

ON ψ -DIRECT SUMS OF BANACH SPACES AND CONVEXITY

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Dedicated to Maestro Ivry Gitlis on his 80th birthday with deep respect and affection

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Abstract

Let X_1, X_2, \dots, X_N be Banach spaces and ψ a continuous convex function with some appropriate conditions on a certain convex set in \mathbb{R}^{N-1} . Let $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_\psi$ be the direct sum of X_1, X_2, \dots, X_N equipped with the norm associated with ψ . We characterize the strict, uniform, and locally uniform convexity of $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_\psi$ by means of the convex function ψ . As an application these convexities are characterized for the $\ell_{p,q}$ -sum $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_{p,q}$ ($1 < q \leq p \leq \infty, q < \infty$), which includes the well-known facts for the ℓ_p -sum $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_p$ in the case $p = q$.

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1. Introduction and preliminaries

A norm $\|\cdot\|$ on \mathbb{C}^N is called *absolute* if $\|(z_1, \dots, z_N)\| = \||z_1|, \dots, |z_N|\|$ for all $(z_1, \dots, z_N) \in \mathbb{C}^N$, and *normalized* if $\|(1, 0, \dots, 0)\| = \dots = \|(0, \dots, 0, 1)\| = 1$ (see for example [3, 2]). In case of $N = 2$, according to Bonsall and Duncan [3] (see also [12]), for every absolute normalized norm $\|\cdot\|$ on \mathbb{C}^2 there corresponds a unique continuous convex function ψ on the unit interval $[0, 1]$ satisfying

$$\max\{1 - t, t\} \leq \psi(t) \leq 1$$

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under the equation $\psi(t) = \|(1 - t, t)\|$. Recently in [11] Saito, Kato and Takahashi presented the N -dimensional version of this fact, which states that for every absolute normalized norm $\|\cdot\|$ on \mathbb{C}^N there corresponds a unique continuous convex function ψ satisfying some appropriate conditions on the convex set

$$\Delta_N = \left\{ t = (t_1, \dots, t_{N-1}) \in \mathbb{R}^{N-1} : \sum_{j=1}^{N-1} t_j \leq 1, t_j \geq 0 \right\}$$

under the equation $\psi(t) = \|(1 - \sum_{j=1}^{N-1} t_j, t_1, \dots, t_{N-1})\|$.

For an arbitrary finite number of Banach spaces X_1, X_2, \dots, X_N , we define the ψ -direct sum $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_\psi$ to be their direct sum equipped with the norm

$$\|(x_1, x_2, \dots, x_N)\|_\psi = \|(\|x_1\|, \|x_2\|, \dots, \|x_N\|)\|_\psi \quad \text{for } x_j \in X_j,$$

where $\|\cdot\|_\psi$ term in the right-hand side is the absolute normalized norm on \mathbb{C}^N with the corresponding convex function ψ . This extends the notion of ℓ_p -sum of Banach spaces. The aim of this paper is to characterize the strict, and uniform convexity of $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_\psi$. The locally uniform convexity is also included. For the case $N = 2$, the first two have been recently proved in Takahashi-Kato-Saito [13] and Saito-Kato [10], respectively. However the proof of the uniform convexity for the 2-dimensional case given in [10] seems difficult to be extended to the N -dimensional case, though it is of independent interest as it is of real analytic nature and maybe useful for estimating the modulus of convexity. Our proof for the N -dimensional case is essentially different from that in [10]. As an application we shall consider the $\ell_{p,q}$ -sum $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_{p,q}$ and show that $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_{p,q}$ is uniformly convex if and only if all X_j are uniformly convex, where $1 < q \leq p \leq \infty, q < \infty$. The same is true for the strict and locally uniform convexity. These results include the well-known facts for the ℓ_p -sum $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_p$ as the case $p = q$.

Let us recall some definitions. A Banach space X or its norm $\|\cdot\|$ is called *strictly convex* if $\|x\| = \|y\| = 1$ ($x \neq y$) implies $\|(x + y)/2\| < 1$. This is equivalent to the following statement: if $\|x + y\| = \|x\| + \|y\|, x \neq 0, y \neq 0$, then $x = \lambda y$ with some $\lambda > 0$ (see for example [9, page 432], [1]). X is called *uniformly convex* provided for any ϵ ($0 < \epsilon < 2$) there exists $\delta > 0$ such that whenever $\|x - y\| \geq \epsilon, \|x\| = \|y\| = 1$, one has $\|(x + y)/2\| \leq 1 - \delta$, or equivalently, provided for any ϵ ($0 < \epsilon < 2$) one has $\delta_X(\epsilon) > 0$, where δ_X is the *modulus of convexity* of X , that is,

$$\delta_X(\epsilon) := \inf\{1 - \|(x + y)/2\|; \|x - y\| \geq \epsilon, \|x\| = \|y\| = 1\} \quad (0 \leq \epsilon \leq 2).$$

We also have the following restatement: X is uniformly convex if and only if, whenever $\|x_n\| = \|y_n\| = 1$ and $\|(x_n + y_n)/2\| \rightarrow 1$, it follows that $\|x_n - y_n\| \rightarrow 0$. X is called *locally uniformly convex* (see for example [9, 4]) if for any $x \in X$ with $\|x\| = 1$ and

for any ϵ ($0 < \epsilon < 2$) there exists $\delta > 0$ such that if $\|x - y\| \geq \epsilon$, $\|y\| = 1$, then $\|(x + y)/2\| \leq 1 - \delta$. Clearly the notion of locally uniform convexity is between those of uniform and strict convexities.

2. Absolute norms on \mathbb{C}^N and ψ -direct sums $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_\psi$

Let AN_N denote the family of all absolute normalized norms on \mathbb{C}^N . Let

$$\Delta_N = \{(s_1, s_2, \dots, s_{N-1}) \in \mathbb{R}^{N-1} : s_1 + s_2 + \dots + s_{N-1} \leq 1, s_j \geq 0(\forall j)\}.$$

For any $\|\cdot\| \in AN_N$ define the function ψ on Δ_N by

$$(1) \quad \psi(s) = \|(1 - s_1 - \dots - s_{N-1}, s_1, \dots, s_{N-1})\| \quad \text{for } s = (s_1, \dots, s_{N-1}) \in \Delta_N.$$

Then ψ is continuous and convex on Δ_N , and satisfies the following conditions:

$$(A_0) \quad \psi(0, \dots, 0) = \psi(1, 0, \dots, 0) = \dots = \psi(0, \dots, 0, 1) = 1,$$

$$(A_1) \quad \psi(s_1, \dots, s_{N-1}) \geq (s_1 + \dots + s_{N-1})\psi\left(\frac{s_1}{\sum_{i=1}^{N-1} s_i}, \dots, \frac{s_{N-1}}{\sum_{i=1}^{N-1} s_i}\right),$$

$$(A_2) \quad \psi(s_1, \dots, s_{N-1}) \geq (1 - s_1)\psi\left(0, \frac{s_2}{1 - s_1}, \dots, \frac{s_{N-1}}{1 - s_1}\right),$$

.....

$$(A_N) \quad \psi(s_1, \dots, s_{N-1}) \geq (1 - s_{N-1})\psi\left(\frac{s_1}{1 - s_{N-1}}, \dots, \frac{s_{N-2}}{1 - s_{N-1}}, 0\right).$$

Note that from (A_0) it follows that $\psi(s_1, \dots, s_{N-1}) \leq 1$ on Δ_N as ψ is convex. Denote Ψ_N be the family of all continuous convex functions ψ on Δ_N satisfying (A_0) , (A_1) , \dots , (A_N) . Then the converse holds true: For any $\psi \in \Psi_N$ define

$$(2) \quad \|(z_1, \dots, z_N)\|_\psi = \begin{cases} \left(\sum_{i=1}^N |z_i|\right) \psi\left(|z_2|/(\sum_{i=1}^N |z_i|), \dots, |z_N|/(\sum_{i=1}^N |z_i|)\right) & \text{if } (z_1, \dots, z_N) \neq (0, \dots, 0), \\ 0 & \text{if } (z_1, \dots, z_N) = (0, \dots, 0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_N$ and $\|\cdot\|_\psi$ satisfies (1). Thus the families AN_N and Ψ_N are in one-to-one correspondence under equation (1) (Saito-Kato-Takahashi [11, Theorem 4.2]).

The ℓ_p -norms

$$\|(z_1, \dots, z_N)\|_p = \begin{cases} \{\{z_1|^p + \dots + |z_N|^p\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{|z_1|, \dots, |z_N|\} & \text{if } p = \infty \end{cases}$$

are typical examples of absolute normalized norms, and for any $\|\cdot\| \in AN_N$ we have

$$(3) \quad \|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1$$

([11, Lemma 3.1], see also [3]). The functions corresponding to ℓ_p -norms on \mathbb{C}^N are

$$\psi_p(s_1, \dots, s_{N-1}) = \begin{cases} \left\{ \left(1 - \sum_{j=1}^{N-1} s_j \right)^p + s_1^p + \dots + s_{N-1}^p \right\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \max \left\{ 1 - \sum_{j=1}^{N-1} s_j, s_1, \dots, s_{N-1} \right\} & \text{if } p = \infty \end{cases}$$

for $(s_1, \dots, s_{N-1}) \in \Delta_N$.

Let X_1, X_2, \dots, X_N be Banach spaces. Let $\psi \in \Psi_N$ and let $\|\cdot\|_\psi$ be the corresponding norm in AN_N . Let $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_\psi$ be the direct sum of X_1, X_2, \dots, X_N equipped with the norm

$$(4) \quad \|(x_1, x_2, \dots, x_N)\|_\psi := \|(\|x_1\|, \|x_2\|, \dots, \|x_N\|)\|_\psi \quad \text{for } x_j \in X_j.$$

As is it immediately seen, $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_\psi$ is a Banach space.

EXAMPLE. Let $1 \leq q \leq p \leq \infty, q < \infty$. We consider the Lorentz $\ell_{p,q}$ -norm $\|z\|_{p,q} = \left\{ \sum_{j=1}^N j^{(q/p)-1} z_j^{*q} \right\}^{1/q}$ for $z = (z_1, \dots, z_N) \in \mathbb{C}^N$, where $\{z_j^*\}$ is the non-increasing rearrangement of $\{|z_j|\}$, that is, $z_1^* \geq z_2^* \geq \dots \geq z_N^*$. (Note that in case of $1 \leq p < q \leq \infty, \|\cdot\|_{p,q}$ is not a norm but a quasi-norm (see [6, Proposition 1], [14, page 126])). Evidently $\|\cdot\|_{p,q} \in AN_N$ and the corresponding convex function $\psi_{p,q}$ is obtained by

$$(5) \quad \psi_{p,q}(s) = \|(1 - s_1 - \dots - s_{N-1}, s_1, \dots, s_{N-1})\|_{p,q}$$

(for $s = (s_1, \dots, s_{N-1}) \in \Delta_N$), that is, $\|\cdot\|_{p,q} = \|\cdot\|_{\psi_{p,q}}$. Let $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_{p,q}$ be the direct sum of Banach spaces X_1, X_2, \dots, X_N equipped with the norm

$$\|(x_1, \dots, x_N)\|_{p,q} := \|(\|x_1\|, \dots, \|x_N\|)\|_{p,q},$$

we call it the $\ell_{p,q}$ -sum of X_1, X_2, \dots, X_N . If $p = q$ the $\ell_{p,p}$ -sum is the usual ℓ_p -sum $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_p$.

For some other examples of absolute norms on \mathbb{C}^N we refer the reader to [11] (see also [12]).

3. Strict convexity of $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_\psi$

A function ψ on Δ_N is called *strictly convex* if for any $s, t \in \Delta_N$ ($s \neq t$) one has $\psi((s + t)/2) < (\psi(s) + \psi(t))/2$. For absolute norms on \mathbb{C}^N , we have

LEMMA 3.1 (Saito-Kato-Takahashi [11, Theorem 4.2]). *Let $\psi \in \Psi_N$. Then $(\mathbb{C}^N, \|\cdot\|_\psi)$ is strictly convex if and only if ψ is strictly convex.*

The following lemma concerning the monotonicity property of the absolute norms on \mathbb{C}^N is useful in the sequel.

LEMMA 3.2 (Saito-Kato-Takahashi [11, Lemma 4.1]). *Let $\psi \in \Psi_N$. Let $z = (z_1, \dots, z_N)$, $w = (w_1, \dots, w_N) \in \mathbb{C}^N$.*

(i) *If $|z_j| \leq |w_j|$ for all j , then $\|z\|_\psi \leq \|w\|_\psi$.*

(ii) *Let ψ be strictly convex. If $|z_j| \leq |w_j|$ for all j and $|z_j| < |w_j|$ for some j , then $\|z\|_\psi < \|w\|_\psi$.*

THEOREM 3.3. *Let X_1, X_2, \dots, X_N be Banach spaces and let $\psi \in \Psi_N$. Then $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_\psi$ is strictly convex if and only if X_1, X_2, \dots, X_N are strictly convex and ψ is strictly convex.*

PROOF. Let $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_\psi$ be strictly convex. Then, each X_j and $(\mathbb{C}^N, \|\cdot\|_\psi)$ are strictly convex since they are isometrically imbedded into $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_\psi$. According to Lemma 3.1, ψ is strictly convex.

Conversely, let each X_j and ψ be strictly convex. Take arbitrary $x = (x_j)$, $y = (y_j)$, $x \neq y$, in $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_\psi$ with $\|x\|_\psi = \|y\|_\psi = 1$. Let first $(\|x_1\|, \dots, \|x_N\|) = (\|y_1\|, \dots, \|y_N\|)$. Then, if $\|x + y\|_\psi = 2$,

$$\begin{aligned} 2 &= \|x + y\|_\psi = \|(\|x_1 + y_1\|, \dots, \|x_N + y_N\|)\|_\psi \\ &\leq \|(\|x_1\| + \|y_1\|, \dots, \|x_N\| + \|y_N\|)\|_\psi \leq \|x\|_\psi + \|y\|_\psi = 2, \end{aligned}$$

from which it follows that $\|x_j + y_j\| = \|x_j\| + \|y_j\|$ for all j by Lemma 3.2. As each X_j is strictly convex, $x_j = k_j y_j$ with $k_j > 0$. Since $\|x_j\| = \|y_j\|$, we have $k_j = 1$ and hence $x_j = y_j$ for all j , or $x = y$, which is a contradiction. Therefore we have $\|x + y\|_\psi < 2$. Let next $(\|x_1\|, \dots, \|x_N\|) \neq (\|y_1\|, \dots, \|y_N\|)$. Since ψ is strictly convex, $(\mathbb{C}^N, \|\cdot\|_\psi)$ is strictly convex by Lemma 3.1. Consequently we have

$$\begin{aligned} \|x + y\|_\psi &= \|(\|x_1 + y_1\|, \dots, \|x_N + y_N\|)\|_\psi \\ &\leq \|(\|x_1\| + \|y_1\|, \dots, \|x_N\| + \|y_N\|)\|_\psi \\ &= \|(\|x_1\|, \dots, \|x_N\|) + (\|y_1\|, \dots, \|y_N\|)\|_\psi < 2, \end{aligned}$$

as is desired. □

Now we see that the function $\psi_{p,q}$ in the above example is strictly convex if $1 < q \leq p \leq \infty$, $q < \infty$. We need the next lemma.

LEMMA 3.4 ([5]). Let $\{\alpha_j\}, \{\beta_j\} \in \mathbb{R}^N$ and $\alpha_j \geq 0, \beta_j \geq 0$. Let $\{\alpha_j^*\}, \{\beta_j^*\}$ be their non-increasing rearrangements, that is, $\alpha_1^* \geq \alpha_2^* \geq \dots \geq \alpha_N^*$ and $\beta_1^* \geq \beta_2^* \geq \dots \geq \beta_N^*$. Then $\sum_{j=1}^N \alpha_j \beta_j \leq \sum_{j=1}^N \alpha_j^* \beta_j^*$.

PROPOSITION 3.5. Let $1 < q \leq p \leq \infty, q < \infty$. Then the function $\psi_{p,q}$ given by (5) is strictly convex on Δ_N .

PROOF. Let $s = (s_j), t = (t_j) \in \Delta_N, s \neq t$. Without loss of generality we may assume that

$$2 - (s_1 + t_1) - \dots - (s_{N-1} + t_{N-1}) \geq s_1 + t_1 \geq \dots \geq s_{N-1} + t_{N-1} \geq 0.$$

Put

$$\begin{aligned} \sigma &= (1 - s_1 - \dots - s_{N-1}, 2^{1/p-1/q} s_1, \dots, N^{1/p-1/q} s_{N-1}), \\ \tau &= (1 - t_1 - \dots - t_{N-1}, 2^{1/p-1/q} t_1, \dots, N^{1/p-1/q} t_{N-1}). \end{aligned}$$

Then by Lemma 3.4 we have

$$\begin{aligned} \|\sigma\|_q &= \left\{ (1 - s_1 - \dots - s_{N-1})^q + 2^{q/p-1} s_1^q + \dots + N^{q/p-1} s_{N-1}^q \right\}^{1/q} \\ &\leq \|(1 - s_1 - \dots - s_{N-1}, s_1, \dots, s_{N-1})\|_{p,q} = \psi_{p,q}(s) \end{aligned}$$

and $\|\tau\|_q \leq \psi_{p,q}(t)$. On the other hand,

$$\begin{aligned} \psi_{p,q}\left(\frac{s+t}{2}\right) &= \left\{ \left(1 - \sum_{i=1}^{N-1} \frac{s_i + t_i}{2}\right)^q + \sum_{i=1}^{N-1} (i+1)^{q/p-1} \left(\frac{s_i + t_i}{2}\right)^q \right\}^{1/q} \\ &= \left[\left(\frac{1}{2} \left\{ \left(1 - \sum_{i=1}^{N-1} s_i\right) + \left(1 - \sum_{i=1}^{N-1} t_i\right) \right\}\right)^q \right. \\ &\quad \left. + \sum_{i=1}^{N-1} \left(\frac{1}{2} \{(i+1)^{1/p-1/q} s_i + (i+1)^{1/p-1/q} t_i\}\right)^q \right]^{1/q} = \left\| \frac{\sigma + \tau}{2} \right\|_q. \end{aligned}$$

Since ℓ_q -norm $\|\cdot\|_q$ ($1 < q < \infty$) is strictly convex and $s \neq t$, we have $\|\sigma + \tau\|_q < \|\sigma\|_q + \|\tau\|_q$. Indeed, if $\|\sigma + \tau\|_q = \|\sigma\|_q + \|\tau\|_q$, then $\sigma = k\tau$ with some $k > 0$ (note that $\sigma \neq 0, \tau \neq 0$). Hence $s_j = kt_j$ for all j , and $1 - \sum_{i=1}^{N-1} s_i = k(1 - \sum_{i=1}^{N-1} t_i)$. Therefore, $k = 1$ and we have $s = t$, which is a contradiction. Consequently,

$$\psi_{p,q}\left(\frac{s+t}{2}\right) = \left\| \frac{\sigma + \tau}{2} \right\|_q < \frac{\|\sigma\|_q + \|\tau\|_q}{2} \leq \frac{\psi_{p,q}(s) + \psi_{p,q}(t)}{2},$$

or $\psi_{p,q}$ is strictly convex. □

By Theorem 3.3 and Proposition 3.5 we have the following result for the $\ell_{p,q}$ -sum of Banach spaces.

COROLLARY 3.6. *Let $1 < q \leq p \leq \infty$, $q < \infty$. Then, $\ell_{p,q}$ -sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{p,q}$ is strictly convex if and only if X_1, X_2, \dots, X_N are strictly convex.*

In particular, the ℓ_p -sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_p$, $1 < p < \infty$, is strictly convex if and only if X_1, X_2, \dots, X_N are strictly convex.

4. Uniform convexity of $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$

Let us characterize the uniform convexity of $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$.

THEOREM 4.1. *Let X_1, X_2, \dots, X_N be Banach spaces and let $\psi \in \Psi_N$. Then $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ is uniformly convex if and only if X_1, X_2, \dots, X_N are uniformly convex and ψ is strictly convex.*

PROOF. The necessity assertion is proved in the same way as the proof of Theorem 3.3. Assume that X_1, X_2, \dots, X_N are uniformly convex and ψ is strictly convex. Take an arbitrary $\epsilon > 0$ and put

$$\delta := 2\delta_X(\epsilon) = \inf\{2 - \|x + y\|_\psi : \|x - y\|_\psi \geq \epsilon, \|x\|_\psi = \|y\|_\psi = 1\}.$$

We show that $\delta > 0$. There exist sequences $\{x_n\}$ and $\{y_n\}$ in $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ so that

$$(6) \quad \begin{aligned} \|x_n - y_n\|_\psi &\geq \epsilon, \\ \|x_n\|_\psi &= \|y_n\|_\psi = 1 \end{aligned}$$

and

$$(7) \quad \lim_{n \rightarrow \infty} \|x_n + y_n\|_\psi = 2 - \delta.$$

Let $x_n = (x_1^{(n)}, \dots, x_N^{(n)})$ and $y_n = (y_1^{(n)}, \dots, y_N^{(n)})$. Since for each $1 \leq j \leq N$, $\|x_j^{(n)}\| = \|(0, \dots, 0, x_j^{(n)}, 0, \dots, 0)\|_\psi \leq \|x_n\|_\psi = 1$ and $\|y_j^{(n)}\| \leq \|y_n\|_\psi = 1$ for all n , the sequences $\{\|x_j^{(n)}\|\}_n$ and $\{\|y_j^{(n)}\|\}_n$ have a convergent subsequence respectively. So we may assume that $\|x_j^{(n)}\| \rightarrow a_j$, $\|y_j^{(n)}\| \rightarrow b_j$ as $n \rightarrow \infty$. Further, in the same way, we may assume that

$$(8) \quad \|x_j^{(n)} - y_j^{(n)}\| \rightarrow c_j \quad \text{as } n \rightarrow \infty$$

and

$$(9) \quad \|x_j^{(n)} + y_j^{(n)}\| \rightarrow d_j \quad \text{as } n \rightarrow \infty.$$

Put $K_n = \sum_{j=1}^N \|x_j^{(n)}\|$. Then $\|x_n\|_\psi = K_n \psi(\|x_2^{(n)}\|/K_n, \dots, \|x_N^{(n)}\|/K_n) = 1$. Letting $n \rightarrow \infty$, as ψ is continuous, we have

$$(10) \quad \|(a_1, \dots, a_N)\|_\psi = \left(\sum_{j=1}^N a_j \right) \psi \left(\frac{a_2}{\sum_{j=1}^N a_j}, \dots, \frac{a_N}{\sum_{j=1}^N a_j} \right) = 1.$$

Also we have

$$(11) \quad \|(b_1, \dots, b_N)\|_\psi = \left(\sum_{j=1}^N b_j \right) \psi \left(\frac{b_2}{\sum_{j=1}^N b_j}, \dots, \frac{b_N}{\sum_{j=1}^N b_j} \right) = 1.$$

Next let $n \rightarrow \infty$ in (6), or in

$$\begin{aligned} \|x_n - y_n\|_\psi &= \left(\sum_{j=1}^N \|x_j^{(n)} - y_j^{(n)}\| \right) \\ &\quad \times \psi \left(\frac{\|x_2^{(n)} - y_2^{(n)}\|}{\sum_{j=1}^N \|x_j^{(n)} - y_j^{(n)}\|}, \dots, \frac{\|x_N^{(n)} - y_N^{(n)}\|}{\sum_{j=1}^N \|x_j^{(n)} - y_j^{(n)}\|} \right) \geq \epsilon. \end{aligned}$$

Then we have

$$(12) \quad \|(c_1, \dots, c_N)\|_\psi = \left(\sum_{j=1}^N c_j \right) \psi \left(\frac{c_2}{\sum_{j=1}^N c_j}, \dots, \frac{c_N}{\sum_{j=1}^N c_j} \right) \geq \epsilon$$

by (8). In the same way, according to (7) and (9), we have

$$(13) \quad \|(d_1, \dots, d_N)\|_\psi = 2 - \delta.$$

Now, assume that $(a_1, \dots, a_N) \neq (b_1, \dots, b_N)$. Then, according to (10), (11) and the strict convexity of ψ we obtain that

$$2 - \delta = \|(d_1, \dots, d_N)\|_\psi \leq \|(a_1 + b_1, \dots, a_N + b_N)\|_\psi < 2,$$

which implies $\delta > 0$. Next, let $(a_1, \dots, a_N) = (b_1, \dots, b_N)$. Since $(c_1, \dots, c_N) \neq (0, \dots, 0)$ from (12), we may assume that $c_1 > 0$ without loss of generality. Then as

$$c_1 = \lim_{n \rightarrow \infty} \|x_1^{(n)} - y_1^{(n)}\| \leq \lim_{n \rightarrow \infty} (\|x_1^{(n)}\| + \|y_1^{(n)}\|) = a_1 + b_1 = 2a_1,$$

we have $a_1 > 0$ and

$$(14) \quad 0 < \frac{c_1}{a_1} = \lim_{n \rightarrow \infty} \left\| \frac{x_1^{(n)}}{\|x_1^{(n)}\|} - \frac{y_1^{(n)}}{\|x_1^{(n)}\|} \right\| = \lim_{n \rightarrow \infty} \left\| \frac{x_1^{(n)}}{\|x_1^{(n)}\|} - \frac{y_1^{(n)}}{\|y_1^{(n)}\|} \right\|.$$

Indeed, we have the latter identity because

$$\begin{aligned} &\left\| \left\| \frac{x_1^{(n)}}{\|x_1^{(n)}\|} - \frac{y_1^{(n)}}{\|x_1^{(n)}\|} \right\| - \left\| \frac{x_1^{(n)}}{\|x_1^{(n)}\|} - \frac{y_1^{(n)}}{\|y_1^{(n)}\|} \right\| \right\| \\ &\leq \|y_1^{(n)}\| \left| \frac{1}{\|x_1^{(n)}\|} - \frac{1}{\|y_1^{(n)}\|} \right| \rightarrow b_1 \left| \frac{1}{a_1} - \frac{1}{b_1} \right| = 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Since X_1 is uniform convex, it follows from (14) that

$$\frac{d_1}{a_1} = \lim_{n \rightarrow \infty} \left\| \frac{x_1^{(n)}}{\|x_1^{(n)}\|} + \frac{y_1^{(n)}}{\|y_1^{(n)}\|} \right\| = \lim_{n \rightarrow \infty} \left\| \frac{x_1^{(n)}}{\|x_1^{(n)}\|} + \frac{y_1^{(n)}}{\|y_1^{(n)}\|} \right\| < 2,$$

whence $d_1 < 2a_1$. Accordingly, by (13) and Lemma 3.2 we obtain that

$$\begin{aligned} 2 - \delta &= \|(d_1, d_2, \dots, d_N)\|_\psi \\ &< \|(2a_1, a_2 + b_2, \dots, a_N + b_N)\|_\psi \\ &= \|(a_1 + b_1, a_2 + b_2, \dots, a_N + b_N)\|_\psi \\ &\leq \|(a_1, \dots, a_N)\|_\psi + \|(b_1, \dots, b_N)\|_\psi = 2, \end{aligned}$$

which implies $\delta > 0$. This completes the proof. \square

The parallel argument works for the locally uniform convexity and we obtain the next result.

THEOREM 4.2. *Let $\psi \in \Psi_N$. Then $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_\psi$ is locally uniformly convex if and only if X_1, X_2, \dots, X_N are locally uniformly convex and ψ is strictly convex.*

Indeed, for the sufficiency, take an arbitrary $x \in (X_1 \oplus X_2 \oplus \dots \oplus X_N)_\psi$ with $\|x\|_\psi = 1$ and merely let $x_n = x$ in the above proof. By Theorem 4.1 and Theorem 4.2 combined with Proposition 3.5 we obtain the following corollary.

COROLLARY 4.3. *Let $1 < q \leq p \leq \infty, q < \infty$. Then, $\ell_{p,q}$ -sum $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_{p,q}$ is uniformly convex (locally uniformly convex) if and only if X_1, X_2, \dots, X_N are uniformly convex (locally uniformly convex).*

In particular, the ℓ_p -sum $(X_1 \oplus X_2 \oplus \dots \oplus X_N)_p, 1 < p < \infty$, is uniformly convex (locally uniformly convex) if and only if X_1, X_2, \dots, X_N are uniformly convex (locally uniformly convex).

References

- [1] B. Beauzamy, *Introduction to Banach spaces and their geometry*, 2nd edition (North-Holland, Amsterdam, 1985).
- [2] R. Bhatia, *Matrix analysis* (Springer, Berlin, 1997).
- [3] F. F. Bonsall and J. Duncan, *Numerical ranges II*, London Math. Soc. Lecture Note Ser. 10 (Cambridge University Press, Cambridge, 1973).
- [4] J. Diestel, *Geometry of Banach spaces*, Lecture Notes in Math. 485 (Springer, Berlin, 1975).

- [5] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities* (Cambridge University Press, Cambridge, 1967).
- [6] M. Kato, 'On Lorentz spaces $\ell_{p,q}\{E\}$ ', *Hiroshima Math. J.* **6** (1976), 73–93.
- [7] D. Kutzarova and T. Landes, 'Nearly uniform convexity of infinite direct sums', *Indiana Univ. Math. J.* **41** (1992), 915–926.
- [8] ———, 'NUC and related properties of finite direct sums', *Boll. Un. Math. Ital. (7)* **8A** (1994), 45–54.
- [9] R. E. Megginson, *An introduction to Banach space theory* (Springer, New York, 1998).
- [10] K.-S. Saito and M. Kato, 'Uniform convexity of ψ -direct sums of Banach spaces', *J. Math. Anal. Appl.* **277** (2003), 1–11.
- [11] K.-S. Saito, M. Kato and Y. Takahashi, 'On absolute norms on \mathbb{C}^n ', *J. Math. Anal. Appl.* **252** (2000), 879–905.
- [12] ———, 'von Neumann-Jordan constant of absolute normalized norms on \mathbb{C}^2 ', *J. Math. Anal. Appl.* **244** (2000), 515–532.
- [13] Y. Takahashi, M. Kato and K.-S. Saito, 'Strict convexity of absolute norms on \mathbb{C}^2 and direct sums of Banach spaces', *J. Inequal. Appl.* **7** (2002), 179–186.
- [14] H. Triebel, *Interpolation theory, function spaces, differential operators* (North-Holland, Amsterdam, 1978).

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