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ON NOETHERIAN RINGS WITH ESSENTIAL SOCLE

JIANLONG CHEN, NANQING DING and MOHAMED F. YOUSIF

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Abstract

It is shown that if R is a right Noetherian ring whose right socle is essential as a right ideal and is contained in the left socle, then R is right Artinian. This result may be viewed as a one-sided version of a result of Ginn and Moss on two-sided Noetherian rings with essential socle. This also extends the work of Nicholson and Yousif where the same result is obtained under a stronger hypothesis. We use our work to obtain partial positive answers to some open questions on right CF, right FGF and right Johns rings.

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1. Introduction

All rings are associative with identity and all modules are unitary. If R is a ring, we denote by $\mathrm{Soc}(R_R) = S_r$, $\mathrm{Soc}({}_RR) = S_l$, $Z(R_R) = Z_r$ and $J(R) = J$ for the right socle, the left socle, the right singular ideal and the Jacobson radical of R, respectively. The left and right annihilators of a subset X of R are denoted by $\mathbf{l}(X)$ and $\mathbf{r}(X)$, respectively. We use $K \leq_e N$ to indicate that K is an essential submodule of N. General background material can be found in [1].

It is well known that over a commutative ring, every Noetherian module with essential socle is Artinian. This is not true for arbitrary right Noetherian rings ([7, 8]). However a result of Ginn and Moss [8, Theorem] asserts that a two-sided Noetherian ring with essential right socle is right and left Artinian. Recently, Nicholson and Yousif [15] obtained a one-sided version of this theorem, by showing that a right Noetherian, right minsymmetric ring (whenever xR is a minimal right ideal of R, then Rx is a minimal left ideal, for every $x \in R$) with essential right socle is right Artinian.

We extend this result by replacing the right minsymmetric condition by the weaker condition $S_r \subseteq S_l$.

Recall that a ring R is called *right Johns* if R is right Noetherian and every right ideal of R is a right annihilator, and that R is *strongly right Johns* if $M_n(R)$ is right Johns for every $n \geq 1$. In [7] an example of a right Johns ring which is not right Artinian is given. However it is not known whether a strongly right Johns ring is right Artinian. Here we prove that a right Johns ring with $S_r \subseteq S_l$ is right Artinian. It is also shown that a right Johns left coherent ring is right Artinian. Hence a strongly right Johns and left coherent ring is QF.

A ring R is called *right FGF* (CF) if every finitely generated (cyclic) right R-module embeds in a free right R-module. It is still open whether a right FGF (CF) ring is QF (right Artinian). We show that if R is a semilocal, right CF ring with $S_r \subseteq S_l$, then R is right Artinian. In particular, a semilocal right FGF ring with $S_r \subseteq S_l$ is QF.

Finally, it is shown that if R satisfies the condition $\mathbf{l}(\mathbf{r}(a) \cap T) = Ra + \mathbf{l}(T)$ for every $a \in R$ and any right ideal T of R, then R is a right weakly continuous ring. As a corollary, some conditions are given to force a right CF ring to be QF.

2. The results

A ring R is called *left Kasch* if every simple left R-module can be embedded in ${}_RR$.

LEMMA 2.1. *Let R be a ring such that R/J is left Kasch and $J = \mathbf{l}(a_1, a_2, \ldots, a_n)$, where $a_i \in R$, $i = 1, 2, \ldots, n$. Then R is a left Kasch ring.*

PROOF. Let K be a simple left R-module. Then K is a simple left R/J-module. Since R/J is left Kasch, there is an R/J-monomorphism $\phi : K \to R/J$. Clearly, ϕ is a monic R-homomorphism. By hypothesis, $J = \mathbf{l}(a_1, a_2, \ldots, a_n)$, and so there is a monomorphism $\psi : R/J \to R^n$. Hence $f = \psi\phi$ is monic. Let $\pi_i : R^n \to R$ be the ith projection, $i = 1, 2, \ldots, n$. Then it is easy to see $\pi_i f$ is monic for some i. So K embeds in ${}_RR$. □

A ring R is said to be a *right $C2$-ring* if every right ideal that is isomorphic to a direct summand of R_R is itself a direct summand, R is called *right finitely cogenerated* if S_r is a finitely generated right ideal and $S_r \leq_e R_R$. The following lemma is a key to our results.

LEMMA 2.2. *Let R be a right finitely cogenerated ring with $S_r \subseteq S_l$. Then the following are equivalent*:

(1) *R is left Kasch.*

(2) *R is a right $C2$-ring.*
(3) $Z_r \subseteq J$.

In this case, R is semilocal and $Z_r = J = \mathbf{l}(S_r) = \mathbf{l}(S_l)$.

PROOF. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) holds in every ring R without any additional hypotheses by [16, Proposition 4.1].

Now we assume that (3) holds.

(i) We claim that R is semilocal. First we have $\mathbf{l}(S_r) \subseteq J$. In fact, let $a \in \mathbf{l}(S_r)$, then $S_r \subseteq \mathbf{r}(a)$. Since R is right finitely cogenerated, $S_r \leq_e R_R$. Thus $\mathbf{r}(a) \leq_e R_R$, and so $a \in Z_r \subseteq J$.

Next, we prove that, for any simple right ideal kR of R, $\mathbf{l}(k) = \bigcap_{i=1}^{s} \mathbf{l}(k_i)$ for some positive integer s, where each $k_i \in R$ and $\mathbf{l}(k_i)$ is a maximal left ideal, $i = 1, 2, \ldots, s$. As a matter of fact, since $k \in kR \subseteq S_r \subseteq S_l$, $Rk \subseteq S_l$. So Rk is semisimple. Now, without loss of generality, we may assume that s is the smallest integer such that $Rk = Rl_1 \oplus Rl_2 \oplus \cdots \oplus Rl_s$, where each Rl_i is simple, $i = 1, 2, \ldots, s$. Let $k = r_1 l_1 + r_2 l_2 + \cdots + r_s l_s$, then $r_i l_i \neq 0$, $i = 1, 2, \ldots, s$, by the choice of s. Let $k_i = r_i l_i$, then $k = k_1 + k_2 + \cdots + k_s$, and so $\mathbf{l}(k) = \bigcap_{i=1}^{s} \mathbf{l}(k_i)$. Since Rl_i is simple and $Rk_i = Rr_i l_i$, Rk_i is simple. Hence $\mathbf{l}(k_i)$ is maximal for each $i = 1, 2, \ldots, s$.

Finally, since R is right finitely cogenerated, S_r is finitely generated. Let $S_r = a_1 R + a_2 R + \cdots + a_n R$, where each $a_i R$ is a simple right ideal, $i = 1, 2, \ldots, n$. By the preceding proof, we have $\mathbf{l}(a_i) = \bigcap_{j=1}^{t_i} \mathbf{l}(a_{ij})$, where $\mathbf{l}(a_{ij})$ is a maximal left ideal for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, t_i$. Thus

$$J \supseteq \mathbf{l}(S_r) = \mathbf{l}\left(\sum_{i=1}^{n} a_i R\right) = \bigcap_{i=1}^{n} \mathbf{l}(a_i) = \bigcap_{i=1}^{n} \bigcap_{j=1}^{t_i} \mathbf{l}(a_{ij}).$$

Clearly, $J \subseteq \bigcap_{i=1}^{n} \bigcap_{j=1}^{t_i} \mathbf{l}(a_{ij})$. Thus $J = \bigcap_{i=1}^{n} \bigcap_{j=1}^{t_i} \mathbf{l}(a_{ij})$, and so R is semilocal.

(ii) $Z_r = J = \mathbf{l}(S_r) = \mathbf{l}(S_l)$. Since R is semilocal by (i), $S_l = \mathbf{r}(J)$. Hence $JS_l = 0$. By hypothesis, $S_r \subseteq S_l$, and so $JS_r = 0$. But $S_r \leq_e R_R$ by hypothesis, and hence $J \subseteq Z_r$. Therefore, $J = Z_r$.

On the other hand, for any ring R, we have $Z_r S_r = 0$, and so $Z_r \subseteq \mathbf{l}(S_r)$. Hence $J = Z_r \subseteq \mathbf{l}(S_r)$. However $\mathbf{l}(S_r) \subseteq J$ by the proof of (i). So $\mathbf{l}(S_r) = J$.

Note that $J \subseteq \mathbf{l}(\mathbf{r}(J)) = \mathbf{l}(S_l)$ (for $S_l = \mathbf{r}(J)$), and $\mathbf{l}(S_l) \subseteq \mathbf{l}(S_r) = J$ (for $S_r \subseteq S_l$), then $J = \mathbf{l}(S_l)$.

(iii) R is left Kasch. $J = \mathbf{l}(S_r)$ by (ii). But S_r is finitely generated, and so R is left Kasch by Lemma 2.1. □

COROLLARY 2.3. *Let R be a right finitely cogenerated ring with $S_r \subseteq S_l$ and ACC on right annihilators. Then R is a semiprimary ring with $J = Z_r$.*

PROOF. Since R has ACC on right annihilators, Z_r is nilpotent by [5, Lemma 18.3] or [10, Proposition 3.31]. So $Z_r \subseteq J$. By Lemma 2.2, R is semilocal and $J = Z_r$. Thus R is semiprimary. □

Recall that a ring R is *semiregular* if R/J is von Neumann regular and idempotents can be lifted modulo J. R is *right weakly continuous* ([17]) if R is semiregular and $J = Z_r$. R is *semiperfect* in case R/J is semisimple Artinian and idempotents lift modulo J. R is called *left P-injective* if every left R-homomorphism from a principal left ideal into R extends to an endomorphism of R.

LEMMA 2.4. *Let R be a semiperfect ring such that $S_l \leq_e R_R$. Then*:

(1) $J = Z_r$.

(2) *R is right weakly continuous.*

PROOF. (1). Suppose $x \in J \subseteq \mathbf{l}(S_l)$, then $xS_l = 0$. Since $S_l \leq_e R_R$, $x \in Z_r$. So $J \subseteq Z_r$. On the other hand, R is left Kasch by [16, Lemma 3.11], and hence $Z_r \subseteq J$ by [16, Proposition 4.1]. Thus (1) follows.

(2). This follows from (1) and the hypothesis. □

In general, a right Noetherian ring with essential right socle need not be right Artinian as shown by Faith-Menal's example ([7]). The following theorem shows that the condition $S_r \subseteq S_l$ is strong enough to force a right Noetherian ring with essential right socle to be right Artinian.

THEOREM 2.5. *The following are equivalent for a ring R*:

(1) *R is a right Noetherian ring such that $S_r \subseteq S_l$ and $S_r \leq_e R_R$.*

(2) *R is right Artinian with $J = Z_r$.*

(3) *R is right Artinian and right weakly continuous.*

PROOF. (1) $\Rightarrow$ (2). Since R is right Noetherian, S_r is finitely generated. Thus R is right finitely cogenerated by hypothesis and so R is a semiprimary ring by Corollary 2.3. So R is a right Artinian ring by Hopkin's theorem. Since $S_r \subseteq S_l$ and $S_r \leq_e R_R$, $S_l \leq_e R_R$. Thus $J = Z_r$ by Lemma 2.4.

(2) $\Rightarrow$ (1). Let $x \in S_r$, then $Z_r x = 0$. Thus $x \in \mathbf{r}(Z_r) = \mathbf{r}(J) = S_l$ (for R is semilocal). So $S_r \subseteq S_l$.

(2) $\Leftrightarrow$ (3). Every right Artinian ring is semiperfect, and hence semiregular. So (2) $\Leftrightarrow$ (3) follows. □

A ring R is called *right minsymmetric* if, whenever kR is a simple right ideal of R, then Rk is also simple, for every $k \in R$. If R is right minsymmetric, then $S_r \subseteq S_l$. The condition that $S_r \subseteq S_l$ simply means that, whenever kR is a simple right ideal, then Rk is a semisimple left ideal. The next example shows that a ring satisfying $S_r \subseteq S_l$ need not be right minsymmetric.

EXAMPLE 1. Let $F = \mathbb{Z}_2 = \{0, 1\}$ be the field of two elements and

$$R = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & d \end{pmatrix} : a, b, c, d \in F \right\}.$$

Then R is a ring under usual addition and multiplication of matrices. It can be easily checked that

$$\begin{pmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F \end{pmatrix},$$
$$R \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d & 0 & d \end{pmatrix} : d \in F \right\}$$

are all simple left ideals of R, and

$$\begin{pmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} R = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ a & 0 & 0 \end{pmatrix} : a \in F \right\}$$

are all simple right ideals of R. Hence

$$S_l = \begin{pmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ F & 0 & F \end{pmatrix} \quad \text{and} \quad S_r = \begin{pmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ F & 0 & 0 \end{pmatrix}.$$

Clearly, $S_r \subseteq S_l$.

Let $x = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right) \in R$. Then

$$xR = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ a & 0 & 0 \end{pmatrix} : a \in F \right\}$$

is a simple right ideal, but $Rx = \left(\begin{smallmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ F & 0 & 0 \end{smallmatrix}\right)$ is a semisimple left ideal which is not simple. So R is not right minsymmetric.

REMARK 1. Example 1 above shows that Lemma 2.2, Corollary 2.3 and Theorem 2.5 are non-trivial extensions of the work in [15, Lemma 1], [15, Theorem 1] and [15, Theorem 2], respectively.

COROLLARY 2.6. *The following are equivalent for a ring R:*

(1) *R is a right Johns ring with $S_r \subseteq S_l$.*
(2) *R is a right Artinian ring and every right ideal is a right annihilator of R.*

PROOF. (1) $\Rightarrow$ (2). Since R is a right Johns ring, $S_r \leq_e R_R$ by [16, Lemma 5.7 (4)]. Thus R is a right Artinian ring by Theorem 2.5.

(2) $\Rightarrow$ (1). This follows from [14, Theorem 3.7]. □

Recall that a ring R is *right finite dimensional* provided that R contains no infinite independent families of nonzero right ideals. R is said to be a *right Goldie ring* [10] if it is a right finite dimensional ring with ACC on right annihilators. We need the following lemma proved in [2, Lemma 6].

LEMMA 2.7. *Let R be a semiprimary ring with ACC on left annihilators, in which $S_r = S_l$ is finite dimensional as a right R-module. Then R is right Artinian.*

The following theorem extends the results in [6, Theorem], [12, Corollary 9] and [15, Theorem 3].

THEOREM 2.8. *Let R be a ring such that $S_r = S_l$. Then*:

(1) *If R is a right finitely cogenerated ring with ACC on right and left annihilators then R is right Artinian.*

(2) *If R is right and left Goldie with essential right socle then R is left and right Artinian.*

PROOF. (1) R is semiprimary by Corollary 2.3. Since R is a right finitely cogenerated, $S_l = S_r$ is a finitely generated right R-module. Note that R has ACC on left annihilators. So R is right Artinian by Lemma 2.7.

(2) Since R is right and left Goldie, R has ACC on right and left annihilators and R is right finite dimensional by definition. But $S_r \leq_e R_R$, and so R is right finitely cogenerated. Hence R is right Artinian by (1). In particular, R is right perfect, and so $S_l \leq_e {}_R R$. Note that R is left finite dimensional by hypothesis. Hence R is left finitely cogenerated. Thus R is left Artinian by (1). □

A ring R is called *left mininjective* if every R-homomorphism from a simple left ideal to R is given by right multiplication by an element of R. Left mininjective rings are always left minsymmetric.

THEOREM 2.9. *The following are equivalent for a ring R*:

(1) *R is left Noetherian and left mininjective such that $S_r \subseteq S_l$ and $S_l \leq_e {}_R R$.*

(2) *R is a left finitely cogenerated and left mininjective ring with $S_r \subseteq S_l$ and ACC on left annihilators.*

(3) *R is a left Artinian and left mininjective ring with $S_r = S_l$.*

PROOF. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). Since R is left mininjective, $S_l \subseteq S_r$. So $S_r = S_l$. By Corollary 2.3, R is semiprimary. In particular, R is semiperfect, and hence S_l is a finitely generated right ideal of R by [14, Proposition 3.3]. So R is right Artinian by Lemma 2.7. Thus R is right Noetherian, and hence R has ACC on right annihilators. Therefore, R is left Artinian by the left version of Theorem 2.8 (1).

(3) $\Rightarrow$ (1) is obvious. $\square$

REMARK 2. We note that a ring satisfying the equivalent conditions in Theorem 2.9 need not be QF. For example, let K be a field, ρ an isomorphism of K onto a subfield L and $R = K[X; \rho]/(X^2)$ the ring in [19, Example 1, pages 208–209]. Let $n = [K : L]$ be the vector space dimension of K over L such that $1 < n < \infty$. Then R is a left Artinian and left P-injective ring with $S_r = S_l$, but R is not QF.

The next theorem extends the results in [16, Theorem 5.9 (6)] and [16, Theorem 5.8 (1) (6) (7)].

THEOREM 2.10. *The following are equivalent for a ring R:*

(1) *R is semilocal and right CF with $S_r \subseteq S_l$.*
(2) *R is a right Artinian ring and every right ideal is a right annihilator of R.*

PROOF. (1) $\Rightarrow$ (2). Since R is right CF, R is left P-injective. Thus $S_l \subseteq S_r$, and so $S_l = S_r$. Since R is right Kasch (for R is right CF) and semilocal, we may assume that $a_1R, a_2R, \ldots, a_nR$ are the representatives for the isomorphism classes of simple right R-modules, where $a_i \in R$. $i = 1, 2, \ldots, n$. Note that $a_i \in a_iR \subseteq S_r = S_l$, and so $Ra_i \subseteq S_l = \mathrm{Soc}({}_RR)$. Thus there exists a simple left R-module Rm_i such that $Rm_i \subseteq Ra_i$, $i = 1, 2, \ldots, n$. Since R is left P-injective, there exists an epimorphism $\phi_i : a_iR \to m_iR$ by [13, Theorem 1.1 (1)]. Note that m_iR is simple (for Rm_i is simple). It follows that ϕ_i is an isomorphism, $i = 1, 2, \ldots, n$. Thus $m_1R, m_2R, \ldots, m_nR$ are the representatives for the isomorphism classes of simple right R-modules. If $Rm_i \cong Rm_j$, then $m_iR \cong m_jR$ by [13, Theorem 1.1 (3)], and so $i = j$. Therefore, $Rm_1, Rm_2, \ldots, Rm_n$ are representatives for the isomorphism classes of simple left R-modules. Hence R is left Kasch. So R is right Artinian by [9, Corollary 2.6].

(2) $\Rightarrow$ (1). Let R/A be a cyclic right R-module, where A is a right ideal of R. Then R/A is torsionless (for $A = \mathbf{r}(\mathbf{l}(A))$) and finitely cogenerated because R is right Artinian. Hence R/A embeds in a free right R-module. So R is right CF. $S_r \subseteq S_l$ follows from Corollary 2.6. $\square$

A ring R is called *left* 2-*injective* ([4, 13]) if R-maps from 2-generated left ideals to R are all given by right multiplication. We need the following result of Rutter.

LEMMA 2.11 ([19, Corollary 3]). *If a ring R is left 2-injective and has ACC on left annihilators, then R is QF.*

A ring R is called *right* 2-*GF* if every 2-generated right R-module embeds in a free right R-module. R is called *right* 2-*Johns* if $M_2(R)$ is right Johns.

THEOREM 2.12. *The following are equivalent for a ring R:*

(1) *R is QF.*
(2) *R is right 2-GF, semilocal and $S_r \subseteq S_l$.*
(3) *R is right 2-Johns and $S_r \subseteq S_l$.*

In particular, every semilocal right FGF ring with $S_r \subseteq S_l$ is QF, and every strongly right Johns ring with $S_r \subseteq S_l$ is QF.

PROOF. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear.

(2) $\Rightarrow$ (1). Since R is right 2-GF, it is right CF. Then R is right Artinian by (2) and Theorem 2.10. Hence R satisfies ACC on left annihilators. If $R_R \to R_R^2 \to N_R \to 0$ is an exact sequence of right R-modules, then N_R is 2-generated, and so it is torsionless (for R is right 2-GF). Thus R is left 2-injective by [4, Theorem 2.17], and hence R is QF by Lemma 2.11.

(3) $\Rightarrow$ (1). Since R is a right 2-Johns ring, it is not difficult to see that R is also right Johns. Thus R is right Artinian by (3) and Corollary 2.6, and so R has ACC on left annihilators. By hypothesis, $M_2(R)$ is left P-injective, and so R is left 2-injective by [13, Theorem 4.2]. Thus R is QF by Lemma 2.11. □

We end this paper with the following results which are of independent interest. Recall that a ring R is called *left coherent* if any direct product of copies of R is flat as a right R-module.

THEOREM 2.13. *Let R be a right Johns and left coherent ring. Then R is right Artinian.*

PROOF. By [11, Theorem 6.1.2], R is right Artinian if and only if every cyclic right R-module is finitely cogenerated. Since every right ideal is a right annihilator, R/I is torsionless for every right ideal I of R. Let $f : R/I \to \prod R$ be a monomorphism from R/I to a product of copies of R. Note that R is a right Noetherian ring, and so R/I is finitely presented. Since R is left coherent, $\prod R$ is a flat right R-module. Hence f factors through a finitely generated free module R^n, that is, there exist $g : R/I \to R^n$ and $h : R^n \to \prod R$ such that $f = hg$. Since f is monomorphic, so is g. This shows that every cyclic right R-module R/I embeds in a free module R^n for some positive integer n (that is, R is right CF). Since R is right Johns, R is right finitely cogenerated. Thus R^n is finitely cogenerated, and so is R/I. This completes the proof. □

REMARK 3. A right Johns and left coherent ring need not be QF because there is a two-sided Artinian right Johns ring which is not QF as shown by Rutter [19, Example 1].

THEOREM 2.14. *If R satisfies one of the following two conditions, then R is a right weakly continuous ring.*

(1) $\mathbf{l}(\mathbf{r}(a) \cap T) = Ra + \mathbf{l}(T)$ *for every* $a \in R$ *and any right ideal* T *of* R.
(2) *R is right P-injective and every complement right ideal of R is principal.*

PROOF. In either case, R is right P-injective, and so $Z_r = J$ by [13, Theorem 2.1]. It suffices to show that, for any $a \in R$, aR has an additive complement in R. Let T be an (intersection) complement of $\mathbf{r}(a)$, that is, T is a right ideal maximal with respect to $\mathbf{r}(a) \cap T = 0$.

If condition (1) holds, $R = \mathbf{l}(\mathbf{r}(a) \cap T) = Ra + \mathbf{l}(T)$. Let $K \subseteq \mathbf{l}(T)$ such that $R = Ra + K$. Then $1 = ra + k$ for some $r \in R$ and $k \in K$, and so $R = Ra + Rk$. Thus $0 = \mathbf{r}(Ra + Rk) = \mathbf{r}(a) \cap \mathbf{r}(k)$. Note that $\mathbf{r}(k) \supseteq \mathbf{r}(K) \supseteq \mathbf{r}(\mathbf{l}(T)) \supseteq T$. The choice of T gives $T = \mathbf{r}(k)$. Therefore, $\mathbf{l}(T) = \mathbf{l}(\mathbf{r}(k)) = Rk \subseteq K \subseteq \mathbf{l}(T)$, and so $K = \mathbf{l}(T)$. This shows that $\mathbf{l}$(T) is the additive complement of Ra. So R is semiregular.

If condition (2) holds, then $T = bR$ for some $b \in R$. Thus $\mathbf{r}(a) \cap bR = 0$, and so $R = \mathbf{l}(\mathbf{r}(a) \cap bR) = Ra + \mathbf{l}(b)$ by [13, Lemma 1.1]. Thus $\mathbf{l}(b)$ is the additive complement of Ra by the foregoing proof, and so R is semiregular. □

In general, a right CF ring need not be QF even if it is left (and right) Artinian (see [19, 18]). Next we give some conditions which guarantee that a right CF ring is QF. Recall that a ring R is called *right CS* ([5]) if every nonzero right ideal is essential in a direct summand of R.

COROLLARY 2.15. *The following are equivalent for a right CF ring R:*

(1) $\mathbf{l}(\mathbf{r}(a) \cap T) = Ra + \mathbf{l}(T)$ *for every* $a \in R$ *and any right ideal* T *of* R.
(2) $I = \mathbf{l}(\mathbf{r}(I))$ *for every finitely generated left ideal* I *of* R.
(3) *R is right P-injective and every complement right ideal of R is principal.*
(4) *R is right CS and left 2-injective.*
(5) *R is left Kasch and left 2-injective.*
(6) *R is left Kasch and right mininjective.*
(7) *R is QF.*

PROOF. It is clear that (7) implies (1) through (6).

(2) ⇒ (1). Let T be a right ideal of R. Then $T = \mathbf{r}(K)$ for a finitely generated left ideal K of R since R is right CF, and so $\mathbf{l}(T) = \mathbf{l}(\mathbf{r}(K)) = K$ by (2). Let $a \in R$.

Then $\mathbf{l}(\mathbf{r}(a) \cap T) = \mathbf{l}(\mathbf{r}(a) \cap \mathbf{r}(K)) = \mathbf{l}(\mathbf{r}(Ra + K)) = Ra + K = Ra + \mathbf{l}(T)$ by (2), as required.

(1) or (3) $\Rightarrow$ (7). By Theorem 2.14, R is right weakly continuous. Hence R is right Artinian by [21, Proposition 1.22]. Note that R is right P-injective, and so it is right mininjective. Thus R is QF by [3, Theorem 3.1].

(4) $\Rightarrow$ (7). R is right Artinian by [9, Corollary 3.10]. Thus R has ACC on left annihilators, and so R is QF by Lemma 2.11.

(5) $\Rightarrow$ (6) follows since a left Kasch and left 2-injective ring is right P-injective by [13, Lemma 2.2] or [4, Corollary 2.8 (2)].

(6) $\Rightarrow$ (7). R is right Artinian by [9, Corollary 2.6]. So R is QF by [14, Corollary 4.8]. □

REMARK 4. (i) Corollary 2.15 (2) was obtained in [20, Corollary 15].

(ii) In [19] there is an example of a right CF, right CS and left Kasch ring which is not QF.

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Department of Mathematics
Nanjing University
Nanjing 210093
China
and
Department of Mathematics
Southeast University
Nanjing 210096
China
e-mail: jlchen@seu.edu.cn

Department of Mathematics
Nanjing University
Nanjing 210093
China
e-mail: nqding@nju.edu.cn

Department of Mathematics
Ohio State University
Lima, Ohio 45804
USA
e-mail: yousif.1@osu.edu