract

The higher Lie characters of the symmetric group S_n arise from the Poincaré-Birkhoff-Witt basis of the free associative algebra. They are indexed by the partitions of n and sum up to the regular character of S_n. A combinatorial description of the multiplicities of their irreducible components is given. As a special case the Kraśkiewicz-Weyman result on the multiplicities of the classical Lie character is obtained.

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1. Introduction

At the beginning of the last century Schur studied the structure of the tensor algebra $T(V)$ over a finite dimensional K-vector space V as a $\mathrm{GL}(V)$-module. In his thesis ([13]) and a famous subsequent paper ([14]) he was able to describe the decomposition of the homogeneous components

$$T_n(V) := \underbrace{V \otimes \cdots \otimes V}_{n}$$

of degree n in $T(V)$ into irreducible $\mathrm{GL}(V)$-modules using the irreducible representations of the symmetric group S_n. The usual Lie bracketing $[x, y] := xy - yx$ turns $T(V)$ into a Lie algebra. The Lie subalgebra $L(V)$ generated by V is free over any basis of V by a classical result of Witt ([17]), and $L_n(V) := T_n(V) \cap L(V)$ is a $\mathrm{GL}(V)$-submodule of $T_n(V)$ for all n. Let $q = q_1. \dots .q_k$ be a partition of n, that is, $q_1 \geq \cdots \geq q_k$ and $q_1 + \cdots + q_k = n$. Then we define

$$L_q(V) := \left\langle \sum_{\pi \in S_k} P_{1\pi} \cdots P_{k\pi} \;\middle|\; P_i \in L_{q_i}(V) \text{ for } 1 \leq i \leq k \right\rangle_K .$$

By the Poincaré-Birkhoff-Witt theorem, $T_n(V)$ is the direct sum of these subspaces:

$$T_n(V) = \bigoplus_{q \vdash n} L_q(V), \tag{1}$$

and this decomposition is $\mathrm{GL}(V)$-invariant.

Meanwhile, different families of idempotents e_q in the group algebra KS_n indexed by partitions have been introduced such that $L_q(V) \cong e_q T_n(V)$ for all q (see, for example, [2, 3, 11]). For any decomposition $e_q KS_n = \bigoplus_p a_{q,p} M_p$ into irreducible S_n-modules, we now have

$$L_q(V) = e_q T_n(V) \cong e_q KS_n \otimes_{KS_n} T_n(V) = \bigoplus_p a_{q,p}(M_p \otimes_{KS_n} T_n(V)).$$

In this decomposition, by Schur's fundamental result, $M_p \otimes_{KS_n} T_n(V)$ is either 0 or an irreducible $\mathrm{GL}(V)$-module. Hence the $\mathrm{GL}(V)$-module structure of $L_q(V)$ is completely determined by the multiplicities $a_{q,p}$ of the *higher Lie module* $e_q KS_n$ of S_n. In this vein, for the special case of $q = n$, the problem of describing the $\mathrm{GL}(V)$-module structure of $L_n(V)$ formulated by Thrall ([16]) could finally be solved in a satisfying way by works of Klyachko ([8]) and Kraśkiewicz and Weyman ([9]).

The *higher Lie characters* λ_q of S_n corresponding to the modules $e_q KS_n$ sum up to the regular character of S_n, by (1), and it is natural to ask for their multiplicities for arbitrary q. In this paper, a combinatorial description of these multiplicities is given in terms of alternating sums of numbers of standard tableaux with certain major index properties (Section 3). For $q = n$, we obtain the Kraśkiewicz-Weyman result mentioned above. Our approach is based on a generalization of Klyachko's result (Section 2) combined with the calculus of noncommutative character theory introduced in [6] (Section 4).

2. The reduction to partitions of block type

Let q be a partition of n. The higher Lie character λ_q is induced by a certain linear character of the centralizer of an element of cycle type q in S_n. For $q = n$, this result is due to Klyachko ([8]). In full generality, it is implicitly contained in [1] for the first time (for details, see [12, Section 8.5]) and will be briefly recalled in two steps in this section.

Let $\mathbb{N}$ ($\mathbb{N}_0$, respectively) be the set of all positive (nonnegative, respectively) integers and $\underline{n} := \{k \in \mathbb{N} \mid k \leq n\}$ for all $n \in \mathbb{N}_0$. Let $\mathbb{N}^*$ be a free monoid over the alphabet $\mathbb{N}$. We write $q.r$ for the concatenation product of $q, r \in \mathbb{N}^*$ in order to avoid confusion with the ordinary product in $\mathbb{N}$. Accordingly, we denote by $d^{.k}$ the k-th power of a letter $d \in \mathbb{N}$ in $\mathbb{N}^*$, for all $k \in \mathbb{N}_0$. If $n \in \mathbb{N}$ and $q = q_1. \ldots .q_k \in \mathbb{N}^*$ such that

$q_1 + \cdots + q_k = n$, we say that q is a *composition* of n of *length* $|q| := k$, and write $q \models n$. If, additionally, $q_1 \geq \cdots \geq q_k$ and hence q is a partition of n, we write $q \vdash n$.

Let K be a field of characteristic 0 containing a primitive n-th root of unity ε_n for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}_0$, we denote by $\mathrm{Cl}_K(S_n)$ the ring of class functions of the symmetric group S_n. Let C_q be the conjugacy class consisting of all permutations π whose cycle partition $z(\pi)$ is a rearrangement of q, for all $q \in \mathbb{N}^*$. Let $\mathrm{ch}_q \in \mathrm{Cl}_K(S_n)$ such that $(\chi, \mathrm{ch}_q)_{S_n} = \chi(C_q)$ is the value of χ on any element $\pi \in C_q$ for all $\chi \in \mathrm{Cl}_K(S_n)$. Then, up to a certain factor, ch_q is the characteristic function of C_q in $\mathrm{Cl}_K(S_n)$, and we have $C_q = C_r$ and $\mathrm{ch}_q = \mathrm{ch}_r$ whenever q is a rearrangement of r, for all $q, r \in \mathbb{N}^*$. The *outer product* $\bullet$ on the direct sum $\mathrm{Cl} := \bigoplus_{n\in\mathbb{N}_0} \mathrm{Cl}_K(S_n)$ may now be defined by

$$\mathrm{ch}_q \bullet \mathrm{ch}_r := \mathrm{ch}_{q.r} \tag{2}$$

for all $q, r \in \mathbb{N}^*$. It corresponds via Frobenius' characteristic mapping to the ordinary multiplication of symmetric functions.

Our starting point is the following part of [12, Theorem 8.23], which already occurs in [16, Section 8].

LEMMA 2.1. *Let $n \in \mathbb{N}$ and $q \vdash n$. Denote by a_i the multiplicity of the letter i in q, for all $i \in \underline{n}$. Then we have $\lambda_q = \lambda_{n^{.a_n}} \bullet \cdots \bullet \lambda_{1^{.a_1}}$.*

Hence, with ζ^p denoting the irreducible character of S_n corresponding to p for $p \vdash n$, the problem of describing the multiplicities

$$a_{q,p} := (\lambda_q, \zeta^p)_{S_n}$$

may be reduced to the case that q is of *block type*, that is, $q = d^{.k}$ is the k-th power of a single letter d. Indeed, for partitions $q = q_1. \ldots .q_k \vdash x$, $r = r_1. \ldots .r_l \vdash y$ such that $q_k > r_1$ and $x + y = n$, we have

$$(\lambda_{q.r}, \zeta^p)_{S_n} = (\lambda_q \bullet \lambda_r, \zeta^p)_{S_n} = \sum_{s\vdash x} \sum_{t\vdash y} c^p_{s,t} a_{q,s} a_{r,t} \tag{3}$$

by Lemma 2.1, where $c^p_{s,t} = (\zeta^s \bullet \zeta^t, \zeta^p)_{S_n}$ is the well-known Littlewood-Richardson coefficient.

For all $n, m \in \mathbb{N}_0$, $\psi \in S_n$ and $\sigma \in S_m$, we define $\psi \# \sigma \in S_{n+m}$ by

$$i(\psi \# \sigma) := \begin{cases} i\psi & i \leq n; \\ (i-n)\sigma + n & i > n \end{cases}$$

for all $i \in \underline{n+m}$. Furthermore, for $d, k \in \mathbb{N}$, $n := dk$ and $\pi \in S_k$, we define $\pi^{[d^{.k}]} \in S_n$ by

$$(dj - i)\pi^{[d^{.k}]} := d(j\pi) - i$$

for all $j \in \underline{k}$, $i \in \underline{d-1} \cup \{0\}$. That is, $\pi^{[d\cdot k]}$ is permuting the k successive blocks of length d in $\underline{n}$ according to π. Now let $\tau_d := (1, \ldots, d) \in S_d$ be the standard cycle of length d in S_d and put

$$\sigma_{d\cdot k} := \underbrace{\tau_d \# \cdots \# \tau_d}_{k} \in C_{d\cdot k} \subseteq S_n.$$

Then the centralizer of $\sigma_{d\cdot k}$ in S_n is a wreath product of the cyclic group generated by τ_d with S_k and may be described as

$$C^{d\cdot k} := C_{S_n}(\sigma_{d\cdot k}) = \left\{\pi^{[d\cdot k]}(\tau_d^{i_1} \# \cdots \# \tau_d^{i_k}) \,\middle|\, \pi \in S_k, i_1, \ldots, i_k \in \underline{d}\right\}.$$

([5, Section 4.1]). With these notations, the remaining part of Theorem 8.23 in [12], transferred to Cl, reads as follows.

THEOREM 2.2. *Let $d, k \in \mathbb{N}$ and $n := dk$. Then*

$$\psi_{d\cdot k} : C^{d\cdot k} \longrightarrow K, \quad \pi^{[d\cdot k]}(\tau_d^{i_1} \# \cdots \# \tau_d^{i_k}) \longmapsto \varepsilon_d^{-(i_1+\cdots+i_k)}$$

is a linear representation of $C^{d\cdot k}$, and $(\psi_{d\cdot k})^{S_n} = \lambda_{d\cdot k}$.

3. Multiplicities

In order to state our main result (Theorem 3.1), we need the notion of a standard Young tableau and its multi major index corresponding to a composition. Let $n \in \mathbb{N}$ and $p = p_1. \ldots .p_l \vDash n$. The frame $R(p) := \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i \in \underline{l}, j \in \underline{p_i}\}$ corresponding to p may be visualized by its Ferrers diagram, an array of boxes with p_1 boxes in the first (top) row, p_2 boxes in the second row and so on. For example, we have

$$R(3.2) \sim \begin{array}{|c|c|c|}\hline & & \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

The images $1\pi, \ldots, n\pi$ of any permutation $\pi \in S_n$ may be entered into $R(p)$ row by row, starting at bottom left and ending at top right. Let SYT^p be the set of all permutations which are increasing in rows (from left to right) and columns (downwards) when entered into $R(p)$ in this way. The elements of SYT^p are called *standard Young tableaux* of shape p. In the above example, the elements of $\mathrm{SYT}^{3.2}$, entered into $R(3.2)$, are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

Accordingly, we obtain

$$\mathrm{SYT}^{3.2} = \left\{ \begin{pmatrix} 12345 \\ 45123 \end{pmatrix}, \begin{pmatrix} 12345 \\ 35124 \end{pmatrix}, \begin{pmatrix} 12345 \\ 25134 \end{pmatrix}, \begin{pmatrix} 12345 \\ 24135 \end{pmatrix}, \begin{pmatrix} 12345 \\ 34125 \end{pmatrix} \right\} \subseteq S_5.$$

For all $\pi \in S_n$, $D(\pi) := \left\{ i \in \underline{n-1} \mid i\pi > (i+1)\pi \right\}$ is called the *descent set* of π. Let $q = q_1. \dots .q_k \models n$ and put $s_j := q_1 + \cdots + q_j$ for all $j \in \underline{k} \cup \{0\}$. Then the *multi major index* of π corresponding to q is defined as

$$\operatorname{maj}_q \pi := m_1. \dots .m_k \in \mathbb{N}^*, \tag{4}$$

where

$$m_j := \sum_{\substack{s_{j-1} < i \leq s_j \\ i \in D(\pi)}} (i - s_{j-1}) \tag{5}$$

for all $j \in \underline{k}$. For $q = n$, we obtain the ordinary major index $\operatorname{maj} \pi := \operatorname{maj}_n \pi$ of π. If, additionally, $r = r_1. \dots .r_k \in \mathbb{N}^*$, we define

$$\mathrm{syt}^p_{q,r} := \left| \left\{ \pi \in \mathrm{SYT}^p \mid \forall j \in \underline{k} : (\operatorname{maj}_q(\pi^{-1}))_j \equiv r_j \mod q_j \right\} \right|. \tag{6}$$

Here $(\operatorname{maj}_q(\pi^{-1}))_j$ always denotes the j-th letter of $\operatorname{maj}_q(\pi^{-1})$, for all $j \in \underline{k}$. For arbitrary $r = r_1. \dots .r_l$, $q = q_1. \dots .q_k \in \mathbb{N}^*$ we write $r \mid q$ if and only if $l = k$ and r_i is a divisor of q_i for all $i \in \underline{k}$. In this case, we define furthermore the following extension of the number theoretic Möbius function μ:

$$\mu(q/r) := \prod_{i=1}^{|q|} \mu(q_i/r_i). \tag{7}$$

Finally, for $k \in \mathbb{N}$ and $r = r_1. \dots .r_l \in \mathbb{N}^*$, we put $k \star r := (kr_1). \dots .(kr_l)$.

MAIN THEOREM 3.1. *Let $d, k, n \in \mathbb{N}$ such that $dk = n$. Let $p \vdash n$. Then we have*

$$(\lambda_{d^{\cdot k}}, \zeta^p)_{S_n} = \frac{1}{k!} \sum_{q \vdash k} |C_q| \sum_{r \mid q} \mu(q/r)\, \mathrm{syt}^p_{d \star q, r}\,.$$

The proof will be given in Section 5. A description of the multiplicity $(\lambda_q, \zeta^p)_{S_n}$ for arbitrary $q \vdash n$ may be obtained from Theorem 3.1 via (3). For $k \leq 3$, we obtain the following specializations of Theorem 3.1, the first of which is due to Kraśkiewicz and Weyman (see the Remark at the end of this section).

COROLLARY 3.2. *Let $d \in \mathbb{N}$.*

(a) *For all $p \vdash d$, we have $(\lambda_d, \zeta^p)_{S_d} = \mathrm{syt}^p_{d,1}$.*
(b) *For all $p \vdash 2d$, we have $(\lambda_{d.d}, \zeta^p)_{S_{2d}} = 1/2(\mathrm{syt}^p_{d.d,1.1} + \mathrm{syt}^p_{2d,2} - \mathrm{syt}^p_{2d,1})$.*

TABLE 1.

π	π^{-1}	$\operatorname{maj}_6 \pi^{-1}$	$\operatorname{maj}_{3.3} \pi^{-1}$	$\operatorname{maj}_{2.2.2} \pi^{-1}$	$\operatorname{maj}_{4.2} \pi^{-1}$
$\begin{matrix} 1 & \underline{2} \\ 3 & \underline{4} \\ 5 & 6 \end{matrix}$	$\begin{pmatrix} 123456 \\ 563412 \end{pmatrix}$	6	2.1	0.0.0	2.0
$\begin{matrix} 1 & \underline{2} \\ \underline{3} & \underline{5} \\ 4 & 6 \end{matrix}$	$\begin{pmatrix} 123456 \\ 563142 \end{pmatrix}$	10	2.2	0.1.1	5.1
$\begin{matrix} \underline{1} & \underline{3} \\ 2 & \underline{4} \\ 5 & 6 \end{matrix}$	$\begin{pmatrix} 123456 \\ 536412 \end{pmatrix}$	8	1.1	1.1.0	4.0
$\begin{matrix} \underline{1} & \underline{3} \\ 2 & \underline{5} \\ 4 & 6 \end{matrix}$	$\begin{pmatrix} 123456 \\ 536142 \end{pmatrix}$	9	1.2	1.1.1	4.1
$\begin{matrix} \underline{1} & \underline{4} \\ 2 & \underline{5} \\ 3 & 6 \end{matrix}$	$\begin{pmatrix} 123456 \\ 531642 \end{pmatrix}$	12	3.3	1.0.1	3.1

(c) *For all* $p \vdash 3d$, *we have*

$$(\lambda_{d.d.d}, \zeta^p)_{S_{3d}} = \frac{1}{6}\left(\operatorname{syt}^p_{d.d.d,1.1.1} + 3(\operatorname{syt}^p_{(2d).d,2.1} - \operatorname{syt}^p_{(2d).d,1.1}) + 2(\operatorname{syt}^p_{3d,3} - \operatorname{syt}^p_{3d,1})\right).$$

We will illustrate Corollary 3.2 in the case of $p = 2.2.2$. The standard Young tableaux π of shape p are listed in Table 1 together with their multi major indices in question. The descents of π^{-1} are underlined in each case.

By Corollary 3.2, we obtain $(\lambda_6, \zeta^{2.2.2})_{S_6} = 0$ and furthermore

$$(\lambda_{3.3}, \zeta^{2.2.2})_{S_6} = \frac{1}{2}(1 + 1 - 0) = 1$$

and

$$(\lambda_{2.2.2}, \zeta^{2.2.2})_{S_6} = \frac{1}{6}(1 + 3(0 - 1) + 2(1 - 0)) = 0.$$

For $p \vdash d \in \mathbb{N}$ and $\pi \in \mathrm{SYT}^p$, note that $i \in \underline{d-1}$ is a descent of π^{-1} if and only if i stands strictly above $i + 1$ in π, entered into $R(p)$. Hence Corollary 3.2 (a) indeed coincides with the original result of Kraśkiewicz and Weyman on the Lie character λ_d ([9]).

4. Noncommutative character theory

Let $n \in \mathbb{N}$. The *descent algebra* $\mathscr{D}_n$ is defined as the linear span of the elements $\delta^D := \sum\{\pi \in S_n \mid D(\pi) = D\}$ $\big(D \subseteq \underline{n-1}\big)$ in KS_n. Due to Solomon ([15]), $\mathscr{D}_n$ is a subalgebra of KS_n, and there exists a certain epimorphism of algebras $c_n : \mathscr{D}_n \to \mathrm{Cl}_K(S_n)$, for all n. The direct sum $KS := \bigoplus_{n\in\mathbb{N}} KS_n$ is a graded algebra with respect to the convolution product $\bullet$ (see [6, 1.3] for a combinatorial description), and $\mathscr{D} := \bigoplus_{n\in\mathbb{N}} \mathscr{D}_n$ is a $\bullet$-subalgebra of KS (see [12]). In [6], a (noncommutative) $\bullet$-subalgebra $\mathscr{R}$ of KS and a $\bullet$-homomorphism $c : \mathscr{R} \to \mathrm{Cl}$ are introduced such that $\mathscr{D} \subseteq \mathscr{R}$ and $c|_{\mathscr{D}_n} = c_n$ for all n. Furthermore, a (bilinear) scalar product $(\cdot,\cdot)$ on KS is defined by

$$(\pi,\sigma) := \begin{cases} 1 & \pi = \sigma^{-1}; \\ 0 & \pi \neq \sigma^{-1} \end{cases}$$

for all permutations π, σ, and it is shown that

$$(\varphi,\psi) = (c(\varphi), c(\psi))_S \tag{8}$$

for all $\varphi, \psi \in \mathscr{R}$, where the scalar product on the right hand side is the canonical orthogonal extension of the ordinary scalar products $(\cdot,\cdot)_{S_n}$ on $\mathrm{Cl}_K(S_n)$, $n \in \mathbb{N}$. For any partition $p \in \mathbb{N}^*$, $\mathsf{Z}^p := \sum_{\pi\in\mathrm{SYT}^p} \pi$ is an element of $\mathscr{R}$ such that

$$c(\mathsf{Z}^p) = \zeta^p \tag{9}$$

is the irreducible character of S_n corresponding to p. For example, for $p = 3.2$, we obtain $\mathsf{Z}^{3.2} = \binom{12345}{45123} + \binom{12345}{35124} + \binom{12345}{25134} + \binom{12345}{24135} + \binom{12345}{34125}$. These results provide the following general concept for describing multiplicities: Given an arbitrary character $\chi \in \mathrm{Cl}_K(S_n)$, any inverse image $\varphi \in \mathscr{R}$ of χ under c may be understood as a *noncommutative character* corresponding to χ. By (8) and (9), for each such φ, it follows that

$$(\chi, \zeta^p)_{S_n} = (c(\varphi), c(\mathsf{Z}^p))_{S_n} = (\varphi, \mathsf{Z}^p). \tag{10}$$

The right-hand side of (10) gives different combinatorial descriptions of the multiplicity on the left-hand side, according to the choice of φ, simply by the definition of Z^p and the scalar product on $\mathscr{R}$.

5. Klyachkos's idempotent and Ramanujan sums

In the sequel, following the concept described in Section 4, an inverse image of λ_{d^k} under c in $\mathscr{D}$ is constructed. It leads to a short proof of our main result Theorem 3.1, by means of (10).

Let $n \in \mathbb{N}$. We put $\kappa_n(x) := \sum_{\pi \in S_n} x^{\operatorname{maj}\pi}\pi$ (x a variable) and

$$M_{n,i} := \sum_{\substack{\pi \in S_n \\ \operatorname{maj}\pi \equiv i \mod n}} \pi \in \mathscr{D}_n$$

for all $i \in \mathbb{N}_0$. Then, up to the factor $1/n$, $\kappa_n(\varepsilon_n) = \sum_{i=1}^n \varepsilon_n^i M_{n,i} \in \mathscr{D}_n$ is a Lie idempotent, that is, $\kappa_n^2 = n\kappa_n$ and $L_n(V) = \kappa_n T_n(V)$. This remarkable result is due to Klyachko ([8]).

Lemma 5.1. *Let $n, i \in \mathbb{N}$ and d be the order of ε_n^i. Then we have*

$$\kappa_n(\varepsilon_n^i) = \underbrace{\kappa_d(\varepsilon_n^i) \bullet \cdots \bullet \kappa_d(\varepsilon_n^i)}_{n/d}.$$

In particular, $c(\kappa_n(\varepsilon_n^i)) = \operatorname{ch}_{d^{\cdot n/d}}$.

The main part of the preceding lemma is a special case of [10, Proposition 4.1], while the additional claim on the c-image follows from [7, Proposition 1]. For $n, m \in \mathbb{N}$, we denote by $\gcd(n, m)$ the greatest common divisor of n and m.

Corollary 5.2. *Let $n \in \mathbb{N}$ and $i, j \in \mathbb{N}_0$ such that* $\gcd(i, n) = \gcd(j, n)$. *Then* $c(M_{n,i}) = c(M_{n,j})$.

Proof. As $\gcd(i, n) = \gcd(j, n)$, we can find an integer $m \in \mathbb{N}$ such that $i \equiv jm$ modulo n and $\gcd(m, n) = 1$. For all $k \in \mathbb{N}$, we have $\gcd(km, n) = \gcd(k, n)$ and hence $c(\kappa_n(\varepsilon_n^k)) = c(\kappa_n(\varepsilon_n^{mk}))$, by Lemma 5.1. It follows that

$$\begin{aligned} nc(M_{n,i}) &= c\left(\sum_{l=1}^{n}\sum_{k=1}^{n}(\varepsilon_n^{l-i})^k M_{n,l}\right) = c\left(\sum_{k=1}^{n}\varepsilon_n^{-ik}\kappa_n(\varepsilon_n^k)\right) \\ &= c\left(\sum_{k=1}^{n}\varepsilon_n^{-ik}\kappa_n(\varepsilon_n^{mk})\right) = c\left(\sum_{l=1}^{n}\sum_{k=1}^{n}(\varepsilon_n^{lm-i})^k M_{n,l}\right) \\ &= c\left(\sum_{l=1}^{n}\sum_{k=1}^{n}((\varepsilon_n^{m})^{l-j})^k M_{n,l}\right) = nc(M_{n,j}). \end{aligned}$$ □

Let $n, m \in \mathbb{N}$. The *Ramanujan sum* corresponding to n and m is defined by

$$\varrho(n, m) := \sum \varepsilon^m,$$

where the sum is taken over all primitive n-th roots of unity ε. In the particular case of $m = 1$ ($m = n$, respectively), $\varrho(n, m)$ yields the Möbius function $\mu(n) = \varrho(n, 1)$

(Euler's function $\varphi(n) = \varrho(n, n)$, respectively). We write $x \mid m$, if $x \in \mathbb{N}$ is a divisor of m, and put

$$R(n, m) := \sum_{x|m} \varrho(n, x)\varrho(m/x, 1). \tag{11}$$

Now, for all $d, k \in \mathbb{N}$ and $p = p_1. \ldots .p_l \in \mathbb{N}^*$, let

$$M_d(k) := \sum_{y|dk} R(dk/y, d) M_{dk,y} \tag{12}$$

and

$$M_d(p) := M_d(p_1) \bullet \cdots \bullet M_d(p_l).$$

Note that $M_d(p) \in \mathscr{D}$, as $\mathscr{D}$ is closed under the convolution product.

LEMMA 5.3. *For all $d, k \in \mathbb{N}$, we have*

$$\lambda_{d \cdot k} = c\left(\frac{1}{k!}\sum_{\pi \in S_k} \frac{1}{d^{|z(\pi)|}} M_d(z(\pi))\right).$$

(Recall that $z(\pi)$ denotes the cycle partition of π for any permutation π.)

PROOF. We write

$$z(\pi; i_1, \ldots, i_k) := z(\pi^{[d \cdot k]}(\tau_d^{i_1} \# \cdots \# \tau_d^{i_k}))$$

for all $\pi \in S_k$, $i_1, \ldots, i_k \in \underline{d-1} \cup \{0\}$. By Theorem 2.2, we then have

$$\begin{aligned}\lambda_{d \cdot k} &= \frac{1}{|C^{d \cdot k}|} \sum_{q \vdash dk} \left(\sum_{\substack{\varphi \in C^{d \cdot k} \\ z(\varphi) = q}} \psi_{d \cdot k}(\varphi) \right) \mathrm{ch}_q \\ &= \frac{1}{k!} \sum_{\pi \in S_k} \frac{1}{d^k} \sum_{i_1, \ldots, i_k = 0}^{d-1} \varepsilon_d^{-\sum i_j} \, \mathrm{ch}_{z(\pi; i_1, \ldots, i_k)}.\end{aligned}$$

By induction on the number $z = |z(\pi)|$ of cycles in $\pi \in S_k$, we show that

$$\frac{1}{d^k} \sum_{i_1, \ldots, i_k = 0}^{d-1} \varepsilon_d^{-\sum i_j} \, \mathrm{ch}_{z(\pi; i_1, \ldots, i_k)} = c\left(\frac{1}{d^z} M_d(z(\pi))\right), \tag{$*$}$$

which implies our claim. We will use some basic facts about cycle partitions of elements of $C^{d \cdot k}$ which can be found in [5, 4.2]. Let $z = 1$. Then $\pi \in S_k$ is a long cycle. Putting $\eta := \varepsilon_{kd}$ and applying [5, 4.2.17], Lemma 5.1 and Corollary 5.2, we obtain

$$\begin{aligned}
&\frac{1}{d^k}\sum_{i_1,\dots,i_k=0}^{d-1}\varepsilon_d^{-\sum i_j}\,\mathrm{ch}_{z(\pi;i_1,\dots,i_k)}\\
&=\frac{1}{d}\sum_{i=0}^{d-1}\varepsilon_d^{-i}\,\mathrm{ch}_{k\star z(\tau_d^i)}=\frac{1}{d}\sum_{x|d}\varrho(d/x,1)\,\mathrm{ch}_{k\star z(\tau_d^x)}\\
&=c\left(\frac{1}{d}\sum_{x|d}\varrho(d/x,1)\kappa_{kd}(\eta^x)\right)=c\left(\frac{1}{d}\sum_{x|d}\sum_{j=0}^{dk-1}\varrho(d/x,1)\eta^{jx}M_{dk}^{(j)}\right)\\
&=c\left(\frac{1}{d}\sum_{y|dk}M_{dk}^{(y)}\sum_{x|d}\varrho(d/x,1)\varrho(dk/y,x)\right)=c\left(\frac{1}{d}\sum_{y|dk}M_{dk}^{(y)}R(dk/y,d)\right)\\
&=c(M_d(k)/d).
\end{aligned}$$

Now let $z>1$, say, $\pi=\tilde{\pi}\sigma$ for a cycle σ of length l in π. Then we have, by [5, 4.2.19], (2) and our induction hypothesis,

$$\begin{aligned}
&\frac{1}{d^k}\sum_{i_1,\dots,i_k=0}^{d-1}\varepsilon_d^{-\sum i_j}\,\mathrm{ch}_{z(\pi;i_1,\dots,i_k)}\\
&=\left(\frac{1}{d^{k-l}}\sum_{i_1,\dots,i_{k-l}=0}^{d-1}\varepsilon_d^{-\sum i_j}\,\mathrm{ch}_{z(\tilde{\pi};i_1,\dots,i_{k-l})}\right)\bullet\left(\frac{1}{d^l}\sum_{i_{k-l+1},\dots,i_k=0}^{d-1}\varepsilon_d^{-\sum i_j}\,\mathrm{ch}_{z(\sigma;i_{k-l+1},\dots,i_k)}\right)\\
&=c\left(\frac{1}{d^{z-1}}M_d(z(\tilde{\pi}))\bullet\frac{1}{d}M_d(z(\sigma))\right)=c\left(\frac{1}{d^z}M_d(z(\pi))\right).
\end{aligned}$$

This completes the proof of ($*$). $\square$

The inverse image of λ_{d^k} under c constructed in the preceding lemma may be simplified by means of a short analysis of the numbers $R(n,m)$. This will be done in three steps.

PROPOSITION 5.4. *Let $n_1,n_2,m_1,m_2\in\mathbb{N}$ such that*

$$\gcd(n_1,n_2)=\gcd(m_1,m_2)=\gcd(n_1,m_2)=\gcd(n_2,m_1)=1.$$

Then we have $R(n_1n_2,m_1m_2)=R(n_1,m_1)R(n_2,m_2)$.

PROOF. By [4, Theorem 67], the Ramanujan sums have the following factorizing property: $\varrho(a_1a_2,b)=\varrho(a_1,b)\varrho(a_2,b)$ for all $a_1,a_2,b\in\mathbb{N}$ such that $\gcd(a_1,a_2)=1$. Furthermore, we have $\varrho(a,b_1b_2)=\varrho(a,b_1)$ for all $a,b_1,b_2\in\mathbb{N}$ such that $(a,b_2)=1$, as in this case taking the b_2-th power induces an automorphism of the group of a-th roots of unity. These two observations imply that

$$\begin{aligned}R(n_1n_2, m_1m_2) &= \sum_{x_1|m_1}\sum_{x_2|m_2} \varrho(n_1n_2, x_1x_2)\varrho\left(\frac{m_1}{x_1}\frac{m_2}{x_2}, 1\right)\\ &= \sum_{x_1|m_1}\sum_{x_2|m_2} \varrho(n_1, x_1x_2)\varrho(n_2, x_1x_2)\varrho\left(\frac{m_1}{x_1}, 1\right)\varrho\left(\frac{m_2}{x_2}, 1\right)\\ &= \sum_{x_1|m_1} \varrho(n_1, x_1)\varrho\left(\frac{m_1}{x_1}, 1\right)\sum_{x_2|m_2} \varrho(n_2, x_2)\varrho\left(\frac{m_2}{x_2}, 1\right)\\ &= R(n_1, m_1)R(n_2, m_2).\end{aligned}$$

□

Let $\mathbb{P}$ be the set of all prime numbers.

PROPOSITION 5.5. *For all $a, b \in \mathbb{N}_0$ and $p \in \mathbb{P}$, we have*

$$R(p^a, p^b) = \begin{cases}\mu(p^{a-b})p^b & b \le a;\\ 0 & b > a.\end{cases}$$

PROOF. For all $n, m \in \mathbb{N}$, the Ramanujan sum corresponding to n and m may be expressed in terms of the Möbius and the Euler function as follows:

$$\varrho(n, m) = \mu(n/\gcd(n, m))\frac{\varphi(n)}{\varphi(n/\gcd(n, m))}$$

([4, Theorem 272]). Let $c := \min\{a, b\}$ and $d := \min\{a, b-1\}$. Then

$$\begin{aligned}R(p^a, p^b) &= \sum_{i=0}^{b} \varrho(p^a, p^i)\varrho(p^{b-i}, 1)\\ &= \varrho(p^a, p^b) - \varrho(p^a, p^{b-1})\\ &= \mu(p^{a-c})\frac{\varphi(p^a)}{\varphi(p^{a-c})} - \mu(p^{a-d})\frac{\varphi(p^a)}{\varphi(p^{a-d})}\end{aligned}$$

and hence $R(p^a, p^b) = 0$ for $b > a$, as $c = d = a$ in this case. Let $b \le a$. Then we have $c = b$ and $d = b - 1$, that is,

$$R(p^a, p^b) = \mu(p^{a-b})\frac{\varphi(p^a)}{\varphi(p^{a-b})} - \mu(p^{a-b+1})\frac{\varphi(p^a)}{\varphi(p^{a-b+1})}.$$

For $b < a - 1$, this shows $R(p^a, p^b) = 0$ as asserted. For $b = a - 1$ it follows that $R(p^a, p^b) = -\varphi(p^{b+1})/\varphi(p) = -p^b$, while, for $b = a$, we may conclude that $R(p^a, p^b) = \varphi(p^b) - \varphi(p^b)/\varphi(p) = p^b$. □

LEMMA 5.6. *For all $n, m \in \mathbb{N}$, we have*

$$R(n,m) = \begin{cases} \mu(n/m)m & m \mid n; \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. Choose $a_p, b_p \in \mathbb{N}_0$ for all $p \in \mathbb{P}$ such that $n = \prod_{p\in\mathbb{P}} p^{a_p}$ and $m = \prod_{p\in\mathbb{P}} p^{b_p}$. Applying Propositions 5.4 and 5.5 we obtain

$$\begin{aligned} R(n,m) &= \prod_{p\in\mathbb{P}} R(p^{a_p}, p^{b_p}) \\ &= \begin{cases} \prod_{p\in\mathbb{P}} \mu(p^{a_p-b_p})p^{b_p} & \forall p \in \mathbb{P} : b_p \leq a_p; \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mu(n/m)m & m \mid n; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$ □

COROLLARY 5.7. *Let $d, k \in \mathbb{N}$. Then $M_d(k) = d \sum_{y|k} \mu(k/y) M_{dk,y}$.*

PROOF. Let y be a divisor of dk. Then Lemma 5.6 implies that

$$R(dk/y, d) = \begin{cases} \mu(dk/dy)d & d \mid dk/y; \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \mu(k/y)d & y \mid k; \\ 0 & \text{otherwise.} \end{cases}$$ □

We are now in a position to give the proof of the Main Theorem 3.1.

PROOF OF THE MAIN THEOREM 3.1. By Lemma 5.3 and (10), we have

$$(\lambda_{d^k}, \zeta^p)_{S_n} = \frac{1}{k!} \sum_{\pi \in S_k} \frac{1}{d^{|z(\pi)|}} (M_d(z(\pi)), Z^p).$$

But, for $\pi \in S_k$ and $q = q_1. \ldots .q_k := z(\pi)$, we may conclude from Corollary 5.7 that

$$\begin{aligned} \frac{1}{d^{|z(\pi)|}}(M_d(z(\pi)), Z^p) &= \frac{1}{d^k}(M_d(q_1) \bullet \cdots \bullet M_d(q_k), Z^p) \\ &= \sum_{r_1|q_1} \cdots \sum_{r_k|q_k} \mu(q_1/r_1) \cdots \mu(q_k/r_k)(M_{dq_1,r_1} \bullet \cdots \bullet M_{dq_k,r_k}, Z^p) \\ &= \sum_{r|q} \mu(q/r)(M_{dq_1,r_1} \bullet \cdots \bullet M_{dq_k,r_k}, Z^p). \end{aligned}$$

This completes the proof, as $(M_{dq_1,r_1} \bullet \cdots \bullet M_{dq_k,r_k}, Z^p) = \mathrm{syt}^p_{d\star q,r}$ for all $r \mid q$, simply by definition of the scalar product $(\cdot,\cdot)$ and the convolution product $\bullet$ in [6, 1.3]. □

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