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A NOTE ON NORMALITY AND SHARED VALUES

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Abstract

Let k be a positive integer and b a nonzero constant. Suppose that $\mathscr{F}$ is a family of meromorphic functions in a domain D. If each function $f \in \mathscr{F}$ has only zeros of multiplicity at least $k+2$ and for any two functions $f, g \in \mathscr{F}$, f and g share 0 in D and $f^{(k)}$ and $g^{(k)}$ share b in D, then $\mathscr{F}$ is normal in D. The case $f \neq 0$, $f^{(k)} \neq b$ is a celebrated result of Gu.

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1. Introduction

Let D be a domain in $\mathbb{C}$ and $\mathscr{F}$ a family of meromorphic functions defined in D. $\mathscr{F}$ is said to be normal in D, in the sense of Montel, if each sequence $\{f_n\} \subset \mathscr{F}$ has a subsequence $\{f_{n_j}\}$ which converges spherically locally uniformly in D, to a meromorphic function or ∞ (see Hayman [4], Schiff [7], Yang [12]).

Suppose that f, g are meromorphic functions on D and $a \in \mathbb{C} \cup \{\infty\}$. If $f(z) = a$ if and only if $g(z) = a$, we say that f and g share a in D.

In 1912, Montel [6] proved the following well-known normality criterion.

THEOREM A. *Let $\mathscr{F}$ be a family of meromorphic functions defined in D, and let a, b and c be three distinct values in the extended complex plane. If for each function $f \in \mathscr{F}$, $f \neq a, b, c$, then $\mathscr{F}$ is normal in D.*

In 1994, Sun [8] extended Theorem A as follows (see for example [1]).

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THEOREM B. *Let $\mathscr{F}$ be a family of meromorphic functions defined in D, and let a, b and c be three distinct values in the extended complex plane. If each pair of functions f and g in $\mathscr{F}$ share a, b and c in D, then $\mathscr{F}$ is normal in D.*

In 1979, Gu [2] proved the following result.

THEOREM C. *Let $\mathscr{F}$ be a family of meromorphic functions defined in D, and let k be a positive integer and b a nonzero constant. If for each function $f \in \mathscr{F}$, $f \neq 0$ and $f^{(k)} \neq b$ in D, then $\mathscr{F}$ is normal in D.*

It is natural to ask whether Theorem C can be extended in the same way that Theorem B extends Theorem A. In this note, we offer such an extension. In each of the results below, k is a positive integer and b is a nonzero complex constant.

THEOREM 1. *Let $\mathscr{F}$ be a family of meromorphic functions defined in D, all of whose zeros have multiplicity at least $k+2$. If each pair of functions f and g in $\mathscr{F}$ share 0 in D and $f^{(k)}$ and $g^{(k)}$ share b in D, then $\mathscr{F}$ is normal in D.*

EXAMPLE 1. Let n, k be positive integers. Let $D = \{z : |z| < 1\}$ and $\mathscr{F} = \{f_n\}$, where

$$f_n(z) = \frac{nz^{k+1}}{k!(nz-1)}, \qquad n = 1, 2, 3, \ldots.$$

Each function in $\mathscr{F}$ has a single zero of multiplicity $k+1$. Clearly, for each pair m, n of positive integers, f_m, f_n share 0 in D. Moreover, since

$$f_n(z) = \frac{1}{k!}\left(z^k + \frac{1}{n}z^{k-1} + \cdots + \frac{1}{n^{k-1}}z + \frac{1}{n^k} + \frac{1}{n^k}\frac{1}{nz-1}\right),$$

$$f_n^{(k)}(z) = 1 + \frac{(-1)^k}{(nz-1)^{k+1}} \neq 1.$$

Thus $f_m^{(k)}$ and $f_n^{(k)}$ also share the value 1 in D. But $\mathscr{F}$ clearly fails to be normal on any neighbourhood of 0. This shows that the condition in Theorem 1 that the zeros of functions in $\mathscr{F}$ have multiplicity at least $k+2$ cannot be weakened.

THEOREM 2. *Let $\mathscr{F}$ be a family of meromorphic functions defined in D, all of whose zeros have multiplicity at least $k+1$ and whose poles have multiplicity at least 2. If each pair of functions f and g in $\mathscr{F}$ share 0 in D and $f^{(k)}$ and $g^{(k)}$ share b in D, then $\mathscr{F}$ is normal in D.*

COROLLARY 3. *Let $\mathscr{F}$ be a family of holomorphic functions defined in D, all of whose zeros have multiplicity at least $k+1$. If each pair of functions f and g in $\mathscr{F}$ share 0 in D and $f^{(k)}$ and $g^{(k)}$ share b in D, then $\mathscr{F}$ is normal in D.*

COROLLARY 4. *Let $\mathscr{F}$ be a family of meromorphic functions defined in D. If each pair of functions f and g in $\mathscr{F}$ share 0 in D and $f^m f'$ and $g^m g'$ share b in D, then $\mathscr{F}$ is normal in D.*

To prove Corollary 4, set $\widetilde{\mathscr{F}} = \{f^{m+1}/(m+1) : f \in \mathscr{F}\}$ and apply Theorem 2 to this family with $k = 1$.

EXAMPLE 2. Let $D = \{z : |z| < 1\}$ and $\mathscr{F} = \{f_n\}$, where $f_n(z) = nz^k$, $n = 1, 2, 3, \ldots$. Then the zeros of functions in $\mathscr{F}$ all have multiplicity k. Moreover, any pair of functions f and g in $\mathscr{F}$ clearly share 0 in D and $f^{(k)}$ and $g^{(k)}$ share $1/2$ in D; but $\mathscr{F}$ is not normal in D. This shows that the condition that the zeros of functions in $\mathscr{F}$ have multiplicity at least $k+1$ in Theorem 2 and Corollary 3 is best possible.

2. Some lemmas

For the proofs of Theorem 1 and Theorem 2, we require the following results.

LEMMA 1 ([9, Theorem 7]). *Let $\mathscr{F}$ be a family of meromorphic functions defined in D, all of whose zeros have multiplicity at least $k+2$. If $f^{(k)} \neq b$ for each $f \in \mathscr{F}$, then $\mathscr{F}$ is normal in D.*

LEMMA 2 ([9, Theorem 5]). *Let $\mathscr{F}$ be a family of meromorphic functions defined in D, all of whose zeros have multiplicity at least $k+1$ and whose poles have multiplicity at least 2. If $f^{(k)} \neq b$ for each $f \in \mathscr{F}$, then $\mathscr{F}$ is normal in D.*

Below, we assume the basic results and notation of Nevanlinna Theory [4, 12]. In particular, $S(r, f)$ denotes any function satisfying $S(r, f) = O(\log r T(r, f))$ as $r \to \infty$, possibly outside a set of finite measure, where $T(r, f)$ is Nevanlinna's characteristic function. In fact, the functions for which we use this notation are all of finite order, so the exceptional set does not occur. For such functions, we have $S(r, f) = o(T(r, f))$ [4, page 41].

LEMMA 3 ([4, Theorem 3.2]). *Let f be a nonconstant meromorphic function in the complex plane. Then*

$$T(r, f) \leq \overline{N}(r, f) + N(r, 1/f) + \overline{N}\left(r, 1/(f^{(k)} - b)\right) + S(r, f). \tag{2.1}$$

By [4, page 61], we also have

LEMMA 4. *Let f be a nonconstant meromorphic function in the complex plane. Then*

$$\overline{N}(r, f) \leq \left(1 + \frac{1}{k}\right) N\left(r, \frac{1}{f}\right) + \left(1 + \frac{2}{k}\right) \overline{N}\left(r, \frac{1}{f^{(k)} - b}\right) + S(r, f). \tag{2.2}$$

LEMMA 5. *Let f be a meromorphic function in the complex plane and l a positive integer satisfying $l > k+4+2/k$. If $f \neq 0$ and the zeros of $f^{(k)} - b$ have multiplicity at least l, then f is a constant.*

PROOF. Since $f \neq 0$ and the zeros of $f^{(k)} - b$ have multiplicity at least l, we have by (2.2)

$$\begin{aligned} \overline{N}(r, f) &\leq \left(1+\frac{2}{k}\right)\overline{N}\left(r, \frac{1}{f^{(k)}-b}\right) + S(r, f) \\ &\leq \frac{1+2/k}{l} N\left(r, \frac{1}{f^{(k)}-b}\right) + S(r, f) \\ &\leq \frac{1+2/k}{l} T\left(r, f^{(k)}\right) + S(r, f) \\ &\leq \frac{1+2/k}{l}[T(r, f) + k\overline{N}(r, f)] + S(r, f). \end{aligned} \tag{2.3}$$

Thus by (2.3) we get

$$\overline{N}(r, f) \leq \frac{k+2}{k(l-k-2)} T(r, f) + S(r, f). \tag{2.4}$$

By (2.1) and the facts that $f \neq 0$ and the zeros of $f^{(k)} - b$ have multiplicity at least l, we have

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f^{(k)}-b}\right) + S(r, f) \\ &\leq \overline{N}(r, f) + \frac{1}{l} N\left(r, \frac{1}{f^{(k)}-b}\right) + S(r, f) \\ &\leq \overline{N}(r, f) + \frac{1}{l} T(r, f^{(k)}) + S(r, f) \\ &\leq \overline{N}(r, f) + \frac{1}{l}[T(r, f) + k\overline{N}(r, f)] + S(r, f) \\ &\leq \left(1+\frac{k}{l}\right)\overline{N}(r, f) + \frac{1}{l} T(r, f) + S(r, f). \end{aligned} \tag{2.5}$$

Thus

$$T(r, f) \leq \frac{l+k}{l-1}\overline{N}(r, f) + S(r, f). \tag{2.6}$$

By (2.4) and (2.6), we have

$$T(r, f) \leq \frac{(k+2)(l+k)}{k(l-1)(l-k-2)} T(r, f) + S(r, f),$$

that is, $[k(l-1)(l-k-2)-(k+2)(l+k)]T(r, f) \leq S(r, f)$. Since $l > k+4+2/k$, we have $k(l-1)(l-k-2)-(k+2)(l+k) > 0$. Thus $T(r, f) = S(r, f)$, so f is constant. □

LEMMA 6 ([3, Theorem 3], [4, Corollary to Theorem 3.5]). *Let f be a nonconstant meromorphic function on $\mathbb{C}$, and let b be a nonzero value. Then for each positive integer k, either f or $f^{(k)} - b$ vanishes. If f is transcendental, then for each positive integer k, either f or $f^{(k)} - b$ has infinitely many zeros.*

LEMMA 7 ([10, 13]). *Let $\mathscr{F}$ be a family of functions meromorphic on the unit disc. Suppose that each $f \in \mathscr{F}$, $f \neq 0$. Then if $\mathscr{F}$ is not normal, there exist, for each $\alpha \geq 0$,*

(a) *a number $0 < r < 1$;*
(b) *points z_n, $|z_n| < r$;*
(c) *functions $f_n \in \mathscr{F}$; and*
(d) *positive numbers $\rho_n \to 0$*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) = g_n(\zeta) \to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$.

3. Proof of Theorem 1

PROOF OF THEOREM 1. Let $z_0 \in D$. We show that $\mathscr{F}$ is normal at z_0. Let $f \in \mathscr{F}$. We consider two cases.

Case 1: $f^{(k)}(z_0) \neq b$. Then there exists a disk $D_\delta = \{z : |z - z_0| < \delta\}$ such that $f^{(k)} \neq b$ in D_δ. Thus, for every $g \in \mathscr{F}$, the zeros of g have multiplicity at least $k+2$ and $g^{(k)} \neq b$ in D_δ. By Lemma 1, $\mathscr{F}$ is normal in D_δ. Hence $\mathscr{F}$ is normal at z_0.

Case 2: $f^{(k)}(z_0) = b$. Then, by the condition of the theorem, $f(z_0) \neq 0$. Hence there exists a disk $D_\delta = \{z : |z - z_0| < \delta\}$ such that $f \neq 0$ in D_δ and $f^{(k)} \neq b$ in $D_\delta^o = \{z : 0 < |z - z_0| < \delta\}$. Hence, by Lemma 1, $\mathscr{F}$ is normal in D_δ^o. We complete the proof of the theorem by using the method of Yang [11].

Let $\{f_n\}$ be a sequence in $\mathscr{F}$; then there exists a subsequence of $\{f_n\}$ (which, without loss of generality, we may again denote by $\{f_n\}$) which converges locally spherically uniformly on D_δ^o to a function h. We consider two subcases.

Case 2.1: $h \not\equiv 0$. Then, by Hurwitz's Theorem, $h \neq 0$ in D_δ^o. Therefore,

$$\min_{0 \leq \theta \leq 2\pi} \left| h\left(z_0 + \delta e^{i\theta}/2\right) \right| > A > 0$$

for some constant A.

Hence for sufficiently large n,

$$\min_{0\le\theta\le 2\pi}\left|f_n\left(z_0+\frac{\delta}{2}e^{i\theta}\right)\right| > \frac{A}{2} > 0.$$

Since f_n is meromorphic and $f_n \neq 0$ in D_δ, $1/f_n$ is holomorphic in D_δ. Thus $1/f_n$ is holomorphic in $\overline{D}_{\delta/2} = \{z : |z - z_0| \le \delta/2\}$, and

$$\max_{0\le\theta\le 2\pi}\frac{1}{|f_n(z_0+\delta e^{i\theta}/2)|} < \frac{2}{A}.$$

By the maximum principle, we conclude that

$$\max_{|z-z_0|\le\delta/2}\frac{1}{|f_n(z)|} < \frac{2}{A}, \quad \text{so} \quad \min_{|z-z_0|\le\delta/2}|f_n(z)| > \frac{A}{2} > 0.$$

Hence there exists a subsequence of $\{f_n\}$ which converges locally spherically uniformly in $D_{\delta/2}$.

Case 2.2: $h \equiv 0$. Then $\{f_n\}$ converges locally uniformly to 0 in D_δ^o. Thus $\{f_n^{(k)}\}$ and $\{f_n^{(k+1)}\}$ also converge locally uniformly to 0. Hence, for sufficiently large n, we have by the argument principle

$$\left|N\left(\frac{\delta}{2}, z_0, f_n^{(k)} - b\right) - N\left(\frac{\delta}{2}, z_0, \frac{1}{f_n^{(k)} - b}\right)\right| = \left|\frac{1}{2\pi i}\int_{|z-z_0|=\delta/2}\frac{f_n^{(k+1)}(z)}{f_n^{(k)}(z) - b}\,dz\right| < 1. \tag{3.1}$$

Thus we have

$$N\left(\frac{\delta}{2}, z_0, f_n^{(k)} - b\right) = N\left(\frac{\delta}{2}, z_0, \frac{1}{f_n^{(k)} - b}\right).$$

Since any pole of $f_n^{(k)} - b$ must have multiplicity at least $k+1$, it follows that the zero of $f_n^{(k)} - b$ at z_0 has multiplicity at least $k+1$.

We consider two subcases.

Case 2.2.1. The set S of positive integers n such that the zeros of $f_n^{(k)} - b$ at z_0 have multiplicity greater than $k + 4 + 2/k$ is infinite. We claim that $G = \{f_n : n \in S\}$ is normal in $D_{\delta/2}$.

Indeed, suppose that G is not normal in $D_{\delta/2}$. Then by Lemma 7, we have (renumbering, as we may) $f_n \in G$, $z_n \in D_{\delta/2}$, and $\rho_n \to 0^+$ such that

$$g_n(\xi) = \frac{f_n(z_n + \rho_n\xi)}{\rho_n^k} \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$.

By Hurwitz's Theorem, $g \neq 0$ and any zeros of $g^{(k)} - b$ have multiplicity greater than $k + 4 + 2/k$. Thus, by Lemma 5, g is constant, a contradiction. Hence there exists a subsequence of $\{f_n\}$ which converges locally spherically uniformly in $D_{\delta/2}$.

Case 2.2.2. The set S_l of positive integers n such that the zeros of $f_n^{(k)} - b$ at z_0 have multiplicity l for some positive integer l such that $k + 1 \leq l \leq k + 4 + 2/k$ is infinite. We claim that $G = \{f_n : n \in S_l\}$ is normal in $D_{\delta/2}$.

In fact, suppose that G is not normal in $D_{\delta/2}$. Then by Lemma 7, we have (again renumbering) $f_n \in G$, $z_n \in D_{\delta/2}$, and $\rho_n \to 0^+$ such that

$$g_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n^k} \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$.

By Hurwitz's Theorem, $g \neq 0$ and each zero of $g^{(k)} - b$ has multiplicity at least l. We claim, in addition, that $g^{(k)} - b$ has only a single zero. That $g^{(k)} - b$ must vanish somewhere follows from Lemma 6. Suppose that ξ_1 and ξ_2 are distinct zeros of $g^{(k)} - b$; then the zeros of $g^{(k)} - b$ at ξ_1 and ξ_2 have multiplicity at least l. Let γ be a simple closed curve containing ξ_1 and ξ_2 in its interior and such that g has no zeros on γ and no poles on or inside γ. Then $g_n(\xi)$ converges to $g(\xi)$ uniformly on and inside γ, and so $g_n^{(k)} - b$ converges to $g^{(k)} - b$ uniformly on and inside γ. By the argument principle, $g_n^{(k)} - b$ and $g^{(k)} - b$ have the same number of zeros (counting multiplicity) inside γ for sufficiently large n. But $g_n^{(k)} - b$ has only l zeros (counting multiplicity) while $g^{(k)}$ has at least $2l$ zeros (counting multiplicity) for sufficiently large n, which is a contradiction.

From the above discussion, $g^{(k)} - b$ has only a single zero, whose multiplicity is l. Since $f_n^{(k)}(z_n + \rho_n \xi) = g_n^{(k)}(\xi)$, which converges to $g^{(k)}(\xi)$ uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of g, it follows from the formula after (3.1) that $f_n^{(k)}$ has l poles (counting multiplicity) in $D_{\delta/2}$ and hence $g_n^{(k)}$ has l poles (counting multiplicity) on the disc $\{\xi : z_n + \rho_n \xi \in D_{\delta/2}\}$. We conclude easily from the argument principle that $g^{(k)}$ has at most l poles (counting multiplicity) in $\mathbb{C}$.

Thus

(i) $g \neq 0$;
(ii) $g^{(k)} - b$ has a single zero, whose multiplicity is l;
(iii) $g^{(k)}$ has at most l poles, counting multiplicities.

We claim that no such function exists. By Lemma 6, there is no transcendental function, satisfying (i) and (ii). Clearly, g cannot be a polynomial. We now turn to the somewhat tedious verification that no rational function satisfies conditions (i), (ii), and (iii). We consider three subcases.

Case 2.2.2.1: $k \geq 3$. Since $k+1 \leq l \leq k+4+2/k$, g has only a single pole. Thus $g(\xi) = A/(\xi - a_1)^m$, where A is a nonzero constant, a_1 is a constant, and m is a positive integer.

Obviously, $g^{(k)} - b$ has $m+k$ distinct zeros, which contradicts the fact that $g^{(k)} - b$ has a single zero.

Case 2.2.2.2: $k = 2$. Since $3 \leq l \leq 7$, g has one of the following forms:

(1) $g(\xi) = A/(\xi - a_1)(\xi - a_2)^2, l = 7$;
(2) $g(\xi) = A/(\xi - a_1)(\xi - a_2), l = 6$;
(3) $g(\xi) = A/(\xi - a_1)^m, l = m+2, 1 \leq m \leq 5$,

where A is a nonzero constant, a_1 and a_2 are distinct constants, and m is a positive integer.

If $g(\xi) = A/[(\xi - a_1)(\xi - a_2)^2]$, then

$$g''(\xi) - b = -\frac{A[3(\xi - a_1)(\xi - a_2) - (3\xi - 2a_1 - a_2)(5\xi - 3a_1 - 2a_2)]}{(\xi - a_1)^3(\xi - a_2)^4} - \frac{b(\xi - a_1)^3(\xi - a_2)^4}{(\xi - a_1)^3(\xi - a_2)^4}.$$

Since $g'' - b$ has only a single zero, we have

$$A[3(\xi - a_1)(\xi - a_2) - (3\xi - 2a_1 - a_2)(5\xi - 3a_1 - 2a_2)] + b(\xi - a_1)^3(\xi - a_2)^4 = b(\xi - c)^7. \tag{3.2}$$

Differentiating the two sides of (3.2) three times, we have

$$(\xi - a_2)p(\xi) = 210b(\xi - c)^4, \tag{3.3}$$

where p is a polynomial and c is a constant.

Thus $a_2 = c$. It then follows from (3.2) that $a_1 = a_2$, a contradiction.

If g is of the form (2) or (3), we can similarly get a contradiction.

Case 2.2.2.3: $k = 1$. Since $2 \leq l \leq 7$, g has one of the following forms:

(1) $g(\xi) = A/(\xi - a_1)(\xi - a_2)(\xi - a_3)^2,\ l = 7$;
(2) $g(\xi) = A/(\xi - a_1)(\xi - a_2)(\xi - a_3),\ l = 6$;
(3) $g(\xi) = A/(\xi - a_1)^2(\xi - a_2)^m,\ l = m+4,\ 2 \leq m \leq 3$;
(4) $g(\xi) = A/(\xi - a_1)(\xi - a_2)^m,\ l = m+3,\ 1 \leq m \leq 4$;
(5) $g(\xi) = A/(\xi - a_1)^m, l = m+1,\ 1 \leq m \leq 6$,

where A is a nonzero constant, a_1, a_2 and a_3 are distinct constants, and m is a positive integer.

We deal with case (1). If $g(\xi) = A/[(\xi - a_1)(\xi - a_2)(\xi - a_3)^2]$, then

$$g'(\xi) - b = -\frac{A[(2\xi - a_1 - a_2)(\xi - a_3) + 2(\xi - a_1)(\xi - a_2)]}{(\xi - a_1)^2(\xi - a_2)^2(\xi - a_3)^3} - \frac{b(\xi - a_1)^2(\xi - a_2)^2(\xi - a_3)^3}{(\xi - a_1)^2(\xi - a_2)^2(\xi - a_3)^3}.$$

Since $g' - b$ has only a single zero, we have

$$A[(2\xi - a_1 - a_2)(\xi - a_3) + 2(\xi - a_1)(\xi - a_2)] + b(\xi - a_1)^2(\xi - a_2)^2(\xi - a_3)^3 = b(\xi - c)^7. \tag{3.4}$$

Differentiating the two sides of (3.4), we have

$$b(\xi - a_1)(\xi - a_2)(\xi - a_3)^2[2(2\xi - a_1 - a_2)(\xi - a_3) + 3(\xi - a_1)(\xi - a_2)] + A(8\xi - 3a_1 - 3a_2 - 2a_3) = 7b(\xi - c)^6. \tag{3.5}$$

Setting $\xi = a_3$ in (3.5) gives

$$3A(2a_3 - a_1 - a_2) = 7b(a_3 - c)^6. \tag{3.6}$$

Differentiating the two sides of (3.5), we obtain

$$8A + (\xi - a_3)p(\xi) = 42b(\xi - c)^5, \tag{3.7}$$

where p is a polynomial.

Setting $\xi = a_3$ in (3.7), we get

$$8A = 42b(a_3 - c)^5. \tag{3.8}$$

Thus by (3.6) and (3.8) we have

$$c = -\frac{7}{2}a_3 + \frac{9}{4}a_1 + \frac{9}{4}a_2. \tag{3.9}$$

On the other hand, differentiating both sides of (3.4) six times and putting $\xi = c$, we obtain

$$c = (2a_1 + 2a_2 + 3a_3)/7. \tag{3.10}$$

Comparing (3.9) and (3.10) gives $a_3 = c$, which contradicts (3.8) since $A \neq 0$.

If g has one of the other forms, we obtain a contradiction in a similar fashion.

Thus we have proved that $\{f_n\}$ is normal in $D_{\delta/2}$. Hence, there exists a subsequence of $\{f_n\}$ which converges locally spherically uniformly in $D_{\delta/2}$. It follows that $\mathscr{F}$ is normal at z_0, and so $\mathscr{F}$ is normal in D. The proof of the theorem is complete. □

The proof of Theorem 2, which uses Lemma 2, is similar. We omit the details.

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