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POSITIVE SOLUTIONS OF SOME QUASILINEAR SINGULAR SECOND ORDER EQUATIONS

J. V. GONCALVES and C. A. P. SANTOS

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Abstract

In this paper we study the existence and uniqueness of positive solutions of boundary value problems for continuous semilinear perturbations, say $f : [0, 1) \times (0, \infty) \to (0, \infty)$, of a class of quasilinear operators which represent, for instance, the radial form of the Dirichlet problem on the unit ball of $\mathbb{R}^N$ for the operators: p-Laplacian ($1 < p < \infty$) and k-Hessian ($1 \le k \le N$). As a key feature, $f(r, u)$ is possibly singular at $r = 1$ or $u = 0$. Our approach exploits fixed point arguments and the Shooting Method.

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1. Introduction

We study the existence and uniqueness of solutions for the class of quasilinear problems

$$\begin{cases} -\left(r^\alpha |u'|^\beta u'\right)' = r^\gamma f(r, u) \quad \text{in } (0, 1), \\ u > 0 \quad \text{in } (0, 1), \quad u(1) = u'(0) = 0, \end{cases} \tag{1.1}$$

where α, β, γ are given real numbers, $f : [0, 1) \times (0, \infty) \to (0, \infty)$ is continuous and $u' = du/dr$. The main feature here is that f is possibly singular at $r = 1$ or $u = 0$. The study of (1.1) is motivated by the search of radial solutions for several classes of quasilinear problems. In fact, denoting by B the unit ball of $\mathbb{R}^N$, if f is x-radially symmetric, (1.1) is the radial form of

$$-\Delta_p u = f(x, u) \text{ in } B, \quad u > 0 \text{ in } B, \quad u = 0 \text{ on } \partial B,$$

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where Δ_p $(1 < p < \infty)$ stands for the p-Laplace operator, provided $\alpha = \gamma = N - 1$ and $\beta = p - 2$, and is further the radial form of

$$(-1)^k S_k(\nabla^2 u) = f(x, u) \text{ in } B, \quad u > 0 \text{ in } B, \quad u = 0 \text{ on } \partial B,$$

where $S_k(\nabla^2 u)$ $(1 \leq k \leq N)$ is the k-Hessian operator,

$$S_k(\nabla^2 u) = \sum_{1 \leq i_1 < \cdots < i_k \leq N} \lambda_{i_1} \cdots \lambda_{i_k},$$

λ_{i_j} denoting the eigenvalues of the Hessian of u, namely $\nabla^2 u = (\partial^2 u / \partial x_i \partial x_j)$ where, in the present case, $\alpha = N - k$, $\gamma = N - 1$ and $\beta = k - 1$. We also remark that $S_1(\nabla^2 u) = \Delta$, (the Laplacian), and $S_N(\nabla^2 u)$ is the Monge-Ampére operator. We refer the reader to Tso [20, 19] and its references for properties of the k-Hessian. It is worth recalling that singular problems are also motivated by questions in the physical sciences. The reader is referred to Nachman and Callegari [2] for the problem

$$-(r^{N-1} u')' = \frac{k r^N}{u^{1/k}} \text{ in } (0, 1), \quad u'(0) = u(1) = 0,$$

with $k \in (0, 1)$, which appears in the theory of pseudoplastic fluids and Fulks and Maybee [11] for singular equations driven by questions in the theory of heat conduction in electrically conducting materials.

In the present article we shall exploit the following conditions:

(1.2) $$\beta > -1, \quad \gamma > \max\{-1, \alpha - 1\},$$

(1.3) $f(r, \cdot)$ is locally Lipschitz continuous in $(0, \infty)$, uniformly with respect to $r \in [0, 1)$,

(1.4) $$\frac{f(r, s)}{s^{\beta+1}} \text{ is decreasing in } s, \text{ for each } r,$$

(1.5) $$\lim_{s \to \infty} \frac{f(r, s)}{s^{\beta+1}} = 0, \quad \text{uniformly in } r.$$

Our main result is

THEOREM 1.1. *Assume (1.2)–(1.5) hold. Then there is $u \in C^2((0, 1)) \cap C^1([0, 1)) \cap C([0, 1])$ solution of (1.1) provided either*

(1.6a) $$\alpha \leq 0 \quad \text{and} \quad \lim_{s \to 0} \frac{f(r, s)}{s^{\beta+1}} = \infty \quad \textit{uniformly in } r$$

or

(1.6b) $$\alpha > 0 \quad \textit{and} \quad f(r, s) \geq \eta_\delta(r), \quad 0 < r < 1/2, \quad s \leq \delta,$$

for some $\delta > 0$ and $\eta_\delta \in C((0, 1/2))$ with $\eta_\delta > 0$. Moreover, $u \in C^2([0, 1))$ if and only if $\beta \leq \gamma - \alpha$ and further u is uniquely determined if $f(r, \cdot)$ is nonincreasing for each r.

REMARK. Condition (1.6b) holds if $f(r,s) \overset{s\to 0}{\longrightarrow} \infty$ uniformly with respect to $r \in [0, 1/2)$.

A few examples of terms $f(r,s)$ to which Theorem 1.1 applies are,

$$(r+1)^3(r-1)^2 s^{-p},$$
$$\sin(r)s^{-p} + \cos(r)s^q, \quad 0 < q < \beta + 1,$$
$$\left[2 + \sin\left(\frac{1}{1-r}\right)\right](s^{-p} + s^q), \quad 0 < q < \beta + 1,$$

provided either $p \ge 0$ and $\alpha > 0$ or $p > -1-\beta$ and $\alpha \le 0$. Moreover, by our theorem, (1.1) is uniquely solvable in the case of the first example, provided $p > 0$.

Theorem 1.1 improves the main result of Hai and Oppenheimer [12] on equations like

$$-(p(r)\varphi(u'))' = p(r) f(r,u) \quad \text{in } (0,1), \tag{1.7}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with concave inverse φ^{-1}, for instance, $\varphi(r) = |r|^\beta r$ with $\beta \ge 0$ and the main result in Wong [22].

Concerning singular problems, we would like to refer to Crandall, Rabinowitz and Tartar [8], Taliaferro [18], Kuzano and Swanson [17], Chabrowski [3], Choi, Lazer and McKenna [5], Lair and Shaker [15], Choi and Kim [4], Zhang [23], Wong [21] and their references.

For problems involving the operator in (1.7) or $(r^\alpha |u'|^\beta u')'$, but with nonsingular term $f(r,u)$, see Hai, Schmitt and Shivaji [14, 13], Clement, Figueiredo and Mitidieri [6], Clement, Manasevich and Mitidieri [7], Figueiredo, Goncalves and Miyagaki [10].

2. Auxiliary results

One basic tool in the proof of Theorem 1.1 is the shooting method. Consider the following family of initial value problems,

$$\begin{cases} -(r^\alpha |u'|^\beta u')' = r^\gamma f(r,u) & \text{in } (0,1); \\ u(0) = a, \quad r^\alpha |u'(r)|^{\beta+1} \overset{r\to 0}{\longrightarrow} 0, \end{cases} \tag{2.1}$$

where $a > 0$ is the shooting parameter. We point out that solving (2.1) is equivalent to solve the integral equation,

$$u(r) = a - \int_0^r \left[s^{-\alpha} \int_0^s t^\gamma f(t, u(t))\, dt \right]^{1/(\beta+1)} ds \tag{2.2}$$

and that a solution u of (2.1) has zero derivative at $r = 0$. Letting

$$\mathscr{F}u(r) = a - \int_0^r \left[s^{-\alpha} \int_0^s t^\gamma f(t, u(t))\, dt \right]^{1/(\beta+1)} ds \tag{2.3}$$

it follows that the eventual solutions of (2.2) are the fixed points of $\mathscr{F}$ in a suitable function space. We state next a crucial result on (2.1).

THEOREM 2.1. *Assume* (1.2) – (1.4) *hold. Then for each* $a > 0$ *there is some* $T(a) \in (0, 1]$ *and a unique solution* $u(\cdot, a) \in C^2((0, T(a))) \cap C^1([0, T(a)))$ *of* (2.1) *satisfying*:

$$u(r, a) \to 0 \quad \text{as } r \to T(a) \text{ provided } T(a) < 1; \tag{2.4}$$

$$u(\cdot, a) \in C^2([0, T(a))) \quad \text{if and only if } \beta \le \gamma - \alpha. \tag{2.5}$$

The proof of Theorem 2.1 uses Banach's Fixed Point Theorem. The technical lemmas below will be used in the proof of Theorem 1.1. In order to state the first lemma we establish some notations. Given $T \in (0, 1)$ and $h > 0$ set

$$X \equiv \left\{ w \in C^1([0, T]) \mid w \ge h,\ r^\alpha |w'(r)|^{\beta+1} \xrightarrow{r\to 0} 0 \right\}.$$

If $w_1, w_2 \in X$ let $H : [0, T] \to \mathbb{R}$ be a continuous function defined by

$$H(r) = r^\alpha \left[\left| \left(w_2^{1/(\beta+2)} \right)' \right|^\beta \left(w_2^{1/(\beta+2)} \right)' w_2^{-(\beta+1)/(\beta+2)} - \left| \left(w_1^{1/(\beta+2)} \right)' \right|^\beta \left(w_1^{1/(\beta+2)} \right)' w_1^{-(\beta+1)/(\beta+2)} \right] (w_1 - w_2)(r),$$

for $r \in (0, T]$ and $H(0) = 0$. The first lemma is

LEMMA 2.2. *If* $w_1, w_2 \in X$, *then*

$$H(T) \le \int_0^T \left[\left(r^\alpha \left| \left(w_2^{1/(\beta+2)} \right)' \right|^\beta \left(w_2^{1/(\beta+2)} \right)' \right)' w_2^{-(\beta+1)/(\beta+2)} - \left(r^\alpha \left| \left(w_1^{1/(\beta+2)} \right)' \right|^\beta \left(w_1^{1/(\beta+2)} \right)' \right)' w_1^{-(\beta+1)/(\beta+2)} \right] (w_1 - w_2)\, dr.$$

Now, the second lemma

LEMMA 2.3. *Assume* $a < b$ *and let* $u(\cdot, a)$, $u(\cdot, b)$ *be the corresponding solutions given by Theorem* 2.1. *Then* $u(\cdot, a) < u(\cdot, b)$ *in* $[0, T(a))$ *and moreover* $T(a) \le T(b)$.

The third one is

LEMMA 2.4. *Assume* (1.2)–(1.4) *hold. Let* $\{a_n\}$ *be a sequence in* $(0, \infty)$ *such that* $a_n \nearrow a$ *or* $a_n \searrow a$ *for some* $a > 0$ *and let* $u(\cdot, a_n)$, $u(\cdot, a)$ *be the solutions given by Theorem* 2.1. *If* $K \in (0, \min\{T(a), \sup_n T(a_n)\})$ *then*

$$\|u(\cdot, a_n) - u(\cdot, a)\|_{C([0,K])} \xrightarrow{n\to\infty} 0 \quad \text{and} \quad u'(r, a_n) \xrightarrow{n\to\infty} u'(r, a), \quad r \in [0, K].$$

3. Proof of Theorem 2.1

Let $a > 0$. By (1.3) there is some $I_a > 1$ such that $f(r, \cdot)$ is Lipschitz continuous on $[a/I_a, a]$ uniformly for $r \in [0, 1)$. Let $\epsilon \in (0, 1)$ small, set

$$X_{a,\epsilon} \equiv \{u \in C([0, \epsilon]) \mid u(0) = a,\ a/I_a \le u(r) \le a,\ r \in [0, \epsilon]\}$$

and notice that $(X_{a,\epsilon}, \|\cdot\|_\infty)$ is a complete metric space. We claim that

$$\text{(3.1)} \qquad \text{(i)} \quad \mathscr{F}(X_{a,\epsilon}) \subset X_{a,\epsilon}, \qquad \text{(ii)} \quad \|\mathscr{F}(u_1) - \mathscr{F}(u_2)\|_\infty \le k\|u_1 - u_2\|_\infty$$

for some $\epsilon > 0$ small enough, for all $u_1, u_2 \in X_{a,\epsilon}$ and for some $k \in (0, 1)$.

We present the proof of (3.1) in Appendix. Assuming it has been done, $\mathscr{F}$ has an only fixed point $u \in X_{a,\epsilon}$ and so (2.1) has a unique local solution. Setting

$$T(a) \equiv \sup\{r \in (0, 1) \mid \text{(2.1) has an only solution in } [0, r]\}$$

and letting $u(\cdot, a) : [0, T(a)) \to \mathbb{R}$ be the solution of (2.1), notice that by (2.2), $u(\cdot, a) \in C([0, T(a)))$ and, in fact,

$$\text{(3.2)} \qquad u'(r, a) = -\left[r^{-\alpha} \int_0^r t^\gamma f(t, u(t, a))\, dt\right]^{1/(\beta+1)}, \quad 0 < r < T(a).$$

Consider the functions

$$\text{(3.3)} \qquad \text{(i)} \quad m(s, x) \equiv \min_{0\le t\le s} \frac{f(t, x)}{x^{\beta+1}}, \qquad \text{(ii)} \quad M(s, x) \equiv \max_{0\le t\le s} \frac{f(t, x)}{x^{\beta+1}}$$

where $0 \le s < 1$ and $0 < x < \infty$. Taking $T < T(a)$, estimating in (3.2) with the use of (3.3) (ii) and (1.4) we have,

$$\text{(3.4)} \qquad \begin{aligned} |u'(r, a)|^{\beta+1} &\le a^{\beta+1} r^{-\alpha} \int_0^r t^\gamma \frac{f(t, u(T, a))}{u(T, a)^{\beta+1}} dt \\ &\le \frac{a^{\beta+1}}{\gamma + 1} M(T, u(T, a)) r^{\gamma-\alpha+1}, \quad 0 < r \le T \end{aligned}$$

so that by (3.4), $u(\cdot, a) \in C^1([0, T])$ and as a consequence, $\nu \equiv \lim_{r\to T} u'(r, a)$ is defined and $\nu \in (-\infty, 0)$. Now, consider the initial value problem

$$\begin{cases} -(r^\alpha |v'|^\beta v')' = r^\gamma f(r, v) \quad \text{in } (T, 1), \\ v(T) = u(T, a), \quad v'(T) = \nu, \end{cases} \tag{3.5}$$

whose solutions are the fixed points of

$$\widehat{\mathscr{F}} v(r) = u(T, a) - \int_T^r \left\{ s^{-\alpha} \left[T^\alpha |\nu|^{\beta+1} + \int_T^s t^\gamma f(t, v(t))\, dt \right] \right\}^{1/(\beta+1)} ds.$$

By the standard fixed point argument again, one infers the existence of a unique solution of (3.5) on some interval $[T, T + \epsilon)$ showing that $u(\cdot, a)$ is uniquely determined. We also have from the arguments above that $u(T(a), a) \equiv \lim_{r\to T(a)} u(r, a)$, $u(\cdot, a) \in C([0, T(a)])$ and further $u(T(a), a) = 0$ when $T(a) < 1$. This shows (2.4).

Next we shall prove (2.5). From (3.2),

$$u''(r, a) = -\frac{r^{\gamma-\alpha} h(r, a)}{\beta + 1} \left[r^{-\alpha} \int_0^r t^\gamma f(t, u(t, a))\, dt \right]^{-\beta/(\beta+1)}, \tag{3.6}$$

where

$$h(r, a) = f(r, u(r, a)) - \alpha r^{-(\gamma+1)} \int_0^r t^\gamma f(t, u(t, a))\, dt$$

and from (3.2) and (3.6), $u \equiv u(\cdot, a) \in C^2((0, T(a))) \cap C^1([0, T(a)))$. Moreover,

$$h(r, a) \overset{r\to 0}{\longrightarrow} \frac{\gamma - \alpha + 1}{\gamma + 1} f(0, a) \tag{3.7}$$

and using (3.3) (i)–(ii) and (1.4),

$$u(r, a)^{\beta+1} m(r, a) \le f(r, u(r, a)) \le a^{\beta+1} M(r, u(r, a)) \tag{3.8}$$

for $r > 0$. Consider the two cases below:

Case 1: $-1 < \beta \le 0$. Integrating from 0 to r in (3.8) we have,

$$\begin{aligned} \left[\frac{u(r, a)^{\beta+1}}{\gamma + 1} m(r, a) r^{\gamma+1} \right]^{-\beta/(\beta+1)} &\le \left(\int_0^r t^\gamma f(t, u(t, a))\, dt \right)^{-\beta/(\beta+1)} \\ &\le \left[\frac{a^{\beta+1}}{\gamma + 1} M(r, u(r, a)) r^{\gamma+1} \right]^{-\beta/(\beta+1)}. \end{aligned}$$

Hence,

$$\begin{aligned}(3.9)\qquad &\left[\frac{m(r,a)}{\gamma+1}\,u(r,a)^{\beta+1}\right]^{-\beta/(\beta+1)} r^{(\gamma-\alpha-\beta)/(\beta+1)}\\ &\quad\le r^{\gamma-\alpha}\left(r^{-\alpha}\int_0^r t^{\gamma} f(t,u(t,a))\,dt\right)^{-\beta/(\beta+1)}\\ &\quad\le\left[\frac{a^{\beta+1}}{\gamma+1}\,M(r,u(r,a))\right]^{-\beta/(\beta+1)} r^{(\gamma-\alpha-\beta)/(\beta+1)}.\end{aligned}$$

From (3.6), (3.7) and (3.9) it follows that $\lim_{r\to 0} u''(r,a)$ exists if and only if $\beta\le\gamma-\alpha$.
Case 2: $\beta>0$. Again, from (3.8), we obtain

$$\begin{aligned}\left[\frac{a^{\beta+1}}{\gamma+1}\,M(r,u(r,a))r^{\gamma+1}\right]^{-\beta/(\beta+1)} &\le\left(\int_0^r t^{\gamma} f(t,u(t))\,dt\right)^{-\beta/(\beta+1)}\\ &\le\left[\frac{u(r,a)^{\beta+1}}{\gamma+1}\,m(r,a)r^{\gamma+1}\right]^{-\beta/(\beta+1)},\end{aligned}$$

and thus,

$$\begin{aligned}(3.10)\qquad &\left[\frac{a^{\beta+1}}{\gamma+1}\,M(r,u(r,a)\right]^{-\beta/(\beta+1)} r^{(\gamma-\alpha-\beta)/(\beta+1)}\\ &\quad\le r^{\gamma-\alpha}\left(r^{-\alpha}\int_0^r t^{\gamma} f(t,u(t,a))\,dt\right)^{-\beta/(\beta+1)}\\ &\quad\le\left[\frac{u(r,a)^{\beta+1}}{\gamma+1}\,m(r,a)\right]^{-\beta/(\beta+1)} r^{(\gamma-\alpha-\beta)/(\beta+1)}.\end{aligned}$$

Therefore, it follows from (3.6), (3.7) and (3.10) that $\lim_{r\to 0} u''(r,a)$ exists if and only if $\beta\le\gamma-\alpha$.

4. Proofs of the lemmas

PROOF OF LEMMA 2.2. We will adapt arguments by Díaz and Saa [9] related to Brézis and Oswald [1]. Consider the functional $J : L^1([0,T])\to\mathbb{R}\cup\{\infty\}$ defined by

$$J(w)=\begin{cases}\dfrac{1}{\beta+2}\displaystyle\int_0^T r^{\alpha}\left|\left(w^{1/(\beta+2)}\right)'\right|^{\beta+2} dr, & w\in X;\\ \infty, & w\notin X.\end{cases}$$

It is straightforward to check that X and J are both convex. Now, letting $w_1, w_2\in X$, $\eta=w_1-w_2$, $p=\beta+2$, remarking that $w_2+t\eta, w_1-t\eta\in X$, $(0\le t\le 1)$, and denoting by $\langle J'(w_i), \eta\rangle$ the directional derivative of J at w_i in the direction η, we claim that

$$\langle J'(w_1), -\eta\rangle = -\frac{1}{p}T^{\alpha}\Big|\big(w_1^{1/p}(T)\big)'\Big|^{p-2}\big(w_1^{1/p}(T)\big)' w_1^{(1-p)/p}(T)\eta(T) + \frac{1}{p}\int_0^T \frac{\big(r^{\alpha}\big|(w_1^{1/p})'\big|^{p-2}(w_1^{1/p})'\big)'}{w_1^{(p-1)/p}}\eta(r)\,dr \tag{4.1}$$

and

$$\langle J'(w_2), \eta\rangle = \frac{1}{p}T^{\alpha}\Big|\big(w_2^{1/p}(T)\big)'\Big|^{p-2}\big(w_2^{1/p}(T)\big)' w_2^{(1-p)/p}(T)\eta(T) - \frac{1}{p}\int_0^T \frac{\big(r^{\alpha}\big|(w_2^{1/p})'\big|^{p-2}(w_2^{1/p})'\big)'}{w_2^{(p-1)/p}}\eta(r)\,dr. \tag{4.2}$$

We will show (4.1) next. Notice that,

$$\langle J'(w_1), -\eta\rangle = \frac{1}{p}\lim_{t\to 0}\int_0^T r^{\alpha}\left[\frac{\big|\big((w_1 - t\eta)^{1/p}\big)'\big|^p - \big|(w_1^{1/p})'\big|^p}{t}\right]dr.$$

By computing we find

$$\langle J'(w_1), -\eta\rangle = \lim_{t\to 0}\int_0^T r^{\alpha}|\theta_t|^{p-2}\theta_t\left[\frac{\big((w_1 - t\eta)^{1/p}\big)' - (w_1^{1/p})'}{t}\right]dr \tag{4.3}$$

where $\min\big\{((w_1 - t\eta)^{1/p})', (w_1^{1/p})'\big\} \le \theta_t \le \max\big\{((w_1 - t\eta)^{1/p})', (w_1^{1/p})'\big\}$. Now, estimating and applying Lebesgue's Theorem to (4.3) we infer that

$$\langle J'(w_1), -\eta\rangle = -\frac{1}{p}\int_0^T r^{\alpha}\big|(w_1^{1/p})'\big|^{p-2}(w_1^{1/p})'(w_1^{(1-p)/p}\eta)'dr,$$

and computing the integral we get (4.1). The verification of (4.2) follows by the same arguments. From (4.1) and (4.2),

$$\langle J'(w_2), \eta\rangle - \langle J'(w_1), \eta\rangle = \frac{1}{p}H(T) - \frac{1}{p}\int_0^T\left[\frac{\big(r^{\alpha}\big|(w_2^{1/(\beta+2)})'\big|^{\beta}(w_2^{1/(\beta+2)})'\big)'}{w_2^{(\beta+1)/(\beta+2)}} - \frac{\big(r^{\alpha}\big|(w_1^{1/(\beta+2)})'\big|^{\beta}(w_1^{1/(\beta+2)})'\big)'}{w_1^{(\beta+1)/(\beta+2)}}\right](w_1 - w_2)\,dr.$$

Since J is convex, $\langle J'(w_1) - J'(w_2), w_1 - w_2\rangle \ge 0$ and Lemma 2.2 follows. □

PROOF OF LEMMA 2.3. Assume, by the contrary, there is some $T > 0$ such that both $u(r,a) < u(r,b)$ for $r \in [0,T)$ and $u(T,a) = u(T,b)$. Setting $w_a \equiv u(\cdot, a)^{\beta+2}$

and $w_b \equiv u(\cdot, b)^{\beta+2}$, notice that $w_a, w_b \in X$, where h in the definition of X is given here by $h \equiv u(T, a)^{\beta+2}$. Notice that

$$\int_0^T \left[\frac{(r^\alpha |u'(\cdot, a)|^\beta u'(\cdot, a))'}{u(\cdot, a)^{\beta+1}} - \frac{(r^\alpha |u'(\cdot, b))|^\beta u'(\cdot, b))'}{u(\cdot, b)^{\beta+1}} \right] (u(\cdot, a)^{\beta+2} - u(\cdot, b)^{\beta+2})\, dr$$
$$= \int_0^T r^\gamma \left[\frac{f(r, u(\cdot, b))}{u(\cdot, b)^{\beta+1}} - \frac{f(r, u(\cdot, a))}{u(\cdot, a)^{\beta+1}} \right] (u(\cdot, a) - u(\cdot, b))\, dr.$$

Now, since $H(T) = 0$, by Lemma 2.2 the first integral just above is nonpositive, while by (1.4), the second one is strictly positive, a contradiction. This proves Lemma 2.3. □

PROOF OF LEMMA 2.4. Assume $a_n \nearrow a$, take $K \in (0, \sup_n T(a_n))$ and an integer $n_K \geq 1$ such that $T(a_{n_K}) > K$. By Lemma 2.3 and taking $n \geq n_K$,

$$T(a_{n_K}) \leq T(a_n) \leq T(a) \quad \text{and} \quad u(\cdot, a_{n_K}) \leq u(\cdot, a_n) \leq u(\cdot, a) \leq a.$$

We claim that $\{u(\cdot, a_n)\}_{n=1}^\infty$ is equibounded and equicontinuous in $C([0, K])$. Indeed, estimating as in (3.4) and using (3.3) (ii) we find

$$|u'(r, a_n)|^{\beta+1} \leq \frac{a^{\beta+1}}{\gamma+1} M(K, u(K, a_{n_K}) K^{\gamma-\alpha+1} \equiv \widehat{K}.$$

Hence there is $\theta_n \in (0, K)$ such that

$$|u(r, a_n) - u(t, a_n)| = |u'(\theta_n, a_n)||r - t| \leq \widehat{K}^{1/(\beta+1)}|r - t|.$$

It follows that $\{u(\cdot, a_n)\}_{n=1}^\infty$ is equibounded as well. So by the Arzéla-Ascoli Theorem there is $v \in C([0, K])$ such that $u(\cdot, a_n) \to v$ uniformly in $[0, K]$, up to a subsequence. Next we remark, by letting $g_n(t) \equiv t^\gamma f(t, u(t, a_n))$, $0 < t \leq K$, that both

$$|g_n(t)| \leq \left[\frac{a}{u(K, a_{n_K})} \right]^{\beta+1} t^\gamma f(t, u(t, a_{n_K})) \equiv h(t), \quad \text{where } h \in L^1[0, K]$$

and $g_n(t) \to t^\gamma f(t, v(t)) \equiv g(t)$, $t \in (0, K]$. So by Lebesgue's Theorem, for $r \in [0, K]$,

$$\int_0^r t^\gamma f(t, u(t, a_n))\, dt \to \int_0^r t^\gamma f(t, v(t))\, dt.$$

Hence,

$$|u'(r, a_n)|^\beta u'(r, a_n) \to -r^{-\alpha} \int_0^r t^\gamma f(t, v(t))\, dt$$

and so $u'(r,a_n) \to w(r)$, where $w(r) \equiv -\big(r^{-\alpha}\int_0^r t^\gamma f(t,v(t))\,dt\big)^{1/(\beta+1)}$. By Lebesgue's Theorem again

$$\int_0^r u'(t,a_n)\,dt \to \int_0^r w(t)\,dt,$$

and as a matter of fact, $v(r) - a = \int_0^r w(t)\,dt$. Since $v' = w$ we get,

$$|v'(r)|^\beta v'(r) = -r^{-\alpha}\int_0^r t^\gamma f(t,v(t))\,dt.$$

Hence v is a solution of (2.1) and by uniqueness provided by Theorem 2.1 it follows that $v \equiv u(\cdot,a)$. We have shown that,

$$\begin{aligned} u(\cdot,a_n) &\to u(\cdot,a) \quad \text{in } C([0,K]),\\ u'(\cdot,a_n) &\to u'(\cdot,a) \quad \text{pointwisely in } [0,K]. \end{aligned}$$

The case $a_n \searrow a$ follows by similar arguments. Lemma 2.4 is proved. □

5. Proof of Theorem 1.1

Setting $\mathscr{A} \equiv \{a > 0 \mid T(a) = 1\}$ we claim that $\mathscr{A} \neq \phi$. Indeed, if $\mathscr{A} = \phi$ then $u(r_a,a) = a/2$ for some $r_a \in (0,T(a))$, since $u(r,a) \overset{r\to T(a)}{\longrightarrow} 0$ by (2.4). Using (2.2) and estimating as in (3.4) we get

$$\frac{a}{2} \le aM\left(r_a,\frac{a}{2}\right)^{1/(\beta+1)}\left(\frac{1}{\gamma+1}\right)^{1/(\beta+1)}\frac{1}{\theta}r_a^\theta, \tag{5.1}$$

where $\theta \equiv (\gamma-\alpha+\beta+2)/(\beta+1)$, and thus

$$\frac{1}{2} \le \left[\frac{f(t_a,a/2)}{(a/2)^{\beta+1}}\right]^{1/(\beta+1)}\left(\frac{1}{\gamma+1}\right)^{1/(\beta+1)}\frac{1}{\theta}$$

for some $t_a \in (0,r_a)$. But this is impossible by (1.5) and so $\mathscr{A} \neq \phi$. Setting $A \equiv \inf\mathscr{A}$ we claim that $0 < A < \infty$. Indeed, at first notice that $A < \infty$ because $\mathscr{A} \neq \phi$. Now, to show that $A > 0$ we consider two cases:
Case 1: $\alpha \le 0$. Set for $r \in [0,1/2]$,

$$U(r,a) = u(r,a) - h(r,a), \quad \text{where } h(r,a) = a - 2ar.$$

We claim that $U(r,a) \ge 0$. Indeed, notice first that $U > 0$ in $(0,r_0)$ for some $r_0 \in (0,1/2)$. If $U(r_2,a) < 0$ for some $r_2 \in (r_0,1/2)$ then we find some $r_1 \in (r_0,r_2)$

with $U'(r_1,a)\le 0$ and further since $U(1/2,a)\ge 0$ we find some $r_3\in(r_2,1/2)$ with $U'(r_3,a)\ge 0$. But this is impossible because since $\alpha\le 0$ it follows using (3.6) that $U''(r,a)<0$ for all $r\in(0,1/2)$. As a consequence, $u(r,a)\ge a-2ar$ for $r\in[0,1/2]$ and hence using (2.2) and (1.4),

$$-u(1/2,a)\ge -a+\int_0^{1/2}\left[s^{-\alpha}\int_0^s t^\gamma \frac{f(t,a)}{a^{\beta+1}}(a-2at)^{\beta+1}dt\right]^{1/(\beta+1)}ds$$
$$=-a+am(1/2,a)^{1/(\beta+1)}\int_0^{1/2}\left[s^{-\alpha}\int_0^s t^\gamma(1-2t)^{\beta+1}dt\right]^{1/(\beta+1)}ds.$$

Hence by (1.6a), $-u(1/2,a)\ge 0$ for some a small enough. But since $u(\cdot,a)$ is a solution of (2.1), it follows that $u(1/2,a)=0$ so that $T(a)=1/2$. So using Lemma 2.3, $A>0$.

Case 2: $\alpha>0$. If $A=0$ it follows using Lemma 2.3 that $\mathscr{A}=(0,\infty)$ so that $u(1,a)>0$ for all $a>0$. Now, since $2(u(1,a)-u(1/2,a))=u'(\theta_a,a)$, for some $\theta_a\in(1/2,1)$ and $0<u(1,a)\le u(1/2,a)\le a$ it follows that

$$\int_0^{\theta_a} t^\gamma f(t,u(t,a))\,dt \xrightarrow{a\to 0} 0.$$

By (1.6b) we get, for small $a>0$,

$$\int_0^{1/2} t^\gamma \eta_\delta(t)\,dt\le\int_0^{\theta_a} t^\gamma f(t,u(t,a))\,dt,$$

impossible. This shows that $A>0$.

In order to prove that $u(\cdot,A)$ is a solution of (1.1) it suffices to show that $A\in\mathscr{A}$ and $u(1,A)=0$. If $T(A)<1$ pick $\epsilon>0$ such that $T(A)+\epsilon<1$ and a sequence $a_n\in\mathscr{A}$ with $a_n\searrow A$. Consider the sequence $u(T(A)+\epsilon/2,a_n)$ which by Lemma 2.3 is decreasing and set $T_{\epsilon,A}\equiv\inf_n\{u(T(A)+\epsilon/2,a_n)\}$. We claim that $T_{\epsilon,A}>0$. Otherwise, it follows remarking that $u(T(A)+\epsilon,a_n)<u(T(A)+\epsilon/2,a_n)$ and

$$u(T(A)+\epsilon,a_n)-u(T(A)+\epsilon/2,a_n)=u'(\theta_n,a_n)(\epsilon/2)$$

for some $\theta_n\in(T(A)+\epsilon/2,T(A)+\epsilon)$ that $u'(\theta_n,a_n)\xrightarrow{n}0$. Now, since

$$(\theta_n)^\alpha|u'(\theta_n,a_n)|^\beta u'(\theta_n,a_n)=-\int_0^{\theta_n} t^\gamma f(t,u(t,a_n))\,dt$$

we get $\int_0^{T(A)/2} t^\gamma f(t,u(t,a_n))\,dt\xrightarrow{n}0$.

By Lemma 2.4 we have

$$\int_0^{T(A)/2} t^\gamma f(t,u(t,a_n))\,dt\to\int_0^{T(A)/2} t^\gamma f(t,u(t,A))\,dt.$$

But this is impossible, because $\int_0^{T(A)/2} t^\gamma f(t,u(t,A))\,dt > 0$. Therefore $T_{\epsilon,A} > 0$. Choose $\delta_0 > 0$ such that $u(r,A) < T_{\epsilon,A}/4$ for $r \in [T(A)-\delta_0, T(A)-\delta_0/2]$. By Lemma 2.4,

$$\|u(\cdot,a_n) - u(\cdot,A)\|_{C([0,T(A)-\delta_0/2])} \stackrel{n}{\to} 0$$

and so there is $n_0 > 1$ such that

$$|u(r,a_{n_0}) - u(r,A)| < T_{\epsilon,A}/4 \quad \text{for all } r \in [0, T(A)-\delta_0/2].$$

Thus, for $r \in [T(A)-\delta_0, T(A)-\delta_0/2]$ we have

$$u(r,a_{n_0}) \le |u(r,a_{n_0}) - u(r,A)| + u(r,A) < T_{\epsilon,A}/2.$$

Since $u(r,a_n) \ge T_{\epsilon,A}$ for $n > 1$ and $r \in [0,T(A)]$, it follows that

$$u(T(A)-\delta_0, a_{n_0}) < T_{\epsilon,A}/2 < T_{\epsilon,A} \le u(T(A), a_{n_0}),$$

impossible. Therefore $A \in \mathscr{A}$. Now assume that $u(1,A) > 0$, and pick a sequence $a_n \nearrow A$. We claim that

$$T(a_n) \stackrel{n}{\to} 1. \tag{5.2}$$

Indeed, notice that $T(a_n) \le T(a_{n+1}) < 1$ and hence $T(a_n) \nearrow T$. If $T < 1$ set $T_A \equiv u(T,A)$. For each n large enough (for instance, such that $a_n > T_A$) take $t_n \in (0,T)$ satisfying $u(t_n,a_n) = T_A/4$.

Since $u(\cdot,a_n)$ is decreasing, consider $0 < \tilde{t}_n < t_n < T$ such that $u(\tilde{t}_n,a_n) = T_A/2$. We will show next that $\tilde{t}_n \to T$. Indeed, noticing that $\tilde{t}_n$ is monotone, $\tilde{t}_n \to \tilde{T} \le T$. If $\tilde{T} < T$ there is $n_0 > 1$ such that $T(a_{n_0}) > \tilde{T}$. Hence $u(r,a_n) \le T_A/2$ for all $n \ge n_0$ and $r \in [\tilde{T}, T(a_{n_0})]$ because otherwise, there would be some $r_{\tilde{n}} \in [\tilde{T}, T(a_{n_0})]$ with $T_A/2 < u(r_{\tilde{n}}, a_{\tilde{n}}) < u(\tilde{t}_{\tilde{n}}, a_{\tilde{n}}) = T_A/2$, impossible.

We infer that $|u(r,a_n) - u(r,A)| \ge T_A/2$ for $r \in [\tilde{T}, \tilde{T}+\epsilon)$ and for some $\epsilon > 0$ with $\tilde{T}+\epsilon < T(a_{n_0})$. But this is impossible because by Lemma 2.4,

$$\|u(\cdot,a_n) - u(\cdot,A)\|_{C([0,\tilde{T}+\epsilon])} \stackrel{n}{\to} 0.$$

Therefore, $\tilde{T} = T$. Now, noticing that

$$u(t_n,a_n) - u(\tilde{t}_n,a_n) = u'(\theta_n,a_n)(t_n - \tilde{t}_n), \quad \tilde{t}_n < \theta_n < t_n,$$

we get

$$|u'(\theta_n,a_n)| = \frac{T_A}{4|t_n - \tilde{t}_n|} \stackrel{n}{\to} \infty.$$

But this is impossible, because estimating as in (3.4) it follows that $u'(r, a_n)$ is bounded for $r \in [0, T]$. Claim (5.2) is proved.

Set $l = u(1, A)$. By (1.4) and (1.5) pick $t_1 \in (0, 1)$ such that

$$\int_{t_1}^{1} \left[s^{-\alpha} \int_0^s t^\gamma f(t, l/2)\, dt \right]^{1/(\beta+1)} ds < l^2/4A. \tag{5.3}$$

Using Lemma 2.4 and (5.2), we have

$$\|u(\cdot, a_n) - u(\cdot, A)\|_{C([0,t_1])} \xrightarrow{n} 0.$$

and as a consequence, $|u(t_1, a_n) - u(t_1, A)| \xrightarrow{n} 0$. But since $u(t_1, A) > l + \epsilon$ for some $\epsilon > 0$,

$$u(t_1, a_n) > u(t_1, A) - \epsilon > l$$

for large n. Now pick $t_2 \in (t_1, 1)$ such that $u(t_2, a_n) = l/2$. We have,

$$u(t_2, a_n) = u(t_1, a_n) - \int_{t_1}^{t_2} \left[s^{-\alpha} \int_0^s t^\gamma f(t, u(t, a_n))\, dt \right]^{1/(\beta+1)} ds.$$

Estimating the above integral as in (3.4) and using (5.3), we get

$$u(t_2, a_n) > l - \frac{2A}{l} \frac{l^2}{4A},$$

contradicting $l/2 = u(t_2, a_n)$. Therefore, $u(1, A) = 0$ and the solution $u \equiv u(\cdot, A)$ given by Theorem 2.1 solves (1.1).

It remains to show uniqueness. Let $\tilde{u} \equiv u(\cdot, B)$ be another solution of (1.1) with $A < B$. By Lemma 2.3, $u(r, A) < u(r, B)$ for $0 \le r < 1$. Set $\omega \equiv \tilde{u} - u$ and let $r_0 \in [0, 1)$ be a point where ω attains a local maximum so that $\omega(r_0) > 0$ and $\omega'(r_0) = 0$.

Integrating the differential equation in (1.1) from r_0 to r with $r \in [r_0, 1]$ and using the fact that $f(r, s)$ is nonincreasing in s, we obtain

$$r^\alpha(|\tilde{u}'(r)|^\beta \tilde{u}'(r) - |u'(r)|^\beta u'(r)) = -\int_{r_0}^{r} t^\gamma [f(t, \tilde{u}(t)) - f(t, u(t))]\, dt \ge 0.$$

Using the following inequality (see Simon [16]),

$$\langle |x|^\beta x - |y|^\beta y, x - y \rangle \ge \begin{cases} c_\beta |x - y|^{\beta+2} & \text{if } \beta \ge 0 \\ c_\beta \dfrac{|x - y|^2}{(1 + |x| + |y|)^{-\beta}} & \text{if } -1 < \beta < 0 \end{cases}$$

for all $x, y \in \mathbb{R}$ and for some $c_\beta > 0$ it follows that $\omega'(r) = \tilde{u}'(r) - u'(r) \ge 0$ for $r \in [r_0, 1]$. Hence $0 = \omega(1) \ge \omega(r_0)$, a contradiction.

6. Appendix

We prove (3.1) (i) first. If $u \in C([0,\epsilon])$ then $\mathscr{F}u \in C((0,\epsilon])$. On the other hand,

$$\int_0^r \left[s^{-\alpha}\int_0^s t^\gamma f(t,u(t))\,dt\right]^{1/(\beta+1)} ds \le aM(r,a/I_a)^{1/(\beta+1)}\left[\frac{1}{\gamma+1}\right]^{1/(\beta+1)} \epsilon^\theta \frac{1}{\theta},$$

where $r \in (0,\epsilon]$, θ as in (5.1). Actually,

$$M(r,a/I_a) \overset{r\to 0}{\longrightarrow} \frac{f(0,a/I_a)}{a/I_a}.$$

As a consequence, $\mathscr{F}(u) \in C([0,\epsilon])$. Picking $\epsilon > 0$ small enough we have

$$\int_0^\epsilon \left[s^{-\alpha}\int_0^s t^\gamma f(t,u(t))\,dt\right]^{1/(\beta+1)} ds < \frac{I_a-1}{I_a}a$$

so that $a/I_a \le \mathscr{F}(u)(r) \le a$, $0 \le r \le \epsilon$, showing (3.1) (i).

Next we show (3.1) (ii). Let $u_i \in C([0,\epsilon])$, $i = 1,2$. We have

$$|\mathscr{F}u_1(r) - \mathscr{F}u_2(r)| \le \int_0^r |X_1(s)^{1/(\beta+1)} - X_2(s)^{1/(\beta+1)}|ds,$$

where $X_i(s) = s^{-\alpha}\int_0^s t^\gamma f(t,u_i(t))\,dt$ $(i = 1,2)$. Using the inequality

$$||x|^{\hat\beta}x - |y|^{\hat\beta}y| \le c_{\hat\beta}(|x|^{\hat\beta} + |y|^{\hat\beta})|x-y| \quad x,y \in \mathbb{R}$$

for some $c_{\hat\beta} > 0$ where $\hat\beta > -1$, we have by making $\hat\beta = -\beta/(\beta+1)$

$$|\mathscr{F}u_1(r) - \mathscr{F}u_2(r)| \le c_{\hat\beta}\int_0^r (|X_1(s)|^{\hat\beta} + |X_2(s)|^{\hat\beta})|X_1(s) - X_2(s)|\,ds.$$

We distinguish two cases.

Case 1: $-1 < \beta \le 0$. Given $\epsilon > 0$ and taking $s \in [0,\epsilon]$ we have

$$|X_i(s)|^{\hat\beta} \le a^{-\beta}\left[\frac{M(s,a/I_a)}{\gamma+1}\right]^{\hat\beta} s^{-\beta(\gamma-\alpha+1)/(\beta+1)}$$

and

$$\begin{aligned}|X_1(s) - X_2(s)| &\le s^{-\alpha}\int_0^s t^\gamma |f(t,u_1(t)) - f(t,u_2(t))|\,dt\\ &\le \frac{\widehat{K}}{\gamma+1}\|u_1 - u_2\|_{C([0,\epsilon])}s^{\gamma-\alpha+1},\end{aligned}$$

where $\widehat{K}$ is the Lipschitz constant of f on $[a/I_a, a]$. From these inequalities we infer that

$$|\mathscr{F}u_1(r) - \mathscr{F}u_2(r)| \le 2c_{\hat{\beta}}\frac{a^{-\beta}}{\theta}\frac{\widehat{K}}{\gamma+1}\left[\frac{M(r, a/I_a)}{\gamma+1}\right]^{\hat{\beta}} \epsilon^{\theta}\|u_1 - u_2\|_{C([0,\epsilon])}$$

and (3.1) (ii) follows by taking ϵ small.

Case 2: $\beta > 0$. As in Case 1, given $\epsilon > 0$ we have,

$$|X_i(s)|^{\hat{\beta}} \le (a/I_a)^{-\beta}\left[\frac{m(s,a)}{\gamma+1}\right]^{\hat{\beta}} \epsilon^{-\beta(\gamma-\alpha+1)/(\beta+1)}, \quad 0 \le s \le \epsilon$$

and

$$|\mathscr{F}u_1(r) - \mathscr{F}u_2(r)| \le 2\frac{c_{\hat{\beta}}}{\theta}\frac{\widehat{K}}{\gamma+1}(a/I_a)^{-\beta}\left[\frac{m(s,a)}{\gamma+1}\right]^{\hat{\beta}} \epsilon^{\theta}\|u_1 - u_2\|_{C([0,\epsilon])}$$

showing (3.1) (ii).

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Universidade de Brasilia
Departamento de Matemática
70910-900 Brasilia(DF)
Brazil
e-mail: jv@mat.unb.br

Universidade Federal de Goiás
Departamento de Matemática
Catalão(GO)
Brazil
e-mail: csantos@unb.br