COMPOSITION OPERATORS IN ORLICZ SPACES

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Communicated by A. Pryde

Abstract

Composition operators C_τ between Orlicz spaces $L^\varphi(\Omega, \Sigma, \mu)$ generated by measurable and nonsingular transformations τ from Ω into itself are considered. We characterize boundedness and compactness of the composition operator between Orlicz spaces in terms of properties of the mapping τ, the function φ and the measure space (Ω, Σ, μ). These results generalize earlier results known for L^p-spaces.

1. Introduction

Let $\Omega = (\Omega, \Sigma, \mu)$ be a σ-finite complete measure space and let $\tau : \Omega \to \Omega$ be a *measurable* transformation, that is, $\tau^{-1}(A) \in \Sigma$ for any $A \in \Sigma$. If $\mu(\tau^{-1}(A)) = 0$ for all $A \in \Sigma$ with $\mu(A) = 0$, then τ is said to be *nonsingular*. This condition means that the measure $\mu \circ \tau^{-1}$, defined by $\mu \circ \tau^{-1}(A) := \mu(\tau^{-1}(A))$ for $A \in \Sigma$, is absolutely continuous with respect to μ (it is usually denoted $\mu \circ \tau^{-1} \ll \mu$). Then the Radon-Nikodým theorem ensures the existence of a non-negative locally integrable function f_τ on Ω such that $\mu \circ \tau^{-1}(A) = \int_A f_\tau(t) d\mu(t)$ for $A \in \Sigma$.

Any measurable nonsingular transformation τ induces a linear operator (*composition operator*) C_τ from $L^0(\Omega)$ into itself defined by

$$(1.1) \qquad C_\tau x(t) = x(\tau(t)), \quad t \in \Omega, \; x \in L^0(\Omega),$$

where $L^0(\Omega)$ denotes the linear space of all equivalence classes of Σ-measurable functions on Ω, where we identify any two functions that are equal μ-almost everywhere on Ω.

Here the nonsingularity of τ guarantees that the operator C_τ is well defined as a mapping of equivalence classes of functions into itself since $x = y$ μ-a.e. implies $C_\tau x = C_\tau y$ μ-a.e.

If C_τ maps an Orlicz space $L^\varphi(\Omega)$ into itself, then we call C_τ a *composition operator in* $L^\varphi(\Omega)$. Note that, in this case, C_τ as a positive linear operator in $L^\varphi(\Omega)$ is bounded.

Boundedness of composition operators in $L^p(\Omega)$-spaces ($1 \le p < \infty$) for finite measures appeared already in the Dunford-Schwartz book [4, Lemma 7, pages 664–665] and for σ-finite measures in Singh's paper [21] (the sequence case in [23]) and also in the book [25]. Namely, a measurable transformation τ induces a bounded composition operator C_τ in $L^p(\Omega)$ for $1 \le p < \infty$ if and only if $\mu \circ \tau^{-1} \ll \mu$ and the Radon-Nikodým derivative $d\mu \circ \tau^{-1}/d\mu = f_\tau$ is essentially bounded on Ω. The last two conditions can be written as one: there exists a constant $K > 0$ such that

(1.2) $$\mu(\tau^{-1}(A)) \le K\mu(A) \quad \text{for all} \ \ A \in \Sigma.$$

Moreover,

$$\|C_\tau\|_{L^p \to L^p} = \left(\sup_{A \in \Sigma, 0 < \mu(A) < \infty} \mu(\tau^{-1}(A))/\mu(A) \right)^{1/p}.$$

Observe that the composition operator C_τ is a bounded operator in $L^\infty(\mu)$ with norm 1, that is, a nonexpansive mapping for any measurable nonsingular transformation τ.

In the case that φ is an N-function, some results on boundedness of composition operators in the Orlicz space $L^\varphi(\Omega)$ were obtained in [9, Theorem 2.1] (see also [16]).

Compactness results of composition operator were proved by Singh [20] and Singh-Kumar [24, 22] in $L^2(\Omega)$-space. Then Petrović [15], Xu [28] and Takagi [26] extended them to the $L^p(\Omega)$-spaces with $1 \le p \le \infty$. It turns out that there is no compact composition operator on $L^p(\Omega)$, when μ is either non-atomic or purely atomic with all atoms of equal measure. But there are some weighted sequence l^p-spaces which do have compact composition operators.

We will present some new results on boundedness and compactness of composition operators in Orlicz spaces.

We need some notions from Orlicz spaces. Let $\varphi : [0, \infty) \to [0, \infty]$ be an *Orlicz function*, that is, a convex function such that $\varphi(0) = 0, \varphi(u) \to \infty$ as $u \to \infty$ and which is not identically zero or infinity on $(0, \infty)$. Assume also that φ is left-continuous at b_φ, where $b_\varphi = \sup\{u > 0 : \varphi(u) < \infty\}$. Note that if $b_\varphi < \infty$, then we can have two cases $\varphi(b_\varphi) < \infty$ or $\varphi(b_\varphi) = \infty$. Another important constant connected with the Orlicz function φ is $a_\varphi = \inf\{u > 0 : \varphi(u) > 0\}$. Of course, $0 \le a_\varphi \le b_\varphi \le \infty$. For $x \in L^0(\Omega)$, define the *modular*

$$I_\varphi(x) = \int_\Omega \varphi(|x(s)|) \, d\mu(s)$$

and the *Orlicz space*

$$L^\varphi(\Omega) = L^\varphi(\Omega, \Sigma, \mu) = \left\{ x \in L^0(\Omega) : I_\varphi(\lambda x) < \infty \text{ for some } \lambda = \lambda(x) > 0 \right\}.$$

This space is a Banach space with two norms: the *Luxemburg-Nakano norm*

$$\|x\|_\varphi = \inf\{\lambda > 0 : I_\varphi(x/\lambda) \leq 1\}$$

and the *Orlicz norm* (in the Amemiya form)

$$\|x\|_\varphi^0 = \inf_{k>0}(1 + I_\varphi(kx))/k.$$

It is well known that $\|x\|_\varphi \leq \|x\|_\varphi^0 \leq 2\|x\|_\varphi$ and $\|x\|_\varphi \leq 1$ if and only if $I_\varphi(x) \leq 1$ (see for example [7, page 80]). Moreover, if $A \in \Sigma$ and $0 < \mu(A) < \infty$, then $\|\chi_A\|_\varphi = 1/\varphi^{-1}(1/\mu(A))$, where $\varphi^{-1}(t) = \inf\{s > 0 : \varphi(s) > t\}$ is the right-continuous inverse of φ. Note that the equality of the Orlicz norm and the Amemiya norm was recently proved in [5]. Before only an estimate of the Orlicz norm from above by the Amemiya norm was known (see [13, Theorem I.2.6]) with equality when both φ and its complementary function φ^* are N-functions (see [7, Theorem III.10.4], [11, Theorem 8.6] and [17, Theorem III.3.13]).

The Orlicz space $L^\varphi(\Omega)$ with each of the above two norms is a rearrangement-invariant space, that is, a symmetric space with the Fatou property (see [1, Theorem 8.9], [8, page 104]).

We say that an Orlicz function φ satisfies the *condition Δ_2 for all u* (respectively, *for large u*; *for small u*) if there exists a positive constant K (respectively, a positive K and $u_0 > 0$ with $\varphi(u_0) < \infty$; a positive constant K and $u_0 > 0$ with $\varphi(u_0) > 0$) such that $\varphi(2u) \leq K\varphi(u)$ for all $u > 0$ (respectively, for all $u \geq u_0$; for all $0 < u \leq u_0$).

If φ satisfies the condition Δ_2 for all u, then for any σ-finite measure space the dual of the Orlicz space $L^\varphi(\Omega)$ is the Orlicz space $L^{\varphi^*}(\Omega)$ generated by the complementary function φ^* of φ, defined by $\varphi^*(u) := \sup_{v>0}(uv - \varphi(v))$. Moreover,

$$\left(L^\varphi(\Omega), \|\cdot\|_\varphi\right)^* = \left(L^{\varphi^*}(\Omega), \|\cdot\|_{\varphi^*}^0\right) \quad \text{and} \quad \left(L^\varphi(\Omega), \|\cdot\|_\varphi^0\right)^* = \left(L^{\varphi^*}(\Omega), \|\cdot\|_{\varphi^*}\right).$$

The paper is organized as follows. In Section 2 we study modular continuity and norm continuity of composition operators C_τ in Orlicz spaces $L^\varphi(\Omega)$. We were able to completely characterize modular continuity of C_τ. It is surprising that the necessary and sufficient conditions for modular continuity do not depend on Orlicz functions φ but only on the transformation τ. Our results on norm continuity of C_τ are also complete. The form of the necessary and sufficient conditions for the norm continuity established in this paper depends on the regularity of the Orlicz function φ. In the case when φ satisfies the condition Δ_2 for all $u > 0$, the conditions have the same form as for the modular continuity, so they do not depend on φ, for any measure μ. If μ is

nonatomic and finite the same is true if φ satisfies the condition Δ_2 for large values of $u > 0$. If the measure μ is nonatomic and infinite, then the condition Δ_2 for large values of $u > 0$ is enough to explain the necessary and sufficient condition for the continuity of C_τ from $L^\varphi(\Omega)$ into itself in terms of φ. Without any regularity condition for φ, and without any restriction on the measure μ it is established that the composition operator C_τ generated by a nonsingular transformation τ is continuous from an Orlicz spaces $L^\varphi(\Omega)$ into itself if and only if $L^\varphi(\Omega)$ is contained in the Musielak-Orlicz space $L^\psi(\Omega)$, where $\psi(t, u) = \varphi(u)w(t)$ and w is the Radon-Nikodým derivative $d\mu \circ \tau^{-1}/d\mu$. It means that in the general case we were not able to explain the conditions directly in terms of the generating function φ.

In Section 3, we investigate compactness of composition operators in Orlicz spaces. As in the L^p-case, compactness of a composition operator C_τ in the Orlicz space $L^\varphi(\Omega)$ implies that the underlying measure μ is purely atomic. Under some assumptions on the function φ and the atoms of μ, we prove necessary and sufficient conditions for C_τ to be compact. In particular, we can have compactness of the composition operator in weighted Orlicz sequence spaces.

2. Modular and norm continuity of composition operators

For the modular continuity of the composition operator C_τ in an Orlicz space $L^\varphi(\Omega)$, we present necessary and sufficient conditions for any Orlicz function φ and any σ-finite measure space (Ω, Σ, μ). For any Orlicz function φ which satisfies the condition Δ_2 for all u, the same is done for norm continuity of the composition operator C_τ in $L^\varphi(\Omega)$. If φ satisfies the condition Δ_2 for large u, then the problem of continuity of the composition operator C_τ in $L^\varphi(\Omega)$ is completely solved if the measure space is nonatomic of finite or infinite measure. Without any regularity condition on φ, the conditions for continuity of C_τ from $L^\varphi(\Omega)$ into itself are explained in terms of the Radon-Nikodým derivative $d\mu \circ \tau^{-1}/d\mu$.

THEOREM 2.1. *Assume that $\tau : \Omega \to \Omega$ is a measurable nonsingular transformation.*

(a) *If $0 < a_\varphi = b_\varphi < \infty$, then $I_\varphi(C_\tau x) = I_\varphi(x)$ whenever $I_\varphi(x) < \infty$.*
(b) *If $0 \leq a_\varphi < b_\varphi \leq \infty$, then the inequality*

$$(2.1) \qquad\qquad I_\varphi(C_\tau x) \leq K I_\varphi(x)$$

holds for all x such that $I_\varphi(x) < \infty$ with some $K > 0$ independent of x if and only if

$$(2.2) \qquad\qquad \mu(\tau^{-1}(A)) \leq K \mu(A)$$

for all $A \in \Sigma$ with $\mu(A) < \infty$.

PROOF. (a) In this case the function φ is 0 on the interval $[0, a_\varphi)$ and ∞ on (a_φ, ∞). Therefore, $I_\varphi(x) < \infty$ if and only if $\|x\|_\infty \leq a_\varphi$ and so $\|C_\tau x\|_\infty \leq a_\varphi$, which finally gives $I_\varphi(C_\tau x) = 0 = I_\varphi(x)$.

(b) Assume that $0 \leq a_\varphi < b_\varphi \leq \infty$.

Sufficiency. Assume that condition (2.2) is satisfied. It gives that $\mu \circ \tau^{-1} \ll \mu$ and by the Radon-Nikodým theorem, $\mu \circ \tau^{-1}(A) = \int_A f_\tau(t) d\mu(t)$ for $A \in \Sigma$ and for some function f_τ locally integrable on Ω. Notice that $f_\tau \in L^\infty(\Omega)$ and $\|f_\tau\|_\infty \leq K$. Otherwise, there is $A \in \Sigma$ with $0 < \mu(A) < \infty$ such that $f_\tau(t) > K$ for any $t \in A$. This yields $\mu \circ \tau^{-1}(A) = \int_A f_\tau(t) d\mu(t) > K\mu(A)$, which contradicts condition (2.2).

Therefore,

$$
\begin{aligned}
I_\varphi(C_\tau x) &= \int_\Omega \varphi(|C_\tau x(s)|) \, d\mu(s) = \int_\Omega \varphi(|x(\tau(s))|) \, d\mu(s) \\
&= \int_{\tau(\Omega)} \varphi(|x(t)|) \, d(\mu \circ \tau^{-1})(t) \leq \int_\Omega \varphi(|x(t)|) \, d(\mu \circ \tau^{-1})(t) \\
&= \int_\Omega \varphi(|x(t)|) f_\tau(t) \, d\mu(t) \leq K \int_\Omega \varphi(|x(t)|) \, d\mu(t) = K I_\varphi(x).
\end{aligned}
$$

Necessity. Assume that condition (2.1) holds. If $A \in \Sigma$ and $\mu(A) = 0$, then the nonsingularity of τ yields $\mu(\tau^{-1}(A)) = 0$ and we have equality in (2.2). Thus suppose that $A \in \Sigma$ and $0 < \mu(A) < \infty$. Take $a \in (a_\varphi, b_\varphi)$ and set $x = a\chi_A$. Then

$$
I_\varphi(x) = \int_A \varphi(a) \, d\mu(s) = \varphi(a)\mu(A) < \infty.
$$

As $C_\tau \chi_A = \chi_{\tau^{-1}(A)}$, it follows by (2.1) that

$$
\varphi(a)\mu(\tau^{-1}(A)) = I_\varphi(C_\tau x) \leq K I_\varphi(x) = K\varphi(a)\mu(A).
$$

Since $0 < \varphi(a) < \infty$, this gives (2.2). $\qquad\square$

THEOREM 2.2. *Assume that $\tau : \Omega \to \Omega$ is a measurable nonsingular transformation. Then the composition operator C_τ is bounded from an Orlicz space $L^\varphi(\Omega)$ into itself, that is, there exists $M > 0$ such that*

$$
(2.3) \qquad\qquad \|C_\tau x\|_\varphi \leq M \|x\|_\varphi \quad \text{for all } x \in L^\varphi(\Omega)
$$

if condition (2.2) holds. If, in addition, φ satisfies the condition Δ_2 for all $u > 0$, then (2.3) and (2.2) are equivalent.

PROOF. *Sufficiency.* By Theorem 2.1 we know that if (2.2) is satisfied with $K \geq 1$, then (2.1) holds and

$$
I_\varphi\left(\frac{C_\tau x}{K\|x\|_\varphi}\right) \leq \frac{1}{K} I_\varphi\left(\frac{C_\tau x}{\|x\|_\varphi}\right) \leq I_\varphi\left(\frac{x}{\|x\|_\varphi}\right) \leq 1,
$$

whence $\|C_\tau x\|_\varphi \leq K\|x\|_\varphi$ for all $0 \neq x \in L^\varphi(\Omega)$, that is, (2.3) holds with $M = K$. Note that this part of the theorem holds without the Δ_2-condition.

Necessity. Setting in inequality (2.3) $x = \chi_A$, where $A \in \Sigma$ and $0 < \mu(A) < \infty$, we get the estimate

$$\frac{1}{\varphi^{-1}(1/\mu(\tau^{-1}(A)))} \leq \frac{M}{\varphi^{-1}(1/\mu(A))},$$

or equivalently

(2.4) $$\varphi^{-1}\left(\frac{1}{\mu(A)}\right) \leq M\varphi^{-1}\left(\frac{1}{\mu(\tau^{-1}(A))}\right)$$

for all $A \in \Sigma$ with $0 < \mu(A) < \infty$.

Since φ satisfies the condition Δ_2 for all $u > 0$, it follows that

$$L := \sup_{u>0} \frac{\varphi(Mu)}{\varphi(u)} < \infty,$$

and $\varphi(Mu) \leq L\varphi(u)$ for all $u > 0$, which gives for $u = \varphi^{-1}(v)$ that

$$\varphi(M\varphi^{-1}(v)) \leq L\varphi(\varphi^{-1}(v)) \leq Lv$$

and so

$$M\varphi^{-1}(v) \leq \varphi^{-1}\{\varphi(M\varphi^{-1}(v))\} \leq \varphi^{-1}(Lv)$$

for all $v > 0$. Consequently, condition (2.4) yields

$$\varphi^{-1}\left(\frac{1}{\mu(A)}\right) \leq M\varphi^{-1}\left(\frac{1}{\mu(\tau^{-1}(A))}\right) \leq \varphi^{-1}\left(\frac{L}{\mu(\tau^{-1}(A))}\right)$$

or equivalently $\mu(\tau^{-1}(A)) \leq L\mu(A)$ for all $A \in \Sigma$ with $0 < \mu(A) < \infty$, which finishes the proof of the necessity with $K = L$. $\qquad\square$

REMARK 1. Theorem 2.2 was, in fact, already formulated in [9] when φ is an N-function but in case of $M > 1$ the proof was not complete.

REMARK 2. The sufficiency of Theorem 2.2 can be proved (again without the Δ_2-condition) in two different ways, namely by using simple functions and by the Orlicz interpolation theorem which is saying that any Orlicz space $L^\varphi(\Omega)$ is an exact interpolation space between $L^1(\Omega)$ and $L^\infty(\Omega)$ (see [12]). We present below the second alternative proof.

ALTERNATIVE PROOF. Since τ is nonsingular, the operator C_τ is continuous between $L^\infty(\Omega)$ with $\|C_\tau x\|_\infty \leq \|x\|_\infty$ for all $x \in L^\infty(\Omega)$. Condition (2.2) yields that

C_τ is continuous on $L^1(\Omega)$ with $\|C_\tau x\|_1 \leq K\|x\|_1$ for all $x \in L^1(\Omega)$. By the Orlicz interpolation theorem (see [12, Theorem 1'], [11, Theorem 13.2']; see also [1, Theorem 2.2.2] and [8, Theorem II.4.9], where there is even more general version of this theorem, the so called Calderón-Mitjagin interpolation theorem), we get $\|C_\tau x\|_\varphi \leq \max\{K, 1\}\|x\|_\varphi$ for all $x \in L^\varphi(\Omega)$. More precisely,

$$\|C_\tau\|_{L^\varphi \to L^\varphi} \leq \sup_{u>0} \frac{\varphi^{-1}(Ku)}{\varphi^{-1}(u)}$$

(see [12, Theorem 15.13] or [8, estimate (4.24) on page 100]).

REMARK 3. From the above proofs we obtain that if condition (2.2) holds, then we have the estimates

$$\sup_{A \in \Sigma, 0 < \mu(A) < \infty} \frac{\varphi^{-1}(1/\mu(A))}{\varphi^{-1}(1/\mu(\tau^{-1}(A)))}$$
$$\leq \|C_\tau\|_{L^\varphi \to L^\varphi} \leq \sup_{u>0} \frac{\varphi^{-1}\left(\sup_{A \in \Sigma, 0 < \mu(A) < \infty}(\mu(\tau^{-1}(A))/\mu(A))u\right)}{\varphi^{-1}(u)},$$

which is a generalization of the L^p-case into Orlicz spaces.

REMARK 4. Condition (2.2) is sufficient for the continuity of any composition operator from any symmetric space X into itself if X has either the Fatou property or an absolutely continuous norm (see, for example, [1, Theorem 2.2.2] or [8, Theorem II.4.9 and 4.10]), because X is then an interpolation space between L^1 and L^∞.

REMARK 5. If $0 < a_\varphi \leq b_\varphi < \infty$, then the Orlicz space $L^\varphi(\Omega)$ is equal to $L^\infty(\Omega)$ with an equivalent norm. Hence the composition operator C_τ is norm-continuous on $L^\varphi(\Omega)$ for every nonsingular transformation τ. However, in order to obtain modular continuity of C_τ which is stronger than norm continuity, we need the additional assumption (2.2) on τ as shown in Theorem 2.1. In particular, note that $a_\varphi = b_\varphi$ is allowed (see [17, first paragraph on page 52]).

THEOREM 2.3. *Assume that μ is a non-atomic infinite measure, $\tau : \Omega \to \Omega$ is a measurable nonsingular transformation and φ is an Orlicz function satisfying the condition Δ_2 for large u. Then the composition operator C_τ is bounded in $L^\varphi(\Omega)$ if and only if there exists a constant $K \geq 1$ such that*

(2.5)
$$\frac{1}{\varphi^{-1}(1/\mu(\tau^{-1}(A)))} \leq \frac{K}{\varphi^{-1}(1/\mu(A))}$$

for all $A \in \Sigma$ with $0 < \mu(A) < \infty$.

PROOF. The necessity is clear by putting $x = \chi_A$ into the assumption.

Sufficiency. Since φ satisfies the condition Δ_2 for large u it follows that φ is finite-valued and so $\lim_{u\to\infty} \varphi^{-1}(u) = \infty$.

Note first that (2.5) guarantee that $\mu \circ \tau^{-1} \ll \mu$. In fact, assume that $\mu(A) \to 0$. Then by the assumption (2.5) and from the assumption on φ we obtain $\mu \circ \tau^{-1}(A) \to 0$, which shows that $\mu \circ \tau^{-1} \ll \mu$ because by the nonsingularity of τ we may assume that $\mu(A) > 0$. Let $g = d\mu \circ \tau^{-1}/d\mu$. Then $g \in L^1 + L^\infty$ and $\mu \circ \tau^{-1}(A) = \int_A g(t)\,d\mu$ for any $A \in \Sigma$.

Now we will prove that $g \in L^\infty(\Omega)$. We can prove in the same way as in the proof of Theorem 2.2 with $2K$ instead of K that $\varphi \in \Delta_2^\infty$ (that is, φ satisfies the condition Δ_2 for large u) yields $2K\varphi^{-1}(v) \le \varphi^{-1}(Lv)$ for $v > 0$ large enough and an absolute constant $L \ge 2K$ or equivalently that $2K\varphi^{-1}(w/L) \le \varphi^{-1}(w)$ for $w \ge a$ with some $a > 0$. We will show that $\|g\|_\infty \le L$. Assume that $\|g\|_\infty > L$. Then, as μ is nonatomic there exists a set $A \in \Sigma$ with $1/\mu(A) > a$ such that $0 < \mu(A) < 1/a$ and $g(t) > L$ for μ-almost all $t \in A$. Consequently,

$$\mu \circ \tau^{-1}(A) = \int_A g(t)\,d\mu > L\mu(A),$$

whence $\varphi^{-1}(1/\mu(\tau^{-1}(A))) \le \varphi^{-1}(1/L\mu(A))$ and so

$$\frac{1}{\varphi^{-1}(1/\mu(\tau^{-1}(A)))} \ge \frac{1}{\varphi^{-1}(1/L\mu(A))} \ge \frac{2K}{\varphi^{-1}(1/\mu(A))} > \frac{K}{\varphi^{-1}(1/\mu(A))},$$

a contradiction. Thus, $\|g\|_\infty \le L$ and the rest of the proof is the same as the proof of the sufficiency in Theorem 2.2. $\square$

REMARK 6. Condition (2.2) implies condition (2.5). Morover, if $\mu(\Omega) < \infty$ and φ satisfies the Δ_2-condition for large u, then conditions (2.5) and (2.2) are equivalent. In fact, the Δ_2-condition of φ on $[u_0, \infty)$ can be extended to the interval $[\varphi^{-1}(1/\mu(\tau^{-1}(\Omega))), \infty)$ and (2.5) is equivalent to

$$1/\mu(A) \le \varphi\big(K\varphi^{-1}\big(1/\mu(\tau^{-1}(A))\big)\big),$$

therefore we get condition (2.2). Note that conditions (2.5) and (2.2), in the case when $\mu(\Omega) = \infty$, are different which will be shown in the next theorem.

In our next theorem, the space $L^\varphi(\Omega) \cap L^\infty(\Omega)$ is considered with the classical norm $\|x\|_{\varphi\cap\infty} = \max(\|x\|_\varphi, \|x\|_\infty)$.

THEOREM 2.4. *Assume that (Ω, Σ, μ) is a non-atomic infinite measure space and $\tau : \Omega \to \Omega$ is a measurable nonsingular transformation. Let φ be an Orlicz function*

satisfying the condition Δ_2 for all $u > 0$. Then the composition operator C_τ is bounded in $L^\varphi(\Omega) \bigcap L^\infty(\Omega)$ if and only if there exists a constant $K \geq 1$ such that

$$(2.6) \qquad \max\{1, \mu(\tau^{-1}(A))\} \leq K \max\{1, \mu(A)\}$$

for all $A \in \Sigma$ with $0 < \mu(A) < \infty$.

PROOF. *Necessity.* Putting $x = \chi_A$, where $A \in \Sigma$ and $0 < \mu(A) < \infty$, we obtain

$$\max\left\{1, \frac{1}{\varphi^{-1}(1/\mu(\tau^{-1}(A)))}\right\} = \max\{\|C_\tau x\|_\infty, \|C_\tau x\|_\varphi\}$$

$$\leq C \max\{\|x\|_\infty, \|x\|_\varphi\}$$

$$= C \max\left\{1, \frac{1}{\varphi^{-1}(1/\mu(A))}\right\},$$

with a constant $C > 0$ independent of A. The function $g(u) = 1/\varphi(1/u)$ is increasing and continuous on $(0, \infty)$ with $\lim_{u \to 0^+} g(u) = 0$ and $g(u) \to \infty$ as $u \to \infty$. In particular, there exists $a > 0$ such that $g(a) = 1$. Moreover, g satisfies the condition Δ_2 for all u since φ is such. Thus

$$\max\{1, \mu(\tau^{-1}(A))\}$$

$$= \max\left\{g(a), g\left(\frac{1}{\varphi^{-1}(1/\mu(\tau^{-1}(A)))}\right)\right\} = g\left(\max\left\{a, \frac{1}{\varphi^{-1}(1/\mu(\tau^{-1}(A)))}\right\}\right)$$

$$\leq g\left(C \max\left\{a, \frac{1}{\varphi^{-1}(1/\mu(A))}\right\}\right) \leq Dg\left(\max\left\{a, \frac{1}{\varphi^{-1}(1/\mu(A))}\right\}\right)$$

$$= D \max\left\{g(a), g\left(\frac{1}{\varphi^{-1}(1/\mu(A))}\right)\right\} = D \max\{1, \mu(A)\},$$

and (2.6) holds.

Sufficiency. We first show that (2.6) implies that C_τ is bounded in $L^1(\Omega) \bigcap L^\infty(\Omega)$ and next using an appropriate interpolation theorem we will extend the boundedness of C_τ into $L^\varphi(\Omega) \bigcap L^\infty(\Omega)$.

Let x be a simple positive function, that is, $x = \sum_{i=1}^n a_i \chi_{A_i}$, where $A_1 \subset \cdots \subset A_n$ and $a_i > 0$, and $A_i \in \Sigma$, $\mu(A_i) < \infty$, $i = 1, 2, \ldots, n$. Then $C_\tau x = \sum_{i=1}^n a_i \chi_{\tau^{-1}(A_i)}$,

$$\|x\|_1 = \int_\Omega \sum_{i=1}^n a_i \chi_{A_i}(t)\, d\mu(t)$$

$$= \sum_{i=1}^n a_i \mu(A_1) + \sum_{i=2}^n a_i \mu(A_2 \setminus A_1) + \cdots + a_n \mu(A_n \setminus A_{n-1}) = \sum_{i=1}^n a_i \mu(A_i),$$

and applying estimate (2.6), we get

$$
\begin{aligned}
\|C_\tau x\|_{1\cap\infty} &= \max\left(\|C_\tau x\|_1, \|C_\tau x\|_\infty\right) \\
&= \max\left(\left\|\sum_{i=1}^{n} a_i \chi_{\tau^{-1}(A_i)}\right\|_1, \left\|\sum_{i=1}^{n} a_i \chi_{\tau^{-1}(A_i)}\right\|_\infty\right) \\
&\le \max\left(\sum_{i=1}^{n} a_i \|\chi_{\tau^{-1}(A_i)}\|_1, \sum_{i=1}^{n} a_i\right) \\
&\le \sum_{i=1}^{n} a_i \max(\mu(\tau^{-1}(A_i)), 1) \le K \sum_{i=1}^{n} a_i \max(\mu(A_i), 1) \\
&\le K \sum_{i=1}^{n} a_i (\mu(A_i) + 1) = K(\|x\|_1 + \|x\|_\infty) \\
&\le 2K \max(\|x\|_1, \|x\|_\infty) = 2K \|x\|_{1\cap\infty}.
\end{aligned}
$$

If x is a simple function in $L^1 \cap L^\infty$, then

$$
\begin{aligned}
\|C_\tau x\|_{1\cap\infty} = \|C_\tau x_+ - C_\tau x_-\|_{1\cap\infty} &\le \|C_\tau x_+\|_{1\cap\infty} + \|C_\tau x_-\|_{1\cap\infty} \\
&\le 2K(\|x_+\|_{1\cap\infty} + \|x_-\|_{1\cap\infty}) \le 4K\|x\|_{1\cap\infty}.
\end{aligned}
$$

Finally, since simple functions are dense in $L^1(\Omega) \cap L^\infty(\Omega)$ (see [8, page 77]), we have $\|C_\tau x\|_{1\cap\infty} \le 4K\|x\|_{1\cap\infty}$ for all $x \in L^1(\Omega) \cap L^\infty(\Omega)$ and we proved the boundedeness of the composition operator C_τ in $L^1(\Omega) \cap L^\infty(\Omega)$.

The composition operator C_τ is obviously bounded in $L^\infty(\Omega)$ and we showed above that it is also bounded in $L^1(\Omega) \cap L^\infty(\Omega)$ when (2.6) holds. Now, we will show using the interpolation theory (or precisely, the Calderón-Lozanovskiĭ interpolation construction) that C_τ is bounded in $L^\varphi(\Omega) \cap L^\infty(\Omega)$.

Since $\rho(L^\infty, L^1 \cap L^\infty) = L^\psi$, where $\psi^{-1}(t) = \rho(1, \min(1, t))$ (see [11, Example 3 or Example 4 and Example 5 on pages 179–181]; see also [14, pages 459–461]) it follows that for $\rho(1, t) = \varphi^{-1}(t)$ we obtain $\psi(u) = \inf\{t > 0 : \psi^{-1}(t) > u\} = \inf\{t > 0 : \min\{\varphi^{-1}(1), \varphi^{-1}(t)\} > u\}$, and so

$$
\psi(u) = \begin{cases} \varphi(u) & \text{if } 0 \le u < \varphi^{-1}(1); \\ \infty & \text{if } u \ge \varphi^{-1}(1). \end{cases}
$$

Thus $L^\psi = L^\varphi \cap L^\infty$ and the interpolation theorem for Orlicz spaces or the Calderón-Lozanovskiĭ construction (see [11, Theorem 14.12 or Theorem 15.14]; see also [14, Theorem 8.2.2]) gives that C_τ is bounded in $\rho(L^\infty, L^1 \cap L^\infty) = L^\varphi \cap L^\infty$. $\qquad\square$

REMARK 7. Under the same assumptions on the measure μ and the Orlicz function φ as in Theorem 2.4 we can prove, similarly as in Theorem 2.4, that the composition

operator C_τ is bounded in $L^\varphi(\Omega) + L^\infty(\Omega)$ if and only if there exists a constant $K \geq 1$ such that $\min\{1, \mu(\tau^{-1}(A))\} \leq K \min\{1, \mu(A)\}$ for all $A \in \Sigma$ with $0 < \mu(A) < \infty$.

We formulate a necessary and sufficient condition for boundedness of composition operators in Orlicz spaces $L^\varphi(\Omega)$ in terms of the embedding of these Orlicz spaces into weighted Orlicz spaces $L_w^\varphi(\Omega)$ with the norm

$$\|x\|_{\varphi,w} = \inf \left\{ \lambda > 0 : \int_\Omega \varphi\left(\frac{|x(t)|}{\lambda}\right) w(t)\, d\mu \leq 1 \right\},$$

where the weight w is the Radon-Nikodým derivative of $\mu \circ \tau^{-1}$ with respect to μ.

THEOREM 2.5. *Let* (Ω, Σ, μ) *be a σ-finite nonatomic measure space and* $\tau : \Omega \to \Omega$ *be a measurable nonsingular transformation with* $\tau(\Omega) = \Omega$. *Denote by w the Radon-Nikodým derivative* $d\mu \circ \tau^{-1}/d\mu$. *Then the following conditions are equivalent*:

(a) *The composition operator C_τ is bounded from $L^\varphi(\Omega)$ into itself.*

(b) *For every $x \in L^\varphi(\Omega)$, there exists $\lambda > 0$ such that*

$$\int_\Omega \varphi(\lambda|x(t)|)w(t)\, d\mu(t) < \infty.$$

(c) *The Orlicz space $L^\varphi(\Omega)$ is embedded continuously into the weighted Orlicz space $L_w^\varphi(\Omega)$.*

(d) *There are $a, b > 0$ and $0 \leq g \in L^1(\Omega)$ such that $\varphi(au)w(t) \leq b\varphi(u) + g(t)$ for all $u > 0$ and $t \in \Omega \setminus A$ with $\mu(A) = 0$.*

PROOF. Since $\tau(\Omega) = \Omega$ it follows that

(2.7) $I_\varphi(C_\tau x) = I_\Phi(x),$

where $\Phi(t, u) = \varphi(u)w(t)$ is the Musielak–Orlicz function (see, for example, [13]). In fact,

$$I_\varphi(C_\tau x) = \int_\Omega \varphi(|C_\tau x(t)|)\, d\mu(t) = \int_\Omega \varphi(|x(\tau(s))|)\, d\mu(s)$$

$$= \int_{\tau(\Omega)} \varphi(|x(t)|)\, d(\mu \circ \tau^{-1}(t)) = \int_{\tau(\Omega)} \varphi(|x(t)|)w(t)\, d(\mu(t))$$

$$= \int_\Omega \varphi(|x(t)|)w(t)\, d\mu(t) = I_\Phi(x).$$

From (2.7) it follows that C_τ is an isometry from $L^\Phi(\Omega)$ into $L^\varphi(\Omega)$. Namely, defining

$$\|x\|_\Phi = \inf \left\{ \lambda > 0 : \int_\Omega \Phi\left(t, \frac{|x(t)|}{\lambda}\right) d\mu \leq 1 \right\},$$

we have for any $x \in L^\Phi(\Omega) \setminus \{0\}$, that $I_\varphi(C_\tau x / \|x\|_\Phi) = I_\Phi(x / \|x\|_\Phi) \le 1$, whence $\|C_\tau x / \|x\|_\Phi\|_\varphi \le 1$, or, equivalently, $\|C_\tau x\|_\varphi \le \|x\|_\Phi$. On the other hand, if $C_\tau x \in L^\varphi \setminus \{0\}$, then $I_\Phi(x / \|C_\tau x\|_\varphi) = I_\varphi(C_\tau x / \|C_\tau x\|_\varphi) \le 1$, whence $\|x\|_\Phi \le \|C_\tau x\|_\varphi$, and the proof that C_τ is an isometry from $L^\Phi(\Omega)$ into $L^\varphi(\Omega)$ is finished.

The continuity of the operator C_τ in $L^\varphi(\Omega)$ means that there exists a constant $K \ge 1$ such that $\|x\|_\Phi \le K\|x\|_\varphi$ for all $x \in L^\varphi(\Omega)$. This gives a continuous embedding of $L^\varphi(\Omega)$ into $L^\Phi(\Omega)$.

It is well known that the last embedding holds if and only if condition (d) is satisfied (see [13, Theorem 8.5] for the finite valued functions φ, Φ and [6, 3, 18] in the arbitrary case). Consequently, we get implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). Moreover, condition (d) yields easily condition (a), so the proof is finished. $\square$

If the measure space (Ω, Σ, μ) is σ-finite and purely atomic, that is, $\Omega = \bigcup_{n=1}^\infty A_n$, where A_n are the atoms with the measures $\mu(A_n) = a_n > 0$ for all $n \in \mathbb{N}$, then the *Orlicz sequence space* $l^\varphi(\{a_n\})$ is defined as the space of all real sequences $x = \{x_n\}_{n=1}^\infty$ such that $I_\varphi(\lambda x, \{a_n\}) < \infty$ for some $\lambda > 0$, where $I_\varphi(x, \{a_n\}) = \sum_{n=1}^\infty \varphi(|x_n|)a_n$. This space is considered with the norm

$$\|x\|_{\varphi, \{a_n\}} = \inf\{\lambda > 0 : I_\varphi(x/\lambda, \{a_n\}) \le 1\}.$$

THEOREM 2.6. *Let (Ω, Σ, μ) be a σ-finite and purely atomic measure space with atoms $\{A_n\}$ of measure $\mu(A_n) = a_n > 0$ for any $n \in \mathbb{N}$. Let $\tau : \Omega \to \Omega$ be a nonsingular transformation with $\tau(\Omega) = \Omega$ and $b_n := \mu(\tau^{-1}(A_n))/\mu(A_n)$. Then the following conditions are equivalent*:

(a) *The composition operator C_τ is bounded from $l^\varphi(\{a_n\})$ into itself.*

(b) *For every $x \in l^\varphi(\{a_n\})$ there exists $\lambda > 0$ such that $\sum_{n=1}^\infty \varphi(\lambda|x_n|)b_n < \infty$.*

(c) *The Orlicz space $l^\varphi(\{a_n\})$ is embedded continuously into the Orlicz space $l^\varphi(\{a_n b_n\})$.*

(d) *There are $a, b, \delta > 0$ and a sequence $\{c_n\}$ in l^1 of nonnegative numbers such that $\varphi(u)a_n < \delta \Rightarrow \varphi(au)a_n b_n \le b\varphi(u)a_n + c_n$ for all $n \in \mathbb{N}$ and all $u > 0$.*

PROOF. It is very similar to the proof of Theorem 2.5, but we present it for the sake of completeness. We have

$$I_\varphi(C_\tau x, \{a_n\}) = \sum_{n=1}^\infty \varphi(|C_\tau x_n|)a_n = \sum_{n=1}^\infty \varphi(|x_{\tau(n)}|)a_n$$

$$= \sum_{n \in \tau(\Omega)} \varphi(|x_n|)\mu(\tau^{-1}(A_n)) = \sum_{n=1}^\infty \varphi(|x_n|)a_n b_n = I_\varphi(x, \{a_n b_n\}).$$

For any $x \in l^\varphi(\{a_n b_n\}) \setminus \{0\}$,

$$I_\varphi\left(\frac{C_\tau x}{\|x\|_{\varphi, \{a_n b_n\}}}, \{a_n\}\right) = I_\varphi\left(\frac{x}{\|x\|_{\varphi, \{a_n b_n\}}}, \{a_n b_n\}\right) \le 1,$$

whence $\|C_\tau x/\|x\|_{\varphi,\{a_n b_n\}}\|_{\varphi,\{a_n\}} \leq 1$ or, equivalently, $\|C_\tau x\|_{\varphi,\{a_n\}} \leq \|x\|_{\varphi,\{a_n b_n\}}$. On the other hand, if $C_\tau x \in l^\varphi(\{a_n\}) \setminus \{0\}$, then

$$I_\varphi\left(\frac{x}{\|C_\tau x\|_{\varphi,\{a_n\}}}, \{a_n b_n\}\right) = I_\varphi\left(\frac{C_\tau x}{\|C_\tau x\|_{\varphi,\{a_n\}}}, \{a_n\}\right) \leq 1,$$

whence $\|x\|_{\varphi,\{a_n b_n\}} \leq \|C_\tau x\|_{\varphi,\{a_n\}}$. Consequently, the equality $\|C_\tau x\|_{\varphi,\{a_n\}} = \|x\|_{\varphi,\{a_n b_n\}}$ is proved and it shows that C_τ is an isometry from $l^\varphi(\{a_n b_n\})$ into $l^\varphi(\{a_n\})$. Taking into account that conditions (c) and (d) are equivalent (see [13, Theorem 8.11, page 51] for the finite valued functions and [19] in the arbitrary case), we can finish the proof in the same way as the proof of Theorem 2.5. □

3. Compactness of composition operators in Orlicz spaces

We start with the following result.

THEOREM 3.1. *Let φ be an Orlicz function vanishing only at zero with finite values, that is, $a_\varphi = 0$ and $b_\varphi = \infty$. Let τ be a measurable nonsingular transformation from Ω into itself such that $\tau(\Omega) = \Omega$. If C_τ is a compact operator from L^φ into itself, then the measure μ is purely atomic.*

PROOF. We can write $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 \cap \Omega_2 = \emptyset$, $\mu|_{\Omega_1}$ is nonatomic and $\mu|_{\Omega_2}$ is purely atomic. Since $\mu \circ \tau^{-1} \ll \mu$, then by the Radon-Nikodým theorem there exists a function h locally integrable on Ω_1 such that $\mu \circ \tau^{-1}(A) = \int_A h(t)\,d\mu$ for any $A \in \Sigma \cap \Omega_1$. Define $A_0 = \{t \in \Omega_1 : h(t) > 0\}$. We will show that $\mu \circ \tau^{-1}(A_0) = 0$. Assume for the contrary that $\mu \circ \tau^{-1}(A_0) > 0$. Then there is $\epsilon > 0$ such that the set $A_1 = \{t \in A_0 : h(t) \geq \epsilon\}$ has positive measure. Take a sequence $\{B_n\}$ of pairwise disjoint subsets of $\Sigma \cap A_1$ with $0 < \mu(B_n) < 1/2^n$ for $n \in \mathbb{N}$ large enough $(n > n_0)$. Define

$$x_n = \varphi^{-1}\left(\frac{1}{\mu(B_n)}\right) \chi_{B_n}, \quad n > n_0.$$

Then $I_\varphi(x_n) = 1$, whence $x_n \in L^\varphi(\Omega)$ and $\|x_n\|_\varphi = 1$ for $n > n_0$. Consequently, we have for $m, n > n_0$ with $m \neq n$,

$$I_\varphi(C_\tau x_m - C_\tau x_n) = \int_\Omega \varphi(|C_\tau x_m(s) - C_\tau x_n(s)|)\,d\mu(s)$$

$$= \int_\Omega \varphi(|x_m(\tau(s)) - x_n(\tau(s))|)\,d\mu(s)$$

$$= \int_{\tau(\Omega)} \varphi(|x_m(t) - x_n(t)|)\,d\mu \circ \tau^{-1}(t)$$

$$= \int_{\Omega} \varphi(|x_m(t) - x_n(t)|) \, d\mu \circ \tau^{-1}(t)$$

$$= \int_{\Omega} \varphi(|x_m(t) - x_n(t)|) h(t) \, d\mu(t)$$

$$= \int_{B_m} \varphi(|x_m(t)|) h(t) \, d\mu(t) + \int_{B_n} \varphi(|x_n(t)|) h(t) \, d\mu(t)$$

$$\geq \frac{1}{\mu(B_m)} \epsilon \mu(B_m) + \frac{1}{\mu(B_n)} \epsilon \mu(B_n) = 2\epsilon.$$

Therefore, $\|C_\tau x_m - C_\tau x_n\|_\varphi \geq 2\epsilon$ for $m, n > n_0$ with $m \neq n$. This means that $\{C_\tau x_n\}$ contains no subsequence which is a Cauchy sequence, that is, $C_\tau(B(L^\varphi(\Omega)))$, where $B(L^\varphi(\Omega))$ denotes the unit ball of $L^\varphi(\Omega)$, is not relatively compact. Consequently, the operator C_τ is not compact, a contradiction. The assumption that the transformation τ is nonsingular yields that $\mu(A_0) = 0$. The proof of the theorem is finished. $\square$

THEOREM 3.2. *Let φ be a finite-valued Orlicz function and let (Ω, Σ, μ) be a σ-finite and purely atomic measure space with the atoms A_n of measure $\mu(A_n) = a_n > 0$. For a measurable nonsingular transformation τ from Ω into itself, denote $b_n := \mu(\tau^{-1}(A_n))/\mu(A_n)$. If C_τ is a compact operator from $l^\varphi(\{a_n\})$ into itself, then $\lim_{n\to\infty} b_n = 0$.*

PROOF. Assume for the contrary that the assumptions are satisfied and $b_n \nrightarrow 0$. We may assume without loss of generality that there is $\epsilon \in (0, 1)$ such that $b_n \geq \epsilon$ for all $n \in \mathbb{N}$. Define

$$x_n = \varphi^{-1}\left(\frac{1}{\mu(\tau^{-1}(A_n))}\right) \chi_{A_n}, \quad n \in \mathbb{N}.$$

Then

$$C_\tau x_n = \varphi^{-1}\left(\frac{1}{\mu(\tau^{-1}(A_n))}\right) \chi_{\tau^{-1}(A_n)}$$

and

$$I_\varphi(x_n) \leq \frac{\mu(A_n)}{\mu(\tau^{-1}(A_n))} = \frac{1}{b_n} \leq \frac{1}{\epsilon} < \infty,$$

whence $\|x_n\|_\varphi \leq 1/\epsilon$ for all $n \in \mathbb{N}$.

On the other hand, we have for all $m \neq n$,

$$I_\varphi(C_\tau x_m - C_\tau x_n) = I_\varphi(C_\tau(x_m - x_n))$$

$$= \frac{1}{\mu(\tau^{-1}(A_m))} \mu(\tau^{-1}(A_m)) + \frac{1}{\mu(\tau^{-1}(A_n))} \mu(\tau^{-1}(A_n)) = 2.$$

Consequently, $\|C_\tau x_m - C_\tau x_n\|_\varphi > 1$ for all $m \neq n$, which means that $\{C_\tau x_n\}$ contains no Cauchy subsequence, that is, C_τ is not compact. This contradiction finishes the proof. $\square$

THEOREM 3.3. *Let φ be an Orlicz function vanishing only at zero. Let (Ω, Σ, μ) be a purely atomic σ-finite measure space withe the atoms A_n of measure $\mu(A_n) = a_n > 0$ and τ be a measurable nonsingular transformation from Ω into itself such that $\lim_{n\to\infty} b_n = 0$. Assume that either φ satisfies condition Δ_2 for all $u > 0$ when $\sum_{n=1}^{\infty} b_n = \infty$ or condition Δ_2 for large $u > 0$ when $\sum_{n=1}^{\infty} b_n < \infty$. Then the composition operator C_τ acts from $l^\varphi(\{a_n\})$ into itself and it is compact.*

PROOF. The assumption on φ implies that (see [2, Theorem 1.39], [13])

(3.1)   for any $\epsilon > 0$ there exists $\delta > 0$ such that $I_\varphi(x) < \delta$ implies $\|x\|_\varphi < \epsilon$.

Let $\{x_n\}$ be a sequence from $B(l^\varphi(\{a_n\}))$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in l^\varphi(\{a_n\})$ such that $\{x_{n_k}\}$ is weakly* convergent to x. We may assume without loss of generality that $x_n \to 0$ weakly* (if $x \neq 0$ we consider the sequence $\{x_n - x\}$ instead of $\{x_n\}$). Since $b_i \to 0$ as $i \to \infty$, there is $j \in \mathbb{N}$ such that $b_i < \delta/2$ for all $i > j$. Then

$$\sum_{i=j+1}^{\infty} b_i \varphi(|x_n(i)|) \mu(A_i) < \delta/2 \sum_{i=j+1}^{\infty} \varphi(|x_n(i)|) \mu(A_i) \leq \delta/2$$

for all $n \in \mathbb{N}$. Note that if $x_n \to 0$ weakly*, then $x_n \to 0$ pointwise. Really, the predual of $l^\varphi(\{a_n\})$ is the subspace h^Ψ of order continuous elements of the Musielak-Orlicz sequence space l^Ψ over the counting measure with the Musielak-Orlicz function $\Psi = \{\psi_n\}_{n=1}^{\infty}$, where $\psi_n(u) = a_n \varphi^\star(u/a_n)$ for all $u > 0$ and $n \in \mathbb{N}$. Take any $k \in \mathbb{N}$. Then $y = e_k \in h^\Psi$. We have $x_n(k) = \langle x_n, y \rangle \to 0$ by assumption, which means that $x_n \to 0$ pointwise.

Thus, there exists $m \in \mathbb{N}$ such that $\sum_{i=1}^{j} b_i \varphi(|x_n(i)|) \mu(A_i) < \delta/2$ for all $n > m$. Combining the above two estimates we get

$$I_\varphi(C_\tau x_n) = \sum_{i=1}^{\infty} \varphi(|x_n(i)|) \mu \tau^{-1}(A_i) = \sum_{i=1}^{\infty} \varphi(|x_n(i)|) b_i \mu(A_i) < \delta$$

for $n > m$. Applying condition (3.1), we obtain $\|C_\tau x_n\|_\varphi < \epsilon$ for all $n > m$, which means that $C_\tau x_n \to 0$. The proof is complete.   □

The following example shows that in the last theorem the assumptions that φ vanishes only at zero and that φ satisfies a suitable Δ_2-condition are important.

EXAMPLE 1. Let $\varphi(u) = 0$ if $0 \leq u \leq 1$ and $\varphi(u) = \infty$ if $u > 1$. Then for any measure space (Ω, Σ, μ), we have $L^\varphi(\Omega, \Sigma, \mu) = L^\infty(\Omega, \Sigma, \mu)$ with equality of the norms $\|x\|_\varphi$ and $\|x\|_\infty$. Thus in the case of the counting measure space, we have

$\|x\|_\varphi = \|x\|_\infty = \sup_{i \in \mathbb{N}} |x(i)|$. It is easy to see that the only compact composition operators $C_\tau : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ are finitely dimensional operators.

Assume for the contrary that C_τ is not finite dimensional. Then we may assume without loss of generality that $C_\tau \chi_{A_n} \neq 0$ for any atom A_n. Thus $\chi_{\tau^{-1}(A_n)} \neq 0$ for any atom A_n, which means that $\mu(\tau^{-1}(A_n)) \neq 0$, whence, $\tau^{-1}(A_n) \neq \emptyset$ for all $n \in \mathbb{N}$. Defining $x_n = \chi_{A_n}$, we have $\|x_n\|_\infty = 1$ for all $n \in \mathbb{N}$ and if $m \neq n$, we get $\|C_\tau x_m - C_\tau x_m\|_\infty = 1$, which means that C_τ is not compact.

THEOREM 3.4. *Let φ be an Orlicz function vanishing only at zero with $b_\varphi < \infty$ and $\varphi'_-(b_\varphi) < \infty$, where φ'_- stands for the left derivative of φ. Assume that (Ω, Σ, μ) is a purely atomic σ-finite measure space with the atoms A_n of measure $\mu(A_n) = a_n > 0$ such that $\liminf_{n \to \infty} \mu(A_n) \geq \alpha > 0$. Assume that τ is a measurable nonsingular transformation from Ω into itself. If C_τ is a compact operator from $l^\varphi(\{a_n\})$ into itself, then $\lim_{n \to \infty} b_n = 0$.*

PROOF. We may assume without loss of generality that $b_\varphi = 1$. Otherwise we may consider the function $\varphi_1(u) = \varphi(b_\varphi u)$, for which $L^{\varphi_1}(\Omega) = L^\varphi(\Omega)$ and $\|\cdot\|_{\varphi_1} = b_\varphi \|\cdot\|_\varphi$. There is a finite-valued Orlicz function ψ such that $\psi(u) = \varphi(u)$ for $0 \leq u \leq 1$. Define

$$\varphi_\infty(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq 1; \\ \infty & \text{if } u > 1. \end{cases}$$

Then $\varphi = \max\{\psi, \varphi_\infty\}$. Therefore (see [11, Theorem 12.2] and [17, page 130]) $L^\varphi(\Omega) = L^\psi(\Omega) \bigcap L^\infty(\Omega)$ and $\|x\|_\varphi = \max\{\|x\|_\psi, \|x\|_\infty\}$ for all $x \in L^\varphi(\Omega)$.

Assume for the contrary that $b_n \not\to 0$ as $n \to \infty$. Then we may assume without loss of generality that there is $\epsilon > 0$ such that $b_n \geq \epsilon$ and $w_n = \mu(A_n) \geq \alpha > 0$ for all $n \in \mathbb{N}$. Define $x_n = \min\{1, \psi^{-1}(1/w_n)\}\chi_{A_n}$, $n = 1, 2, \ldots$. Since

$$I_\varphi(x_n) = I_\psi(x_n) = \psi\left(\min\left\{1, \psi^{-1}(1/w_n)\right\}\right) \mu(A_n) \leq \frac{1}{w_n} w_n = 1,$$

it follows that $\|x_n\|_\varphi \leq 1$ for all $n \in \mathbb{N}$. On the other hand, we have for $m \neq n$,

$$\begin{aligned} I_\varphi(C_\tau x_m - C_\tau x_n) &= I_\psi(C_\tau x_m - C_\tau x_n) \\ &= \psi(\min\{1, \psi^{-1}(1/w_m)\})\mu(\tau^{-1}(A_m)) \\ &\quad + \psi(\min\{1, \psi^{-1}(1/w_n)\})\mu(\tau^{-1}(A_n)) \\ &\geq \epsilon\psi(\min\{1, \psi^{-1}(1/w_m)\})w_m + \epsilon\psi(\min\{1, \psi^{-1}(1/w_n)\})w_n \\ &\geq \epsilon \min\{\psi(1), 1/w_m\}w_m + \epsilon \min\{\psi(1), 1/w_n\}w_n \\ &\geq 2\epsilon \min\{\alpha\psi(1), 1\} > 0. \end{aligned}$$

This means that the operator C_τ is not compact which is a contradiction. $\square$

Acknowledgements

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