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SMALE'S MEAN VALUE CONJECTURE FOR ODD POLYNOMIALS

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Abstract

In this paper, we shall show that the constant in Smale's mean value theorem can be reduced to two for a large class of polynomials which includes the odd polynomials with nonzero linear term.

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1. Introduction and main result

Let P be any polynomial; then b is a critical point of P if and only if $P'(b) = 0$, and v is a critical value of P if and only if $v = P(b)$ for some critical point b of P.

In 1981 Steve Smale proved the following interesting result about critical points and critical values of polynomials.

THEOREM A ([3]). *Let P be a non-linear polynomial and a be any given complex number. Then there exists a critical point b of P such that*

$$\left|\frac{P(a)-P(b)}{a-b}\right| \leq 4|P'(a)| \tag{1}$$

or equivalently, we have

$$\min_{b,P'(b)=0}\left|\frac{P(a)-P(b)}{a-b}\right| \leq 4|P'(a)|. \tag{2}$$

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Smale then asked whether one can replace the factor 4 in the upper bound in (1) by 1, or even possibly by $(d-1)/d$. He also pointed out that the number $(d-1)/d$ would, if true, be the best possible bound here as it is attained (for any nonzero A, B) when $P(z) = Az^d - Bz$ and $a = 0$ in (1). The conjecture has been verified for $d = 2, 3, 4$, and also in some other special circumstances (see [1, 4] and the references therein).

It is easy (see [1]) to show that Smale's conjecture is equivalent to the following:

NORMALISED CONJECTURE. *Let P be a monic polynomial of degree $d \geq 2$ such that $P(0) = 0$ and $P'(0) \neq 0$. Let $b_1, \ldots, b_{d-1}$ be its critical points. Then*

$$\min_i \left| \frac{P(b_i)}{b_i} \right| \leq N|P'(0)| \tag{3}$$

holds for $N = 1$ (or even $(d-1)/d$).

Let M_d be the least possible value of N such that (3) holds for all degree d polynomials. Recently, in [1], it was shown that $M_d \leq 4^{(d-2)/(d-1)}$. In this paper we shall prove that for a very large class of polynomials (which includes the non-linear odd polynomials), one can take $N = 2$ in (3).

THEOREM 1. *Let P be a polynomial of degree $d \geq 2$ such that $P(0) = 0$ and $P'(0) \neq 0$. Let $b_1, \ldots, b_{d-1}$ be its critical points such that $|b_1| \leq |b_2| \leq \cdots \leq |b_{d-1}|$. Suppose that $b_2 = -b_1$, then*

$$\min_i \left| \frac{P(b_i)}{b_i} \right| \leq 2|P'(0)|. \tag{4}$$

COROLLARY 1. *If P is a nonlinear odd polynomial with nonzero linear term, then (4) holds for P.*

PROOF. If P is a non-linear odd polynomial (that is, $P(-z) = -P(z)$), then $P(0) = 0$. Hence, $P(z) = z^k Q(z^2)$ for some odd number $k \geq 1$ and non-constant polynomial Q with $Q(0) \neq 0$. Since the linear term of P is nonzero, $P'(0) \neq 0$. Clearly, $P'(z) = R(z^2)$ for some suitable polynomial R. Therefore, we can take $b_2 = -b_1$ and apply Theorem 1 to complete the proof. □

PROOF OF THEOREM 1. We may assume that $P(b_i) \neq 0$, for all i, for otherwise, we are done. Therefore, $r = \min_i\{|P(b_i)|\} > 0$ as there are only finitely many critical values. Let $\mathbb{D}(0, r)$ be the open disk with center $w = 0$ and radius r. Then $\mathbb{D}(0, r)$ contains no critical values of P. Since $P(0) = 0$ and $P'(0) \neq 0$, by the inverse function theorem, $P^{-1}(z)$ exists in a neighbourhood of 0 with $P^{-1}(0) = 0$. By the Monodromy Theorem, $P^{-1}(z)$ can be extended to a single valued function on the whole $\mathbb{D}(0, r)$.

Let $f : \mathbb{D}(0,1) \to \mathbb{C}$ be defined by $f(z) = P^{-1}(rz)$. Then f is an univalent function and omits all the b_i's. This will give some restrictions on the size of $|f'(0)|$ which is equal to $r/|(P'(0)|$. In fact, we have the following result of Lavrent'ev.

THEOREM B ([2]). *Let $0 \le \theta \le 2\pi$. Suppose $f : \mathbb{D}(0,1) \to \mathbb{C}$ is an univalent function which omits the set $A = \{Re^{\{\theta+(2\pi j)/n\}i} \mid 1 \le j \le n\}$, then $|f'(0)| \le 4^{1/n}R$.*

Recall that $|b_1| \le |b_2| \le \cdots \le |b_{d-1}|$, so $\min_i\{|b_i|\} = |b_1|$. Since $b_2 = -b_1$, we can take $n = 2$ in Theorem B. Now

$$\min_i \left|\frac{P(b_i)}{b_i}\right| \frac{1}{|P'(0)|} \le \frac{\min_i\{|P(b_i)|\}}{\min_i\{|b_i|\}|P'(0)|} = \frac{r}{\min_i\{|b_i|\}|P'(0)|}$$
$$= \frac{|f'(0)|}{\min_i\{|b_i|\}} = \frac{|f'(0)|}{|b_1|} \le \frac{4^{1/2}|b_1|}{|b_1|} \le 2$$

and we are done. □

Note added in proof. From the proof of Theorem 1 and Corollary 1, it is easy to see that if for some kth root of unity λ we have $p(\lambda z) = \lambda p(z)$ identically and $p'(0) \ne 0$ (for example, polynomials of the form $zQ(z^k)$, $Q(0) \ne 0$), then (3) holds with $N = 4^{1/k}$. Of course for k at least 3 there are not so many of these polynomials, but interestingly for the conjectured extremal example of $p(z) = Az^n - Bz$, this holds with $k = n - 1$.

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References

[1] A. F. Beardon, D. Minda and T.W. Ng, 'Smale's mean value conjecture and the hyperbolic metric', *Math. Ann.* **322** (2002), 623–632.

[2] M. A. Lavrent'ev, 'On the theory of conformal mappings', *Trav. Inst. Phys.-Math. Stekloff* **5** (1934), 159–245; English translation: *Transl., II. Ser., Amer. Math. Soc.* **122** (1984), 1–63.

[3] S. Smale, 'The fundamental theorem of algebra and complexity theory', *Bull. Amer. Math. Soc.* **4** (1981), 1–36.

[4] ———, 'Mathematical problems for the next century', in: *Mathematics: frontiers and perspectives* (eds. V. Arnold, M. Atiyah, P. Lax and B. Mazur) (Amer. Math. Soc., Providence, 2000) pp. 271–294.

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