J. Aust. Math. Soc. **75** (2003), 153–161

COMPUTATION OF NONSQUARE CONSTANTS OF ORLICZ SPACES

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(Received 30 July 2001; revised 16 April 2002)

Communicated by J. R. J. Groves

Abstract

In this paper, we present the computation of exact value of nonsquare constants for some types of Orlicz sequence and function spaces. Main results: Let $\Phi(u)$ be an N-function, $\phi(t)$ be the right derivative of $\Phi(u)$, then we have

(i) if $\phi(t)$ is concave, then $1/\alpha'_\Phi \le J(l^{(\Phi)}) \le 1/\tilde{\alpha}_\Phi$, $J(L^{(\Phi)}[0,\infty)) = 1/\bar{\alpha}_\Phi$;
(ii) if $\phi(t)$ is convex, then $2\beta'_\Phi \le J(l^{(\Phi)}) \le 2\tilde{\beta}_\Phi$, $J(L^{(\Phi)}[0,\infty)) = 2\bar{\beta}_\Phi$.

2000 *Mathematics subject classification*: primary 46B45, 46E30.

1. Introduction

The concept of nonsquareness is an important geometric property of Banach spaces which expose the intrinsic construction of a space according to the 'shape' of the unit ball of the spaces. The computation of nonsquare constants in Orlicz spaces has attracted the interest of many researchers and a considerable number of papers on this topic have appeared. However there has been little achievement of it since Gao and Lau [3] studied the value for Banach spaces. This paper is devoted to deriving exact estimates of nonsquare constants of Orlicz spaces which are easy to use in concrete applications.

Let X be a Banach space; $S(X) = \{x : \|x\| = 1, x \in X\}$ denotes the unit sphere of X. The nonsquare constants in the sense of James $J(X)$ and in the sense of Schaffer $g(X)$ are defined as:

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S(X)\}, \tag{1}$$

$$g(X) = \inf\{\max(\|x+y\|, \|x-y\|) : x, y \in S(X)\}. \tag{2}$$

Clearly, if $\dim X \geq 2$, then $1 \leq g(X) \leq \sqrt{2} \leq J(X) \leq 2$. Ji and Wang [5] asserted that

$$g(X)J(X) = 2 \tag{3}$$

for $\dim X \geq 2$. It was proved (Chen [1]) that $J(X) = 2$ if X fails to be reflexive. However, practical calculation for $J(X)$ when X is reflexive except L^p and l^p remains unsolved. In this paper, we extend the results of several authors (for instance, Ren [9], Ji and Wang [5], Ji and Zhan [6]) and deal with the computation of $J(X)$ when X is Orlicz function space $L^{(\Phi)}[0, \infty)$ and a sequence space $l^{(\Phi)}$ equipped with the Luxemburg norm.

Let $\Phi(u) = \int_0^{|u|} \phi(t)\,dt$ be an N-function, that is, $\phi(t)$ is right continuous, $\phi(0) = 0$, and $\phi(t) \nearrow \infty$ as $t \nearrow \infty$. The above two spaces are defined as follows:

$$L^{(\Phi)}[0, \infty) = \left\{ x : \rho_\Phi(\lambda x) = \int_{[0,\infty)} \Phi(\lambda |x(t)|)\,dt < \infty \text{ for some } \lambda > 0 \right\},$$

$$l^{(\Phi)} = \left\{ x = \{x(i)\} : \rho_\Phi(\lambda x) = \sum_{n=1}^{\infty} \Phi(\lambda |x(i)|) < \infty \text{ for some } \lambda > 0 \right\}.$$

The Luxemburg norm is expressed as

$$\|x\|_{(\Phi)} = \inf \{c > 0 : \rho_\Phi(x/c) \leq 1\}.$$

We say that $\Phi \in \Delta_2(0)$ (or Δ_2), if there exist $u_0 > 0$ and $k > 2$ such that $\Phi(2u) \leq k\Phi(u)$ for $0 \leq u \leq u_0$ (or for $u \geq 0$). Later, we will frequently use Semenove indices of $\Phi(u)$:

$$\alpha_\Phi = \liminf_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \beta_\Phi = \limsup_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \tag{4}$$

$$\alpha_\Phi^0 = \liminf_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \beta_\Phi^0 = \limsup_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \tag{5}$$

$$\bar{\alpha}_\Phi = \inf_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \bar{\beta}_\Phi = \sup_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}. \tag{6}$$

We extend the definition of the indices for the sequential usage:

$$\tilde{\alpha}_\Phi = \inf \left\{ \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 \leq u \leq \frac{1}{2} \right\}, \quad \tilde{\beta}_\Phi = \sup \left\{ \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 \leq u \leq \frac{1}{2} \right\}; \tag{7}$$

$$\begin{aligned} \alpha'_\Phi &= \inf \left\{ \frac{\Phi^{-1}(1/(2k))}{\Phi^{-1}(1/k)} : k = 1, 2, \ldots \right\}, \\ \beta'_\Phi &= \sup \left\{ \frac{\Phi^{-1}(1/(2k))}{\Phi^{-1}(1/k)} : k = 1, 2, \ldots \right\}. \end{aligned} \tag{8}$$

2. Lower bounds of $J(l^{(\Phi)})$ and $J(L^{(\Phi)}[0,\infty))$

We first estimate the lower bounds for $l^{(\Phi)}$ and $L^{(\Phi)}[0,\infty)$. The idea is refined from Ren [9]. We improve it so that the lower bounds may meet the upper ones and we obtain the exact values.

THEOREM 2.1. *Let $\Phi(u)$ be an N-function. Then the nonsquare constants of $l^{(\Phi)}$ and $L^{(\Phi)}[0,\infty)$, in the sense of James, satisfy*

$$\max\left(1/\alpha'_\Phi, 2\beta'_\Phi\right) \le J(l^{(\Phi)}) \qquad \text{and} \tag{9}$$

$$\max\left(1/\bar{\alpha}_\Phi, 2\bar{\beta}_\Phi\right) \le J(L^{(\Phi)}[0,\infty)). \tag{10}$$

PROOF. To prove (9), we first show that

$$1/\alpha'_\Phi \le J(l^{(\Phi)}). \tag{11}$$

For any natural number k, put

$$x = (\overbrace{\Phi^{-1}(1/k), \dots, \Phi^{-1}(1/k)}^{k}, 0, 0, \dots),$$

$$y = (\overbrace{0, \dots, 0}^{k}, \overbrace{\Phi^{-1}(1/k), \dots, \Phi^{-1}(1/k)}^{k}, 0, 0, \dots).$$

Then we have $\rho_\Phi(x) = \rho_\Phi(y) = 1$, $\|x\|_{(\Phi)} = \|y\|_{(\Phi)} = 1$ and

$$\|x - y\|_{(\Phi)} = \|x + y\|_{(\Phi)} = \frac{\Phi^{-1}(1/k)}{\Phi^{-1}(1/(2k))}.$$

Therefore,

$$\min(\|x - y\|_{(\Phi)}, \|x + y\|_{(\Phi)}) \ge \frac{\Phi^{-1}(1/k)}{\Phi^{-1}(1/(2k))} \quad (k = 1, 2, \dots).$$

Inequality (11) is proved.

Secondly, we prove that

$$2\beta'_\Phi \le J(l^{(\Phi)}). \tag{12}$$

Given a natural number k, put

$$x = (\overbrace{\Phi^{-1}(1/(2k)), \dots, \Phi^{-1}(1/(2k))}^{k}, \overbrace{\Phi^{-1}(1/(2k)), \dots, \Phi^{-1}(1/(2k))}^{k}, 0, \dots),$$

$$y = (\overbrace{\Phi^{-1}(1/(2k)), \dots, \Phi^{-1}(1/(2k))}^{k}, \overbrace{-\Phi^{-1}(1/(2k)), \dots, -\Phi^{-1}(1/(2k))}^{k}, 0, \dots).$$

Then $\|x\|_{(\Phi)} = \|y\|_{(\Phi)} = 1$ since $\rho_\Phi(x) = \rho_\Phi(y) = 1$, and

$$\|x-y\|_{(\Phi)} = \|x+y\|_{(\Phi)} = \frac{2\Phi^{-1}(1/(2k))}{\Phi^{-1}(1/k)}.$$

Therefore,

$$\min(\|x-y\|_{(\Phi)}, \|x+y\|_{(\Phi)}) \geq \frac{2\Phi^{-1}(1/(2k))}{\Phi^{-1}(1/k)} \quad (k = 1, 2, \dots)$$

and we obtain (12). Finally (9) follows from (11) and (12).

To prove (10), we first show

$$1/\bar{\alpha}_\Phi \leq J(L^{(\Phi)}[0,\infty)). \tag{13}$$

Take a real number $u \in (0,\infty)$, choose G_1 and G_2 in $[0,\infty)$ such that $G_1 \cap G_2 = \emptyset$ and $\mu(G_1) = \mu(G_2) = 1/2u$. Put $x(t) = \Phi^{-1}(2u)\chi_{G_1}(t)$ and $y(t) = \Phi^{-1}(2u)\chi_{G_2}(t)$, where χ_{G_1} is the characteristic function of G_1. Note that

$$\|\chi_{G_1}\|_{(\Phi)} = \|\chi_{G_2}\|_{(\Phi)} = \frac{1}{\Phi^{-1}(1/(\mu(G_1)))} = \frac{1}{\Phi^{-1}(2u)}.$$

We have $\|x\|_{(\Phi)} = \|y\|_{(\Phi)} = 1$ and

$$\|x-y\|_{(\Phi)} = \|x+y\|_{(\Phi)} = \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)}.$$

Take the supremum over $u \in (0,\infty)$. Since the function $G_\Phi(u) = \Phi^{-1}(u)/\Phi^{-1}(2u)$ is right continuous at 0 and takes value on $[1/2, 1]$, we deduce that

$$J(L^{(\Phi)}[0,\infty)) \geq \sup_{u\in(0,\infty)} \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)} = \sup_{u\in[0,\infty)} \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)} = \frac{1}{\bar{\alpha}_\Phi}.$$

Finally, we show

$$2\bar{\beta}_\Phi \leq J(L^{(\Phi)}[0,\infty)). \tag{14}$$

For every real number $v > 0$, choose E_1, E_2 in $[0,\infty)$ such that $E_1 \cap E_2 = \emptyset$ and $\mu(E_1) = \mu(E_2) = 1/2v$. Put

$$x(t) = \Phi^{-1}(v)[\chi_{E_1}(t) + \chi_{E_2}(t)] \quad \text{and} \quad y(t) = \Phi^{-1}(v)[\chi_{E_1}(t) - \chi_{E_2}(t)].$$

Then $\|x\|_{(\Phi)} = \|y\|_{(\Phi)} = 1$ and

$$\|x-y\|_{(\Phi)} = \|x+y\|_{(\Phi)} = \frac{2\Phi^{-1}(v)}{\Phi^{-1}(2v)}.$$

Take the supremum over $v \in (0,\infty)$ (the function $2\Phi^{-1}(v)/\Phi^{-1}(2v)$ is right continuous at 0 and takes value on $[1, 2]$) we also have $J(L^{(\Phi)}[0,\infty)) \geq 2\bar{\beta}_\Phi$. Hence (10) follows from (13) and (14). $\square$

3. Upper bounds of $J(l^{(\Phi)})$ and $J(L^{(\Phi)}[0,\infty))$

Upper bounds for Orlicz spaces remained unsolved (see [1, 9]) until Ji and Wang ([5, Theorem 3]) and Ji and Zhan ([6, Theorem 2]) offered the following equivalent presentation of $J(l^{(\Phi)})$ and $J(L^{(\Phi)}[0,\infty))$:

Assume $\Phi \in \Delta_2(0)$, then ([6])

(i) if $\phi(t)$ is a concave function, then

$$J(l^{(\Phi)}) = \sup\{k_x > 0 : \rho_\Phi(x/k_x) = 1/2,\ \rho_\Phi(x) = 1\}; \tag{15}$$

(ii) if $\phi(t)$ is convex, then

$$g(l^{(\Phi)}) = \inf\{k_x > 0 : \rho_\Phi(x/k_x) = 1/2,\ \rho_\Phi(x) = 1\}. \tag{16}$$

Suppose Φ satisfies the Δ_2-conditions for all u, we have [5]

(i) if $\phi(t)$ is a concave function, then

$$g(L^{(\Phi)}[0,\infty)) = \inf\{k_x > 0 : \rho_\Phi(2x/k_x) = 2,\ \rho_\Phi(x) = 1\}; \tag{17}$$

(ii) if $\phi(t)$ is convex, then

$$J(L^{(\Phi)}[0,\infty)) = \sup\{k_x > 0 : \rho_\Phi(2x/k_x) = 2,\ \rho_\Phi(x) = 1\}. \tag{18}$$

Now we extend these results and get the upper bounds.

THEOREM 3.1. *Suppose $\phi(t)$ is the right derivative of $\Phi(u)$, we have*

(i) *if $\phi(u)$ is concave, then*

$$J(l^{(\Phi)}) \le 1/\tilde{\alpha}_\Phi; \tag{19}$$

$$J(L^{(\Phi)}[0,\infty)) \le 1/\bar{\alpha}_\Phi; \tag{20}$$

(ii) *if $\phi(u)$ is convex, then*

$$J(l^{(\Phi)}) \le 2\tilde{\beta}_\Phi, \tag{21}$$

$$J(L^{(\Phi)}[0,\infty)) \le 2\bar{\beta}_\Phi. \tag{22}$$

PROOF. For the sequence spaces, if $\Phi \notin \Delta_2(0)$, which is equivalent to $\beta_\Phi^0 = 1$, then $l^{(\Phi)}$ is nonreflexive and hence $J(l^{(\Phi)}) = 2$ according to the results in Chen [1] or Hudzik [4]. Since $\phi(t)$ is concave implies $\Phi \in \Delta_2(0)$ (see Krasnoselskiĭ and Rutickiĭ [7, page 26]), we only need to check (21) when $\phi(t)$ is convex, but this is trivial since $J(l^{(\Phi)}) = 2 = 2\beta_\Phi^0 = 2\tilde{\beta}_\Phi$. Similarly we check that (20) and (22) hold when $\Phi \notin \Delta_2$.

Therefore it suffices for us to prove (19) and (21) for $\Phi \in \Delta_2(0)$ and (20) and (22) for $\Phi \in \Delta_2$.

To show (19) when $\Phi \notin \Delta_2(0)$, note that for $x = \{x(i)\} \in l^{(\Phi)}$, $\rho_\Phi(x(i)) = \sum_{n=1}^{\infty} \Phi(|x(i)|) = 1$ we have $u_i = \Phi(|x(i)|) \leq 1$ for $i \geq 1$. Define $G_\Phi(u) = \Phi^{-1}(u)/\Phi^{-1}(2u)$, then $u = \Phi[G_\Phi(u)\Phi^{-1}(2u)]$. Put $u_i = \Phi(|x(i)|)/2$, then $|x(i)| = \Phi^{-1}(2u_i)$ and

$$\frac{1}{2}\Phi(|x(i)|) = \Phi\left[G_\Phi\left(\frac{1}{2}\Phi(|x(i)|)\right)|x(i)|\right]. \tag{23}$$

Therefore, when $0 \leq u_i = \Phi(|x(i)|)/2 \leq 1/2$, we have

$$\tilde{\alpha}_\Phi \leq \frac{\Phi^{-1}(u_i)}{\Phi^{-1}(2u_i)} = G_\Phi(u_i) = G_\Phi[\Phi(|x(i)|)/2],$$

and hence, according to (23),

$$\rho_\Phi\left(\tilde{\alpha}_\Phi \cdot x\right) \leq \sum_{n=1}^{\infty} \Phi\left\{[G_\Phi(u_i)] \cdot |x(i)|\right\} = \frac{1}{2}\sum_{n=1}^{\infty} \Phi(|x(i)|) = \frac{1}{2}.$$

Thus we have $J(l^{(\Phi)}) \leq 1/\tilde{\alpha}_\Phi$ when $\phi(u)$ is concave by (15).

Analogously we prove $g(l^{(\Phi)}) \geq 1/\tilde{\beta}_\Phi$ by (16) when $\phi(u)$ is convex. From (3) we have $J(l^{(\Phi)}) \leq 2\tilde{\beta}_\Phi$.

Finally, we prove (20) for $\Phi(u) \in \Delta_2$, which is equal to

$$g(L^{(\Phi)}[0,\infty)) \geq 2\bar{\alpha}_\Phi \tag{24}$$

when $\phi(t)$ is concave in view of (3) and (17).

Let $H_\Phi(u) = \Phi^{-1}(2u)/\Phi^{-1}(u)$, then $\Phi^{-1}(2u) = H_\Phi(u)\Phi^{-1}(u)$. Put $x = \Phi^{-1}(u)$, then $u = \Phi(x)$ and $2\Phi(x) = \Phi[H_\Phi(\Phi(x))x]$. Therefore, when $u = \Phi(x(t)) \geq 0$ we have

$$\begin{aligned}\rho_\Phi\left(\frac{2x(t)}{2\bar{\alpha}_\Phi}\right) = \rho_\Phi\left(\frac{x(t)}{\bar{\alpha}_\Phi}\right) &\geq \rho_\Phi\left(\frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)}x(t)\right)\\ &= \rho_\Phi[H_\Phi(u)x(t)] = 2\rho_\Phi(x(t)) = 2\end{aligned}$$

for $\rho_\Phi(x(t)) = 1$. It follows that (24) holds and hence (20) holds.

One can prove (22) similarly by (18). The proof is finished. $\square$

4. Examples for computation

With the above bounds for $J(l^{(\Phi)})$ and $J(L^{(\Phi)}[0,\infty))$, we immediately obtain satisfactory estimates which are easy to compute.

THEOREM 4.1. *Let $\Phi(u)$ be an N-function, $\phi(t)$ be the right derivative of $\Phi(u)$. We have*

(i) *if $\phi(t)$ is concave, then $1/\alpha'_\Phi \le J(l^{(\Phi)}) \le 1/\tilde{\alpha}_\Phi$ and $J(L^{(\Phi)}[0,\infty)) = 1/\bar{\alpha}_\Phi$;*

(ii) *if $\phi(t)$ is convex, then $2\beta'_\Phi \le J(l^{(\Phi)}) \le 2\tilde{\beta}_\Phi$ and $J(L^{(\Phi)}[0,\infty)) = 2\bar{\beta}_\Phi$.*

EXAMPLE 1. For $p > 1$, we have $J(L^p) = J(l^p) = \max(2^{1/p}, 2^{1-1/p})$, $(1<p<\infty)$. In fact, let $\Phi = |u|^p$, then $\alpha'_\Phi = \beta'_\Phi = \tilde{\alpha}_\Phi = \bar{\alpha}_\Phi = \bar{\beta}_\Phi = \tilde{\beta}_\Phi = 2^{-1/p}$. Obviously, if $1 < p \le 2$ then $\phi(t) = pt^{p-1}$ is concave, and if $2 \le p < \infty$ then $\phi(t)$ is convex. By Theorem 4.1 we get:

- if $1 < p \le 2$, then $J(L^p) = J(l^p) = 2^{1/p}$;
- if $2 \le p < \infty$, then $J(L^p) = J(l^p) = 2^{1-1/p}$.

REMARK 1. If the index function $G_\Phi(u) = \Phi^{-1}(u)/\Phi^{-1}(2u)$ is decreasing or increasing on an interval, then the indices α_Φ and β_Φ take the values at either end of it. The author [12] found that if $F_\Phi(t) = t\phi(t)/\Phi(t)$ is increasing (decreasing) on $(0, \Phi^{-1}(u_0)]$ then $G_\Phi(u)$ is also increasing (decreasing) on $(0, u_0/2]$, respectively. Rao and Ren [8] found the interrelation between Semenove and Simonenko indices:

$$2^{-1/A_\Phi} \le \alpha_\Phi \le \beta_\Phi \le 2^{-1/B_\Phi}, \quad 2^{-1/A^0_\Phi} \le \alpha^0_\Phi \le \beta^0_\Phi \le 2^{-1/B^0_\Phi},$$

where

$$A_\Phi = \liminf_{t\to\infty} \frac{t\phi(t)}{\Phi(t)}, \qquad B_\Phi = \limsup_{t\to\infty} \frac{t\phi(t)}{\Phi(t)};$$
$$A^0_\Phi = \liminf_{t\to 0} \frac{t\phi(t)}{\Phi(t)}, \qquad B^0_\Phi = \limsup_{t\to 0} \frac{t\phi(t)}{\Phi(t)}.$$

Therefore, when the index function $F_\Phi(t)$ is monotonic, the limits $C_\Phi = \lim_{t\to\infty} F_\Phi(t)$ and $C^0_\Phi = \lim_{t\to 0} F_\Phi(t)$ must exist and we have

$$\alpha_\Phi = \beta_\Phi = \lim_{u\to\infty} G_\Phi(u) = 2^{-1/C_\Phi}, \quad \alpha^0_\Phi = \beta^0_\Phi = \lim_{u\to 0} G_\Phi(u) = 2^{-1/C^0_\Phi}. \tag{25}$$

This makes it easier to calculate the indices in Theorem 4.1.

EXAMPLE 2. Let a pair of complementary N-functions be

$$M(u) = e^{|u|} - |u| - 1 \quad \text{and} \quad N(v) = (1+|v|)\ln(1+|v|) - |v|.$$

Then $p(t) = M'(t) = e^t - 1$ is convex and $q(s) = N'(s) = \ln(1+s)$ is concave on $[0, +\infty)$. It is easy to check that the index function $F_M(t) = t(e^t-1)/(e^t-t-1)$ is increasing and $F_N(t) = t\ln(1+t)/[(1+t)\ln(1+t) - t]$ is decreasing on $[0,\infty)$. In view of Remark 1, the index function G_M is accordingly increasing on $[0,\infty)$, with

G_N decreasing on $[0, \infty)$. Therefore $\tilde{\beta}_M$ and $\tilde{\alpha}_N$ both take their value at the right end of $[0, 1]$, that is,

$$\tilde{\beta}_M = \frac{M^{-1}(1/2)}{M^{-1}(1)} \approx 0.74828; \quad \tilde{\alpha}_N = \frac{N^{-1}(1/2)}{N^{-1}(1)} \approx 0.67250.$$

From Theorem 4.1 we have

$$J(l^{(M)}) = 2\tilde{\beta}_M = 2\beta'_M = \frac{2M^{-1}(1/2)}{M^{-1}(1)} \approx 1.49656;$$
$$J(l^{(N)}) = \frac{1}{\tilde{\alpha}_N} = \frac{1}{\alpha'_N} = \frac{N^{-1}(1)}{N^{-1}(1/2)} \approx 1.48699.$$

Since $C_M = \lim_{t\to\infty} F_M(t) = \infty$, $C_N = \lim_{t\to\infty} F_N(t) = 1$, we have

$$\alpha_M = \beta_M = 2^{-1/C_M} = 1, \quad \alpha_N = \beta_N = 2^{-1/C_N} = 1/2$$

by (25). Then from Theorem 4.1 we have

$$J(L^{(M)}[0,\infty)) = 2\bar{\beta}_M = 2\beta_M = 2; \quad J(L^{(N)}[0,\infty)) = \frac{1}{\bar{\alpha}_N} = \frac{1}{\alpha_N} = 2.$$

This result coincides with the fact that both the spaces $L^{(M)}[0,\infty)$ and $L^{(N)}[0,\infty)$ are nonreflexive.

EXAMPLE 3. Consider the N-function (see Gallardo [2])

$$\Phi_{p,r}(u) = |u|^p \ln^r(1+|u|), \quad 1 \le p < \infty,\ 0 < r < \infty.$$

It is easy to check that $\phi_{p,r}(t)$, the right derivative of $\Phi_{p,r}(u)$, is convex when $1 \le p < \infty$, $2 \le r < \infty$. The index function

$$F_{\Phi_{p,r}}(t) = \frac{t\Phi'_{p,r}(t)}{\Phi_{p,r}(t)} = p + \frac{rt}{(1+t)\ln(1+t)}$$

is decreasing from $p + r$ to p on $[0, \infty)$ since

$$\frac{d}{dt}\Phi_{p,r}(t) = \frac{r[\ln(1+t) - t]}{(1+t)^2 \ln^2(1+t)} < 0.$$

So $C^0_{\Phi_{p,r}}(t) = \lim_{t\to 0} F_{\Phi_{p,r}}(t) = p + r$. According to (25) and Theorem 4.1 we have

$$J(l^{(\Phi_{p,r})}) = J(L^{(\Phi_{p,r})}[0,\infty)) = 2\beta^0_{\Phi_{p,r}} = 2 \cdot 2^{-1/(p+r)} = 2^{1-1/(p+r)}.$$

REMARK 2. The author studied the estimation of $J(L^{(\Phi)}[0, 1])$ in [11] and showed that:

- if $\phi(t)$ is concave then $1/\alpha_{\Phi[1,\infty)} \le J(L^{(\Phi)}[0,1]) \le 1/\bar{\alpha}_\Phi$;
- and if $\phi(t)$ is convex then $2\beta_{\Phi[1,\infty)} \le J(L^{(\Phi)}[0,1]) \le 2\bar{\beta}_\Phi$, where

$$\alpha_{\Phi[1,\infty)} = \inf_{u\in[1,\infty)} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad \beta_{\Phi[1,\infty)} = \sup_{u\in[1,\infty)} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}.$$

Consequently we have the nonsquare constants for the N-functions given in Example 2 and Example 3:

$$J(L^{(M)}[0,1]) = J(L^{(N)}[0,1]) = 2;$$

$$2^{1-1/p} \le \frac{2\Phi_{p,r}^{-1}(1)}{\Phi_{p,r}^{-1}(2)} \le J(L^{(\Phi_{p,r})}[0,1]) \le 2^{1-1/(p+r)}.$$

Acknowledgments The author would like to thank Professors Z. D. Ren, D. H. Ji and J. H. Qiu for their advice and help.

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