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FITTING CLASSES AND LATTICE FORMATIONS II

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Abstract

Given a lattice formation $\mathscr{F}$ of full characteristic, an $\mathscr{F}$- Fitting class is a Fitting class with stronger closure properties involving $\mathscr{F}$-subnormal subgroups. The main aim of this paper is to prove that the associated injectors possess a good behaviour with respect to $\mathscr{F}$-subnormal subgroups.

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1. Introduction

All groups considered are finite and soluble.

In a previous paper [2], $\mathscr{F}$-Fitting classes associated to a lattice formation $\mathscr{F}$ containing $\mathscr{N}$, the class of all nilpotent groups, are introduced and studied. A lattice formation is a class of groups whose elements are the direct product of Hall subgroups corresponding to fixed pairwise disjoint sets of primes. An $\mathscr{F}$-Fitting class is a class of groups which is closed under taking $\mathscr{F}$-subnormal subgroups and the join of $\mathscr{F}$-subnormal subgroups (see Definition 2.3). The classical Fitting classes appear as $\mathscr{N}$-Fitting classes.

In [2, Theorem 3.9] a large family of $\mathscr{F}$-Fitting classes, for every lattice formation $\mathscr{F}$ containing $\mathscr{N}$, is presented. The Fitting classes in this family are also saturated formations. Other examples of a different nature are also shown in [2, Examples I and II].

Since $\mathscr{N}$ is contained in the lattice formation $\mathscr{F}$, the subnormal subgroups are $\mathscr{F}$-subnormal and the $\mathscr{F}$-Fitting classes are Fitting classes. Our main aim in this paper is to prove that the following result, for an $\mathscr{F}$-Fitting class $\mathscr{X}$, holds: If W is an

$\mathscr{X}$-injector of a group G and H is an $\mathscr{F}$-subnormal subgroup of G, then $H \cap W$ is an $\mathscr{X}$-maximal subgroup of H. In fact, this property characterizes $\mathscr{F}$-Fitting classes (see Theorem 3.9 and Proposition 3.3), as the existence of injectors characterizes Fitting classes. The result obtained in [4, Theorem 4.5] appears now as one particular case.

2. Preliminaries

The reader is assumed to be familiar with the theories of saturated formations and Fitting classes and their projectors and injectors subgroups, respectively. We refer to [8] for the relevant definitions, notations and results.

For the sake of completeness we will recall some concepts and results.

A *lattice formation $\mathscr{F}$ of characteristic π* is a saturated formation locally defined by a formation function f given by: $f(p) = \mathscr{S}_{\pi_i}$, if $p \in \pi_i \subseteq \pi$, where $\{\pi_i\}_{i \in I}$ is a partition of the set of primes π, and $f(q) = \emptyset$, the empty formation, if $q \notin \pi$. $\mathscr{S}_{\pi_i}$ denotes the set of all soluble π_i-groups.

In this case, for a prime $p \in \pi$, the set of primes π_i such that $p \in \pi_i$, will be also identified by $\pi(p)$.

LEMMA 2.1 ([5, Remark 3.6], [4, Lemma 3.2]). *Let $\mathscr{F}$ be a lattice formation with characteristic π and $p \in \pi$. Then*:

(a) *The canonical local definition of $\mathscr{F}$ and the smallest local definition of $\mathscr{F}$ are given by setting*:

- *If $|\pi(p)| = 1$, then $F(p) = \mathscr{S}_p$ and $\underline{f}(p) = (1)$.*
- *If $|\pi(p)| \geq 2$, then $F(p) = f(p) = \mathscr{S}_{\pi(p)}$. In particular, for a group G, $G^{F(p)} = G^{\underline{f}(p)} = O^{\pi(p)}(G)$.*

(b) *A group G belongs to $\mathscr{F}$ if and only if G is a soluble π-group with a normal Hall π_i-subgroup, for every $i \in I$.*

Henceforth $\mathscr{F}$ will always denote a lattice formation containing $\mathscr{N}$ and the above notation will be assumed.

In this section, $\mathscr{G}$ denotes a subgroup-closed saturated formation.

DEFINITION 2.2 ([8, III, Definition 4.13, IV, Definition 5.12]). A maximal subgroup M of a group G is said to be *$\mathscr{G}$-normal* in G if $G/\mathrm{Core}_G(M) \in \mathscr{G}$; otherwise, it is called *$\mathscr{G}$-abnormal*.

A subgroup H of a group G is said to be *$\mathscr{G}$-subnormal* in G if either $H = G$ or there exists a chain $H = H_n < H_{n-1} < \cdots < H_0 = G$ such that H_{i+1} is a $\mathscr{G}$-normal maximal subgroup of H_i, for every $i = 0, \ldots, n-1$. We write $H\,\mathscr{G}\text{-sn}\,G$.

DEFINITION 2.3 ([2, Definition 3.1]). A class $\mathscr{X}$ $(\neq \emptyset)$ of groups is called an $\mathscr{F}$-*Fitting class* if the following conditions are satisfied:

(i) If $G \in \mathscr{X}$ and H $\mathscr{F}$-sn G, then $H \in \mathscr{X}$.
(ii) If H, K $\mathscr{F}$-sn $G = \langle H, K\rangle$ with H and K in $\mathscr{X}$, then $G \in \mathscr{X}$.

The Fitting classes are exactly the $\mathscr{N}$-Fitting classes. Moreover, an $\mathscr{F}$-Fitting class is, in particular, a Fitting class.

PROPOSITION 2.4 ([2, Proposition 3.4 (a)]). *Let $\mathscr{X}$ be an $\mathscr{F}$-Fitting class and G a group. The $\mathscr{X}$-radical $G_{\mathscr{X}}$ of G has the form*: $G_{\mathscr{X}} = \langle H \leq G : H$ $\mathscr{F}$-sn $G, H \in \mathscr{X}\rangle$.

DEFINITION 2.5 ([11, Definition], [12, Definition 5.8]). A subgroup H of a group G is said to be $\mathscr{G}$*-abnormal in* G if every link in every maximal chain joining H to G is $\mathscr{G}$-abnormal, that is, H is a $\mathscr{G}$-abnormal subgroup of G if, whenever $H \leq M < L \leq G$ and M is a maximal subgroup of L, then M is a $\mathscr{G}$-abnormal subgroup of L. We write H $\mathscr{G}$-abn G.

In [12, Definition 3.15], $\mathscr{G}$-pronormal subgroups are defined in terms of complement $\mathscr{G}$-basis. They are characterized in the following way:

THEOREM 2.6 ([12, Satz 3.21]). *A subgroup H of a group G is $\mathscr{G}$-pronormal in G if and only if H satisfies the following property*: *'If $g \in G$, then $H^g = H^x$ for some $x \in \langle H, H^g\rangle^{\mathscr{G}}$'. In this case, we write H $\mathscr{G}$-pr G.*

THEOREM 2.7. *For a subgroup H of a group G, the following are equivalent*:

(1) *H $\mathscr{G}$-pr G.*
(2) ([12, Satz 3.26]) *If $H \leq K \trianglelefteq L \leq G$, then $L = K^{\mathscr{G}} N_L(H)$.*
(3) ([6, Theorem 3], [9, Theorem 2.10]) *If $H \leq L \leq G$, then $L = S_L(H, \mathscr{G}) N_L(H)$, where $S_L(H, \mathscr{G})$ is the $\mathscr{G}$-subnormal closure of H in L, that is, the intersection of all $\mathscr{G}$-subnormal subgroups of G containing H.*

By [12, Satz 5.14], a subgroup H of a group G is $\mathscr{G}$-abnormal in G if and only if H is $\mathscr{G}$-pronormal and self-normalizing in G.

THEOREM 2.8 ([12, Satz 3.18, Satz 5.17]). *Let H be a $\mathscr{G}$-pronormal subgroup of a group G and $N \trianglelefteq G$. Then*:

(1) *HN/N is $\mathscr{G}$-pronormal in G/N.*
(2) *$N_G(H)$ contains a $\mathscr{G}$-normalizer of G.*

THEOREM 2.9 ([7, Lemma 5.1], [12, Satz 5.22]). *Let H be a subgroup of a group G. Then H is a $\mathscr{G}$-projector of G if and only if $H \in \mathscr{G}$ and H is $\mathscr{G}$-abnormal in G.*

In particular, the $\mathscr{G}$-projectors of G are also $\mathscr{G}$-pronormal in G.

THEOREM 2.10 ([8, IV, Theorem 5.18]). *Let G be a group whose $\mathscr{G}$-residual $G^{\mathscr{G}}$ is abelian. Then $G^{\mathscr{G}}$ is complemented in G and two complements in G of $G^{\mathscr{G}}$ are conjugate. The complements are the $\mathscr{G}$-projectors of G.*

For a group G, we write $\mathrm{Proj}_{\mathscr{G}}(G)$ to denote the set of all $\mathscr{G}$-projectors of G. $\mathrm{Z}_{\mathscr{G}}(G)$ denotes the $\mathscr{G}$-hypercentre of the group G ([8, IV, Definition 6.8]).

A subgroup H of a group G is called *self-$\mathscr{G}$-normalizing* in G, if whenever H $\mathscr{G}$-sn $T \leq G$, then $H = T$.

THEOREM 2.11 ([2, Theorem 4.2]). *For a subgroup H of a group G, the following statements are equivalent*:

(i) *H is a $\mathscr{G}$-projector of G.*

(ii) *H is a self-$\mathscr{G}$-normalizing $\mathscr{G}$-subgroup of G and H satisfies the following property*: '*if $H \leq K \leq G$, then $H \cap K^{\mathscr{G}} \leq (K^{\mathscr{G}})'$*'.

3. $\mathscr{F}$-Fitting classes and injectors. The main result

In order to prove our main result we proceed in the following way.

DEFINITION 3.1. Let $\mathscr{X}$ be a class of groups and let $\mathscr{F}$ be a lattice formation containing $\mathscr{N}$. An *$(\mathscr{X}, \mathscr{F})$-injector of a group G* is a subgroup V of G with the property that $V \cap K$ is an $\mathscr{X}$-maximal subgroup of K, for all $\mathscr{F}$-subnormal subgroups K of G. We denote the (possibly empty) set of $(\mathscr{X}, \mathscr{F})$-injectors of G by $\mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(G)$.

Obviously, the $\mathscr{X}$-injectors are the $(\mathscr{X}, \mathscr{N})$-injectors.

$\mathrm{Inj}_{\mathscr{X}}(G)$ denotes the (possibly empty) set of $\mathscr{X}$-injectors of a group G.

REMARK 3.2. Let G be a group and $\mathscr{X}$ a class of groups.

(a) If $V \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(G)$ and K $\mathscr{F}$-sn G, then $V \cap K \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(K)$.

(b) If $V \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(G)$ and $\alpha : G \to (G)\alpha$ an isomorphism, then

$$(V)\alpha \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}((G)\alpha),$$

in particular, $\mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(G)$ is a union of G-conjugacy classes.

(c) Let V be an $\mathscr{X}$-maximal subgroup of G, and assume that $V \cap M \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(M)$, for every $\mathscr{F}$-normal maximal subgroup M of G. Then $V \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(G)$.

(d) $\mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(G) \subseteq \mathrm{Inj}_{\mathscr{X}}(G)$. This is because $\mathscr{N} \subseteq \mathscr{F}$, which implies that subnormal subgroups are $\mathscr{F}$-subnormal subgroups.

Moreover, if $\mathscr{X}$ is a Fitting class, then $\mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(G) \neq \emptyset$ if and only if $\mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(G) = \mathrm{Inj}_{\mathscr{X}}(G)$. This is clear by the very well known result of Fischer, Gaschütz and Hartley about the existence and conjugacy of injectors ([8, VIII, Theorem 2.9], [10]).

(We recall that the $\mathscr{F}$-Fitting classes are Fitting classes.)

It is well known that the existence of injectors in every group characterizes Fitting classes. The first corresponding result for $\mathscr{F}$-Fitting classes is the following one. It can be proven by arguing as in the classical result with obvious changes (see [8, IX, Theorem 1.4]). Thus we omit the proof.

PROPOSITION 3.3. *Let $\mathscr{X}$ be a class of groups. If every group has an $(\mathscr{X}, \mathscr{F})$-injector, then $\mathscr{X}$ is an $\mathscr{F}$-Fitting class.*

Our aim is to prove that the converse of this proposition is also true. The proof of our main result (Theorem 3.9) is inspired by the proof of the Fischer, Gaschütz and Hartley classical result ([8, VIII, Theorem 2.9], [10]). We begin with some preparatory lemmas. Also Theorem 2.11 will play an important role.

REMARK 3.4. It is well known that the injectors and the projectors associated to a Fitting class and to a Schunck class (in particular, to a saturated formation), respectively, are pronormal (see [8, III, Corollary 3.22, IX, Theorem 1.5]). Even more, the $\mathscr{G}$-projectors associated to a saturated formation $\mathscr{G}$, are $\mathscr{G}$-pronormal (Theorem 2.9). This is not the case for the injectors, if $\mathscr{G}$ is a saturated Fitting formation. Take for instance $\mathscr{G} = \mathscr{S}_{2'}\mathscr{S}_2$, the class of all 2-nilpotent groups, and $G = \mathrm{Sym}(4)$ the symmetric group of degree 4. The $\mathscr{G}$-injectors of G are the Sylow 2-subgroups of G. Let $P \in \mathrm{Syl}_2(G)$ and let x be a 3-element of G. Then $G^{\mathscr{G}} = \langle P, P^x\rangle^{\mathscr{G}}$ is the normal four-subgroup of G. It is clear that P and P^x are not conjugate in $G^{\mathscr{G}}$. Then P is not $\mathscr{G}$-pronormal in G.

If $\mathscr{X}$ is an $\mathscr{F}$-Fitting class, we will obtain that the $(\mathscr{X}, \mathscr{F})$-injectors are $\mathscr{F}$-pronormal. This means that the $\mathscr{X}$-injectors are $\mathscr{F}$-pronormal, for this Fitting class $\mathscr{X}$. A first step is given by the following result.

LEMMA 3.5. *Let $\mathscr{X}$ be an $\mathscr{F}$-Fitting class and let G be a group. Suppose that U is an $(\mathscr{X}, \mathscr{F})$-injector of G and U satisfies the following property*

$$(*) \qquad \text{if } U \le T \le G, \text{ then } U \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(T).$$

Then U is $\mathscr{F}$-pronormal in G.

PROOF. Let $x \in G$. Since U is an $\mathscr{X}$-injector of G, then U is pronormal in G. Consequently, there exists $t \in \langle U, U^x\rangle$ such that $U^x = U^t$. In particular, $t \in \langle U, U^t\rangle = \langle U, U^x\rangle$.

Assume that $\langle U, U^t\rangle < G$. Since U satisfies the property $(*)$, arguing by induction on the order of G we can assume that U is $\mathscr{F}$-pronormal in $\langle U, U^t\rangle$. Then there exists $r \in \langle U, U^t\rangle^{\mathscr{F}} = \langle U, U^x\rangle^{\mathscr{F}}$ such that $U^r = U^t = U^x$.

Consider now the case $G = \langle U, U^x\rangle$.

If $U\langle U, U^x\rangle^{\mathscr{F}} = G$, then $x = um$ for some $u \in U$ and $m \in \langle U, U^x\rangle^{\mathscr{F}}$. Obviously, we have $U^x = U^m$ with $m \in \langle U, U^x\rangle^{\mathscr{F}}$.

Otherwise we would have that $T = U\langle U, U^x\rangle^{\mathscr{F}} < G$. But in the case under consideration $G^{\mathscr{F}} = \langle U, U^x\rangle^{\mathscr{F}}$, which implies that T is an $\mathscr{F}$-subnormal subgroup of G. Therefore, $T \cap U^x \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(T)$. Thus, $U, T \cap U^x \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(T) = \mathrm{Inj}_{\mathscr{X}}(T)$. Consequently, there would exist $t = ur \in T = U\langle U, U^x\rangle^{\mathscr{F}}$ with $u \in U$ and $r \in \langle U, U^x\rangle^{\mathscr{F}}$, such that $T \cap U^x = U^t = U^r$. In particular, $U^r \leq U^x$. Clearly we would deduce also in this situation that $U^r = U^x$ with $r \in \langle U, U^x\rangle^{\mathscr{F}}$.

Hence U is an $\mathscr{F}$-pronormal subgroup of G. □

LEMMA 3.6. *Let $\mathscr{X}$ be an $\mathscr{F}$-Fitting class and let G be a group. Let K be a normal subgroup of G such that $G/K \in \mathscr{F}$. Suppose that there exists an $\mathscr{X}$-maximal subgroup W of K and an $\mathscr{X}$-maximal subgroup X of G such that $W \trianglelefteq G$ and $X \cap K = W$. Then:*

(a) $X/W \leq \mathrm{Z}_{\mathscr{F}}(N_G(X)/W)$.
(b) $X = (CW)_{\mathscr{X}}$, *for every* $C \in \mathrm{Proj}_{\mathscr{F}}(N_G(X))$.
(c) *If* $CW/W \in \mathrm{Proj}_{\mathscr{F}}(N_G(X)/W)$, *then* CW/W *is a self-$\mathscr{F}$-normalizing $\mathscr{F}$-maximal subgroup of* G/W.

PROOF. Let $N = N_G(X)$.

(a) Let L_1/L_0 be an N-composition factor of X such that $W \leq L_0 \leq L_1 \leq X$. Suppose that L_1/L_0 is a p-group, for a prime p. It is clear that XK/K is N-isomorphic to X/W is such a way that L_1K/K and L_0K/K are N-isomorphic to L_1/W and L_0/W, respectively. Consequently we have that $(L_1K/K)/(L_0K/K)$ is a chief factor of NK/K and $C_N(L_1K/L_0K) = C_N(L_1/L_0)$. Moreover, $[L_1, N \cap K] \leq L_1 \cap K = W \leq L_0$, that is, $N \cap K \leq C_N(L_1/L_0)$. Then we have:

$$\begin{aligned}(NK)/C_{NK}((L_1K)/(L_0K)) &= (NK)/(C_N((L_1K)/(L_0K))K)\\ &= (NK)/(C_N(L_1/L_0)K)\\ &\cong N/(C_N(L_1/L_0)(N \cap K)) = N/C_N(L_1/L_0).\end{aligned}$$

Since $G/K \in \mathscr{F}$, then $NK/K \in \mathscr{F}$ and we can conclude that $N/C_N(L_1/L_0) \in F(p)$. This implies that $X/W \leq \mathrm{Z}_{\mathscr{F}}(N_G(X)/W)$.

(b) We have that $X/W \leq \mathrm{Z}_{\mathscr{F}}(N/W) \leq CW/W$, for all $CW/W \in \mathrm{Proj}_{\mathscr{F}}(N/W)$, with $C \in \mathrm{Proj}_{\mathscr{F}}(N)$, by (a) and [8, IV, Theorem 6.14]. Since $CW/W \in \mathscr{F}$, then X is an $\mathscr{F}$-subnormal subgroup of CW. But X is an $\mathscr{X}$-maximal subgroup of G, which implies that $X = (CW)_{\mathscr{X}}$ because $\mathscr{X}$ is an $\mathscr{F}$-Fitting class.

(c) Assume that CW is $\mathscr{F}$-subnormal in $T \leq G$. Then X is also $\mathscr{F}$-subnormal in T, because X is normal in CW by (b). Again the $\mathscr{X}$-maximality of X in G implies that $X = T_{\mathscr{X}}$. In particular, $T \leq N$. Therefore, CW/W is also an $\mathscr{F}$-projector of T/N (see [8, III, Corollary 3.22]) and then, CW/W is $\mathscr{F}$-abnormal in T/N by Theorem 2.9. Consequently, $CW = T$.

In particular, CW/W is $\mathscr{F}$-maximal in G/N. This is clear because every subgroup of a group in $\mathscr{F}$ is $\mathscr{F}$-subnormal in the group. □

LEMMA 3.7. *Let $\mathscr{X}$ be an $\mathscr{F}$-Fitting class, G a group and K a normal subgroup of G such that $G/K \in \mathscr{F}$. Suppose that W is an $\mathscr{X}$-maximal subgroup of K such that $W \trianglelefteq G$. Suppose also that G has an $(\mathscr{X}, \mathscr{F})$-injector X, which satisfies the following property*

$$(\star) \qquad \text{if } X \leq T \leq G, \text{ then } X \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(T).$$

(*Note that $X \cap K = W$.*)

Then $X = (HW)_{\mathscr{X}}$, for some $H \in \mathrm{Proj}_{\mathscr{F}}(G)$.

PROOF. Lemma 3.6 implies that

$$X/W \leq UW/W, \quad \text{for every} \quad UW/W \in \mathrm{Proj}_{\mathscr{F}}(N_G(X)/W), \quad \text{and}$$
$$X = (UW)_{\mathscr{X}}, \quad \text{for every} \quad U \in \mathrm{Proj}_{\mathscr{F}}(N_G(X)).$$

Note that every subgroup L of G containing X satisfies the hypothesis of the Lemma just with $K \cap L$ instead of K.

Consider the sets:

$$\mathscr{A} = \{L \leq G : X \leq L,\ \mathrm{Proj}_{\mathscr{F}}(N_L(X)/W) \subseteq \mathrm{Proj}_{\mathscr{F}}(L/W)\}, \quad \text{and}$$
$$\mathscr{B} = \{L \leq G : X \leq L\}.$$

Note that $\mathscr{A}$ is non-empty because at least $X \in \mathscr{A}$. We claim that $\mathscr{A} = \mathscr{B}$.

Assume that it is not true and take a subgroup L of minimal order in $\mathscr{B} \setminus \mathscr{A}$. Take $UW/W \in \mathrm{Proj}_{\mathscr{F}}(N_L(X)/W)$, with $U \in \mathrm{Proj}_{\mathscr{F}}(N_L(X))$.

We use the 'bar' notation to denote images under the natural homomorphism $G \to G/W = \overline{G}$.

Whenever $\overline{X} \leq \overline{U} \leq \overline{T} < \overline{L}$, then $\overline{U} \leq N_{\overline{L}}(\overline{X}) \cap \overline{T} = N_{\overline{T}}(\overline{X}) \leq N_{\overline{L}}(\overline{X})$. In particular, $\overline{U} \in \mathrm{Proj}_{\mathscr{F}}(N_{\overline{T}}(\overline{X}))$. The choice of L implies that $\overline{U} \in \mathrm{Proj}_{\mathscr{F}}(\overline{T})$.

By the hypothesis, we can apply Lemma 3.5 to L and X and conclude that X is $\mathscr{F}$-pronormal in L. In particular, $\overline{X}$ is $\mathscr{F}$-pronormal in $\overline{L}$ which implies that $N_{\overline{L}}(\overline{X})$ contains an $\mathscr{F}$-normalizer of $\overline{L}$ by Theorem 2.8. It is clear that $N_{\overline{L}}(\overline{X}) < \overline{L}$ by the choice of L. Thus, there exists $\overline{M}$ a maximal subgroup of $\overline{L}$ such that $N_{\overline{L}}(\overline{X}) \leq \overline{M} < \overline{L}$. Since $\overline{M}$ contains an $\mathscr{F}$-normalizer of $\overline{L}$, $\overline{M}$ is $\mathscr{F}$-abnormal in $\overline{L}$ by [8, V, Lemma 3.4]. In particular, $\overline{L} = \overline{L}^{\mathscr{F}}\overline{M}$. Moreover, $\overline{M} = \overline{M}^{\mathscr{F}}\overline{U}$, because $\overline{U}$ is an $\mathscr{F}$-projector of $\overline{M}$. Then $\overline{L} = \overline{L}^{\mathscr{F}}\overline{U}$. This implies that every maximal subgroup of $\overline{L}$

containing $\overline{U}$ is $\mathscr{F}$-abnormal. Moreover $\overline{U}$ is $\mathscr{F}$-abnormal in every proper subgroup of $\overline{L}$ containing $\overline{U}$, by Theorem 2.9, because $\overline{U}$ is an $\mathscr{F}$-projector of a such subgroup. Consequently $\overline{U}$ is $\mathscr{F}$-abnormal in $\overline{L}$ and $\overline{U} \in \mathscr{F}$. Then $\overline{U}$ is an $\mathscr{F}$-projector of $\overline{L}$ by Theorem 2.9. This contradicts the choice of L and proves that $\mathscr{A} = \mathscr{B}$.

Consequently, if $U \in \mathrm{Proj}_{\mathscr{F}}(N_G(X))$, then $\overline{U} \in \mathrm{Proj}_{\mathscr{F}}(N_{\overline{G}}(\overline{X})) \subseteq \mathrm{Proj}_{\mathscr{F}}(\overline{G})$. Since $U \in \mathrm{Proj}_{\mathscr{F}}(UW)$, we have that $U \in \mathrm{Proj}_{\mathscr{F}}(G)$. Therefore $X = (UW)_{\mathscr{X}}$ with $U \in \mathrm{Proj}_{\mathscr{F}}(N_G(X)) \subseteq \mathrm{Proj}_{\mathscr{F}}(G)$ and we are done. □

LEMMA 3.8. *Let $\mathscr{X}$ be an $\mathscr{F}$-Fitting classes, G a group and K a normal subgroup of G such that $G/K \in \mathscr{F}$. Suppose that G satisfies the following property*

$$(**) \qquad \text{If } H < G, \quad \text{then } \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(H) \neq \emptyset.$$

Suppose that W is an $\mathscr{X}$-maximal subgroup of K and that X is an $\mathscr{X}$-maximal subgroup of G such that $W \trianglelefteq G$ and $X \cap K = W$.

Then $\mathrm{Proj}_{\mathscr{F}}(N_G(X)/W) \subseteq \mathrm{Proj}_{\mathscr{F}}(G/W)$, and consequently it follows that

$$\mathrm{Proj}_{\mathscr{F}}(N_G(X)) \subseteq \mathrm{Proj}_{\mathscr{F}}(G).$$

PROOF. As in Lemma 3.7, we take into consideration the following facts.

By Lemma 3.6, we have that

$$X/W \leq UW/W, \quad \text{for every } UW/W \in \mathrm{Proj}_{\mathscr{F}}(N_G(X)/W),$$

where $U \in \mathrm{Proj}_{\mathscr{F}}(N_G(X))$.

It is clear that every subgroup L of G containing X satisfies the hypothesis of the Lemma with $K \cap L$ instead of K.

Consider the following sets

$$\mathscr{A} = \{L \leq G : X \leq L,\ \mathrm{Proj}_{\mathscr{F}}(N_L(X)/W) \subseteq \mathrm{Proj}_{\mathscr{F}}(L/W)\}, \quad \text{and}$$
$$\mathscr{B} = \{L \leq G : X \leq L\}.$$

Notice that $X \in \mathscr{A} \neq \emptyset$.

Our purpose is to prove that $\mathscr{A} = \mathscr{B}$. The result then follows easily.

Assume that this is not true and take a group L of minimal order in $\mathscr{B} \setminus \mathscr{A}$. Consider $UW/W \in \mathrm{Proj}_{\mathscr{F}}(N_L(X)/W)$, with $U \in \mathrm{Proj}_{\mathscr{F}}(N_L(X))$.

We use the 'bar' notation to denote images in the factor group $G/W = \overline{G}$.

We split the proof into the following steps.

Step 1. *If $\overline{X} \leq \overline{U} \leq \overline{T} < \overline{L}$, then $\overline{U} \in \mathrm{Proj}_{\mathscr{F}}(\overline{T})$.*

It is clear by the choice of L as in Lemma 3.7. Note that $N_L(X) < L$, by the choice of L.

Step 2. *Every maximal subgroup of $\overline{L}$ containing $\overline{U}$ is $\mathscr{F}$-normal in $\overline{L}$.*

Assume that there exists $\overline{M}$ an $\mathcal{F}$-abnormal maximal subgroup of $\overline{L}$ containing $\overline{U}$. Arguing as in Lemma 3.7 we can deduce that $\overline{U}$ is an $\mathcal{F}$-projector of $\overline{L}$, which contradicts the choice of L and proves Step 2.

Step 3. *If $\overline{M} = M/W$ is a maximal subgroup of $\overline{L}$ containing $\overline{U}$, then $X \in \mathrm{Inj}_{(\mathcal{X},\mathcal{F})}(M)$.*

By Step 1, we have that $\overline{U} \in \mathrm{Proj}_{\mathcal{F}}(\overline{M})$. Arguing as in Lemma 3.6 (b), we deduce that $X = (UW)_{\mathcal{X}}$. Moreover, $U \in \mathrm{Proj}_{\mathcal{F}}(M)$ as $U \in \mathrm{Proj}_{\mathcal{F}}(UW)$. Since $M < G$, there exists $Y \in \mathrm{Inj}_{(\mathcal{X},\mathcal{F})}(M)$, by the hypothesis.

Notice that M satisfies the hypothesis of Lemma 3.7. In particular, we claim that whenever $Y \leq T \leq M$, then $Y \in \mathrm{Inj}_{(\mathcal{X},\mathcal{F})}(T)$. The hypothesis implies that $\emptyset \neq \mathrm{Inj}_{(\mathcal{X},\mathcal{F})}(M) = \mathrm{Inj}_{\mathcal{X}}(M)$ and $\emptyset \neq \mathrm{Inj}_{(\mathcal{X},\mathcal{F})}(T) = \mathrm{Inj}_{\mathcal{X}}(T)$. Then $Y \in \mathrm{Inj}_{\mathcal{X}}(T) = \mathrm{Inj}_{(\mathcal{X},\mathcal{F})}(T)$, by [8, IX, Theorem 1.5 (c)].

By Lemma 3.7, we have that $Y = (ZW)_{\mathcal{X}}$ with $Z \in \mathrm{Proj}_{\mathcal{F}}(M)$. Consequently, $X = Y^m \in \mathrm{Inj}_{(\mathcal{X},\mathcal{F})}(M)$, for some $m \in M$.

Step 4. *Let M be a maximal subgroup of L containing UW. Then $L = L^{\mathcal{F}} N_L(X \cap M^{F(p)})$ with p the prime dividing $|L:M|$.*

By Step 2, M is $\mathcal{F}$-normal in L. Then $L^{F(p)} = M^{F(p)} \trianglelefteq L$, because $M^{F(p)} = O^{\pi(p)}(M)$. Step 3 implies that $X \in \mathrm{Inj}_{(\mathcal{X},\mathcal{F})}(M)$, then $X \cap M^{F(p)} = X \cap L^{F(p)} = J \in \mathrm{Inj}_{(\mathcal{X},\mathcal{F})}(L^{F(p)})$. Take $l \in L$. We have that $J^l \in \mathrm{Inj}_{(\mathcal{X},\mathcal{F})}((L^{F(p)})^l) = \mathrm{Inj}_{(\mathcal{X},\mathcal{F})}(L^{F(p)})$.

In particular, J and J^l are $\mathcal{X}$-injectors of $L^{F(p)}$, which implies that $J^l = J^t$, for some $t \in L^{F(p)}$.

Arguing as in Step 3 we can prove that every $(\mathcal{X},\mathcal{F})$-injector of $L^{F(p)}$ is an $(\mathcal{X},\mathcal{F})$-injector of every subgroup of $L^{F(p)}$, containing the $(\mathcal{X},\mathcal{F})$-injector. By Lemma 3.5 we conclude that J is an $\mathcal{F}$-pronormal subgroup of $L^{F(p)}$.

Consequently, there exists $x \in \langle J, J^t\rangle^{\mathcal{F}}$ such that $J^t = J^l = J^x$. Since $\langle J, J^t\rangle^{\mathcal{F}} \leq L^{\mathcal{F}}$, it is clear that $l \in L^{\mathcal{F}} N_G(J)$. Thus $L = L^{\mathcal{F}} N_L(X \cap L^{F(p)}) = L^{\mathcal{F}} N_L(X \cap M^{F(p)})$.

Step 5. *There exists a unique maximal subgroup M of L containing UW. In particular, $N_{\overline{L}}(\overline{X}) \leq \overline{M}$.*

Let $\overline{M}$ be a maximal subgroup of $\overline{L}$ containing $\overline{U}$. By Step 1 and Step 2, it is clear that $\overline{M} = \overline{M}^{\mathcal{F}}\,\overline{U} = \overline{L}^{\mathcal{F}}\,\overline{U}$ and the conlusion is obvious.

Let p be the prime dividing $|L:M|$.

Step 6. $X \cap M^{F(p)} \trianglelefteq L$.

Note that $UW \leq N_L(X) \leq N_L(X \cap M^{F(p)}) = N_L(X \cap L^{F(p)})$. Consequently, if $N_L(X \cap M^{F(p)})$ were a proper subgroup of L, then it would be contained in M, by Step 5. Moreover $L^{\mathcal{F}} \leq M$ by Step 2. Then we could conclude that $L = M$, by Step 4, which is a contradiction.

Step 7. *Let $\overline{R}$ be an $\mathcal{F}$-projector of $\overline{L}$ and $\overline{T} = (\overline{L}^{\mathcal{F}})'$. Let*

$$\overline{C}/\overline{T} = \mathsf{X}_{q \notin \pi(p)} C_{O_q(\overline{L}^{\mathcal{F}}/\overline{T})}\left((((\overline{R} \cap \overline{M})\overline{T})/\overline{T})^{F(q)}\right).$$

Then $(\overline{U}\,\overline{T})/\overline{T}$ *is conjugate to* $\overline{E}/\overline{T} = (\overline{C}(\overline{R} \cap \overline{M}))/\overline{T}$ *in* $\overline{M}/\overline{T}$.

Since $\overline{R}$ is an $\mathscr{F}$-projector of $\overline{L}$, $\overline{L} = \overline{L}^{\mathscr{F}}\,\overline{R}$. By Step 2, $\overline{L}^{\mathscr{F}} \leq \overline{M}$ which implies that $\overline{M} = \overline{L}^{\mathscr{F}}(\overline{M} \cap \overline{R})$. Obviously, $(\overline{U}\,\overline{T})/\overline{T} \in \mathrm{Proj}_{\mathscr{F}}(\overline{M}/\overline{T})$. On the other hand, $\overline{M}/\overline{T} = (\overline{L}^{\mathscr{F}}/\overline{T})(((\overline{M} \cap \overline{R})\overline{T})/\overline{T})$, with $\overline{L}^{\mathscr{F}}/\overline{T}$ an abelian normal subgroup of $\overline{M}/\overline{T}$ and $((\overline{M} \cap \overline{R})\overline{T}/\overline{T}) \in \mathscr{F}$. Consider for a moment a subgroup $\overline{C}/\overline{T}$, constructed as in the statement, but with the primes in the direct product running all the prime numbers. Thus the subgroup $\overline{E}/\overline{T}$ constructed as in the statement is an $\mathscr{F}$-projector of $\overline{M}/\overline{T}$ by [8, IV, Theorem 5.16]. In particular, this subgroup $\overline{E}/\overline{T}$ is conjugate to $(\overline{U}\,\overline{T})/\overline{T}$ in $\overline{M}/\overline{T}$.

We claim that $\pi(p) \cap \pi(\overline{C}/\overline{T}) = \emptyset$, which proves Step 7.

Notice that the group $\overline{L}/\overline{T} = (\overline{L}^{\mathscr{F}}/\overline{T})((\overline{R}\,\overline{T})/\overline{T})$ is a semidirect product because $\overline{L}^{\mathscr{F}} \cap \overline{R} \leq \overline{T}$ by Theorem 2.10. In particular, $\overline{M}/\overline{T} = (\overline{L}^{\mathscr{F}}/\overline{T})(((\overline{M} \cap \overline{R})\overline{T})/\overline{T})$ is also a semidirect product.

Since $\overline{M}/\overline{T}$ is $\mathscr{F}$-normal in $\overline{L}/\overline{T}$, then $(\overline{M}/\overline{T})/(\overline{L}^{\mathscr{F}}/\overline{T})$ is $\mathscr{F}$-normal in $(\overline{L}/\overline{T})/(\overline{L}^{\mathscr{F}}/\overline{T})$, which implies that $((\overline{M} \cap \overline{R})\overline{T})/\overline{T}$ is an $\mathscr{F}$-normal maximal subgroup of $\overline{R}\,\overline{T}/\overline{T}$ with index a p-number. Consequently, $((\overline{R}\,\overline{T})/\overline{T})^{F(p)} = (((\overline{M} \cap \overline{R})\overline{T})/\overline{T})^{F(p)}$.

Since $(\overline{R}\,\overline{T})/\overline{T}$ is an $\mathscr{F}$-projector of $\overline{L}/\overline{T} = (\overline{L}^{\mathscr{F}}/\overline{T})((\overline{R}\,\overline{T})/\overline{T})$ we can deduce by using again [8, IV, Theorem 5.16] and Theorem 2.10, that

$$C_{O_p(\overline{L}^{\mathscr{F}}/\overline{T})}\left((((\overline{M} \cap \overline{R})\overline{T})/\overline{T})^{F(p)}\right) = C_{O_p(\overline{L}^{\mathscr{F}}/\overline{T})}\left(((\overline{R}\,\overline{T})/\overline{T})^{F(p)}\right)$$

is the trivial group. Obviously, the same is true for every prime $r \in \pi(p) = \pi(r)$. Therefore $\pi(p) \cap \pi(\overline{C}/\overline{T}) = \emptyset$, which concludes this proof.

Step 8. $\overline{U} \cap \overline{H}^{\mathscr{F}} \leq (\overline{H}^{\mathscr{F}})'$, *for every subgroup* $\overline{H}$ *such that* $\overline{U} \leq \overline{H} \leq \overline{L}$.

If $\overline{H} < \overline{L}$, $\overline{U}$ is an $\mathscr{F}$-projector of $\overline{H}$ by Step 1 and the result is clear for this subgroup $\overline{H}$ by Theorem 2.10. Thus, it is enough to prove that $\overline{U} \cap \overline{L}^{\mathscr{F}} \leq (\overline{L}^{\mathscr{F}})'$.

By Step 7 and with the same notation, $(\overline{U}\,\overline{T})/\overline{T}$ is conjugate to $\overline{E}/\overline{T}$ in $\overline{M}/\overline{T}$. Then if we prove that $(\overline{E}/\overline{T}) \cap (\overline{L}^{\mathscr{F}}/\overline{T})$ is trivial, the result will be clearly deduced.

Thus, we are going to prove that $(\overline{E}/\overline{T}) \cap (\overline{L}^{\mathscr{F}}/\overline{T})$ is the trivial group. The notation used in Step 7 is assumed.

Since $\overline{E} = (\overline{R} \cap \overline{M})\overline{C}$ and $\overline{R} \cap \overline{L}^{\mathscr{F}} \leq \overline{T} \leq \overline{C}$, then $\overline{E} \cap \overline{L}^{\mathscr{F}} = ((\overline{R} \cap \overline{M})\overline{C}) \cap \overline{L}^{\mathscr{F}} = ((\overline{R} \cap \overline{M}) \cap \overline{L}^{\mathscr{F}})\overline{C} = \overline{C}$.

Assume that $\overline{C}/\overline{T}$ is non-trivial.

We observe that $\overline{E}/\overline{T} = (\overline{C}/\overline{T})(((\overline{R} \cap \overline{M})\overline{T})/\overline{T})$ is a semidirect product, because $\overline{C}/\overline{T}$ is normal in $\overline{E}/\overline{T}$, and the intersection of the subgroups into consideration is trivial.

Since $\overline{X} \trianglelefteq \overline{U}$, then $(\overline{X}\,\overline{T})/\overline{T} \trianglelefteq (\overline{U}\,\overline{T})/\overline{T} = (\overline{E}/\overline{T})^{\bar{m}\overline{T}}$, for some $\bar{m} \in \overline{M}$. Consequently, $(\overline{X}^{\bar{m}^{-1}}\overline{T})/\overline{T} \trianglelefteq \overline{E}/\overline{T}$. Let $\overline{Y} = \overline{X}^{\bar{m}^{-1}}$.

Notice that $\overline{Y} \cap \overline{L}^{\mathscr{F}} \leq \overline{Y} \cap \overline{G}^{\mathscr{F}} \leq \overline{Y} \cap \overline{K} = \bar{1}$, then $((\overline{Y}\,\overline{T})/\overline{T}) \cap (\overline{L}^{\mathscr{F}}/\overline{T})$ is the trivial group. Therefore, $[\overline{C}/\overline{T}, (\overline{Y}\,\overline{T})/\overline{T}] \leq ((\overline{Y}\,\overline{T})/\overline{T}) \cap (\overline{L}^{\mathscr{F}}/\overline{T})$ is trivial.

We claim that $\overline{R}\,\overline{T} < \overline{L}$. Otherwise, $\overline{L}^{\mathscr{F}} = \overline{T}(\overline{R} \cap \overline{L}^{\mathscr{F}}) = \overline{T}$, which would imply $\overline{L}^{\mathscr{F}} = \bar{1}$, that is, $\overline{L} \in \mathscr{F}$. Then X would be $\mathscr{F}$-subnormal in L. Since X is $\mathscr{X}$-maximal in L, we would conclude that $X = L_{\mathscr{X}} \trianglelefteq L$, because $\mathscr{X}$ is an $\mathscr{F}$-Fitting class. But this contradicts the choice of L and proves that $\overline{R}\,\overline{T} < \overline{L}$.

Assume that $(\overline{Y}\,\overline{T})/\overline{T}$ is contained in $((\overline{R} \cap \overline{M})\overline{T})/\overline{T}$. In particular, $\overline{Y} \le \overline{R}\,\overline{T}$. Then $\overline{X} \le \overline{R}^{\bar{m}}\overline{T} < \overline{L}$. By the choice of L, $\overline{X} \le \overline{R}^{\bar{m}\bar{t}} \in \mathrm{Proj}_{\mathscr{F}}(\overline{R}^{\bar{m}}\overline{T})$, for some $\bar{t} \in \overline{T}$. Arguing as above we can obtain that $\overline{R}^{\bar{m}\bar{t}} \le N_{\overline{L}}(\overline{X})$. In particular, $\overline{R}^{\bar{m}\bar{t}}$ would be conjugate to $\overline{U}$, because they are $\mathscr{F}$-projectors of $N_{\overline{L}}(\overline{X})$. Thus $\overline{U}$ is also an $\mathscr{F}$-projector of $\overline{L}$. Then $\overline{U} \cap \overline{L}^{\mathscr{F}} \le (\overline{L}^{\mathscr{F}})'$ and Step 8 would be proved.

Consider now the case when $(\overline{Y}\,\overline{T})/\overline{T}$ is not contained in $((\overline{R} \cap \overline{M})\overline{T})/\overline{T}$. We take into account that $((\overline{Y}\,\overline{T})/\overline{T}) \cap (\overline{C}/\overline{T})$ is trivial and $(\overline{Y}\,\overline{T})/\overline{T} \le \overline{E}/\overline{T}$. Consequently there exists an element $\bar{1} \ne \bar{x}\overline{T} \in (\overline{Y}\,\overline{T})/\overline{T}$ such that $\bar{x}\overline{T} = (\bar{a}\overline{T})(\bar{b}\overline{T})$, with $\bar{1} \ne \bar{a}\overline{T} \in ((\overline{R} \cap \overline{M})\overline{T})/\overline{T}$ and $\bar{1} \ne \bar{b}\,\overline{T} \in \overline{C}/\overline{T}$.

Since $[\overline{C}/\overline{T}, (\overline{Y}\,\overline{T})/\overline{T}] = \bar{1}$, we have that $(\bar{x}\overline{T})^{\bar{b}\,\overline{T}} = \bar{x}\overline{T} = (\bar{a}\overline{T})^{\bar{b}\,\overline{T}}(\bar{b}\,\overline{T})$, which implies that $\bar{a}\overline{T}$ conmutes with $\bar{b}\,\overline{T}$. Notice that $(o(\bar{a}\overline{T}), o(\bar{b}\,\overline{T})) \ne 1$. Otherwise, $\bar{1} \ne (\bar{x}\overline{T})^{o(\bar{a}\overline{T})} = (\bar{b}\,\overline{T})^{o(\bar{a}\overline{T})} \in ((\overline{Y}\,\overline{T})/\overline{T}) \cap (\overline{C}/\overline{T}) = \bar{1}$, which is a contradiction. Thus there exists a prime q dividing $o(\bar{a}\overline{T})$ and $o(\bar{b}\,\overline{T})$ and certainly $q \notin \pi(p)$. We write s to denote the product $o(\bar{a}\overline{T})_{q'}$ and $o(\bar{b}\,\overline{T})_{q'}$, the greatest q'-numbers dividing $o(\bar{a}\overline{T})$ and $o(\bar{b}\,\overline{T})$, respectively. Then $(\bar{x}\overline{T})^s = (\bar{a}\overline{T})^s(\bar{b}\,\overline{T})^s \ne \bar{1}$.

Consequently we can suppose that $\bar{x}\overline{T} = (\bar{a}\overline{T})(\bar{b}\,\overline{T})$ is a q-element of $(\overline{Y}\,\overline{T})/\overline{T}$ with $q \notin \pi(p)$, $\bar{1} \ne \bar{a} \in \overline{R} \cap \overline{M}$ and $1 \ne \bar{b} \in \overline{C}$. Since $\mathscr{F}$ is a lattice formation, we deduce that $\overline{T} \le \overline{L}^{\mathscr{F}} \le \overline{L}^{F(p)} = \overline{M}^{F(p)} = O^{\pi(p)}(\overline{M})$. In particular, we obtain that $\bar{x}\overline{T} \in ((\overline{Y}\,\overline{T})/\overline{T}) \cap (\overline{M}^{F(p)}/\overline{T}) = ((\overline{Y} \cap \overline{M}^{F(p)})\overline{T})/\overline{T}$, which is a normal subgroup in $\overline{L}/\overline{T}$ by Step 6.

We claim that $\bar{b}\,\overline{T} \in C_{O_q(\overline{L}^{\mathscr{F}}/\overline{T})}\big(((\overline{R}\,\overline{T})/\overline{T})^{F(q)}\big)$, but this subgroup is trivial because $(\overline{R}\,\overline{T})/\overline{T}$ is an $\mathscr{F}$-projector of $\overline{L}/\overline{T}$. Thus, we will obtain a contradiction which proves Step 8.

Let $\bar{y}\overline{T} \in ((\overline{R}\,\overline{T})/\overline{T})^{F(q)}$. We recall that $\overline{R} \in \mathscr{F}$, which is a lattice formation. Then $(\bar{x}\overline{T})^{\bar{y}\overline{T}} = (\bar{a}\overline{T})(\bar{b}\,\overline{T})^{\bar{y}\overline{T}} \in (\overline{Y}\,\overline{T})/\overline{T}$. Hence $(\bar{b}\,\overline{T})^{-1}(\bar{b}\,\overline{T})^{\bar{y}\overline{T}} \in ((\overline{Y}\,\overline{T})/\overline{T}) \cap (\overline{L}^{\mathscr{F}}/\overline{T})$, which is the trivial group, and this concludes the proof.

Step 9. $\overline{U} \in \mathrm{Proj}_{\mathscr{F}}(\overline{L})$.

By Lemma 3.6, $\overline{U}$ is a self-$\mathscr{F}$-normalizing $\mathscr{F}$-subgroup of $\overline{L}$. Moreover, Step 8 proves that $\overline{U} \cap \overline{H}^{\mathscr{F}} \le (\overline{H}^{\mathscr{F}})'$, for every subgroup $\overline{H}$ of $\overline{L}$ containg $\overline{U}$. By Theorem 2.11 we obtain that $\overline{U} \in \mathrm{Proj}_{\mathscr{F}}(\overline{L})$, which provides the final contradiction and proves the lemma. □

THEOREM 3.9. *Let $\mathscr{X}$ be an $\mathscr{F}$-Fitting class. For every group G, $\mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(G) = \mathrm{Inj}_{\mathscr{X}}(G)$.*

PROOF. Since $\mathscr{X}$ is a Fitting class and $(\mathscr{X}, \mathscr{F})$-injectors are $\mathscr{X}$-injectors, it is enough to prove that $\mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(G) \ne \emptyset$, for every group G.

Suppose that this result is not true and take a group G of minimal order such that $\mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(G) = \emptyset$. By the choice of G, there exists $W \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(G^{\mathscr{F}}) \neq \emptyset$. Let X be an $\mathscr{X}$-maximal subgroup of G containing W. It is clear that $W = X \cap G^{\mathscr{F}}$.

Let M be an $\mathscr{F}$-normal maximal subgroup of G. The choice of G implies that there exists $I \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(M)$. Since $G^{\mathscr{F}} \leq M$, then $I \cap G^{\mathscr{F}} \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(G^{\mathscr{F}}) = \mathrm{Inj}_{\mathscr{X}}(G^{\mathscr{F}})$ and $I \cap G^{\mathscr{F}}$ is conjugate to W in $G^{\mathscr{F}}$. Without loss of generality, we can assume that $I \cap G^{\mathscr{F}} = W$. Take J an $\mathscr{X}$-maximal subgroup of G containing I. Obviously, $W = J \cap G^{\mathscr{F}}$.

Assume first that $\langle X, J\rangle < G$. It is not difficult to prove that the group $\langle X, J\rangle$ satisfies the hypothesis of Lemma 3.6, so that we can deduce that $X = (H_1W)_{\mathscr{X}}$ with $H_1 \in \mathrm{Proj}_{\mathscr{F}}(N_{\langle X,J\rangle}(X))$ and $J = (H_2W)_{\mathscr{X}}$ with $H_2 \in \mathrm{Proj}_{\mathscr{F}}(N_{\langle X,J\rangle}(J))$. Moreover, the choice of G and Lemma 3.8 imply that $H_1, H_2 \in \mathrm{Proj}_{\mathscr{F}}(\langle X, J\rangle)$. Again by the choice of G, we can deduce that $\langle X, J\rangle$ satisfies the hypothesis of Lemma 3.7. This allow us to conclude that X and J are $(\mathscr{X}, \mathscr{F})$-injectors of $\langle X, J\rangle$.

We observe now that X and J are $\mathscr{F}$-pronormal in $\langle X, J\rangle$ by Lemma 3.5. Then, there exists $m \in \langle X, J\rangle^{\mathscr{F}} \leq G^{\mathscr{F}} \leq M$, such that $X^m = J$.

On the other hand, $J \cap M$ is $\mathscr{F}$-subnormal in $J \in \mathscr{X}$, because $J^{\mathscr{F}} \leq J \cap M$. Since $\mathscr{X}$ is an $\mathscr{F}$-Fitting class, we obtain that $J \cap M \in \mathscr{X}$ and consequently $I = J \cap M$. Therefore, $(X \cap M)^m = J \cap M = I \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(M)$ and clearly $X \cap M \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(M)$.

Consider now the case when $G = \langle X, J\rangle$. In particular, $W \trianglelefteq G$ and the hypothesis of Lemma 3.6 and Lemma 3.8 are satisfied with X and also with J. Therefore, $X = (H_1W)_{\mathscr{X}}$ and $J = (H_2W)_{\mathscr{X}}$ with $H_1 \in \mathrm{Proj}_{\mathscr{F}}(N_G(X)) \subseteq \mathrm{Proj}_{\mathscr{F}}(G)$ and $H_2 \in \mathrm{Proj}_{\mathscr{F}}(N_G(J)) \subseteq \mathrm{Proj}_{\mathscr{F}}(G)$. Thus $H_2 = H_1^x$, for some $x \in G$. Moreover H_1 is $\mathscr{F}$-pronormal in G, then it follows that $H_1^x = H_1^t$, for some $t \in \langle H_1, H_1^x\rangle^{\mathscr{F}} \leq G^{\mathscr{F}} \leq M$. Clearly $J = X^t$. Again we have that $I = J \cap M = (X \cap M)^t \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(M)$ and $X \cap M \in \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(M)$.

Consequently, we can conclude that X is an $(\mathscr{X}, \mathscr{F})$-injector of G. This provides the final contradiction which proves the theorem. □

As a consequence of the above proof we obtain the following result:

COROLLARY 3.10. *Let $\mathscr{X}$ be an $\mathscr{F}$-Fitting class and let G be a group. Let K be a normal subgroup of G such that $G^{\mathscr{F}} \leq K$, and let $W \in \mathrm{Inj}_{\mathscr{X}}(K) = \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(K)$. Then an $\mathscr{X}$-maximal subgroup of G containing W is an $\mathscr{X}$- injector of G.*

COROLLARY 3.11. *Let $\mathscr{X}$ be an $\mathscr{F}$-Fitting class and let G be a group. Let $V \in \mathrm{Inj}_{\mathscr{X}}(G) = \mathrm{Inj}_{(\mathscr{X},\mathscr{F})}(G)$ and let $K \trianglelefteq G$. Then:*

(a) *V is $\mathscr{F}$-pronormal in G. In fact, $V \cap K$ is $\mathscr{F}$-pronormal in G.*
(b) *$G = K^{\mathscr{F}} N_G(V \cap K)$.*
(c) *If $V \leq L \leq G$, then $L = S_L(V, \mathscr{F})N_L(V)$, where $S_L(V, \mathscr{F})$ is the $\mathscr{F}$-subnormal closure of V in L.*

PROOF. (a) Since the $\mathscr{X}$-injectors are the $(\mathscr{X}, \mathscr{F})$-injectors, Lemma 3.5 implies that V is $\mathscr{F}$-pronormal in G. For the rest, argue as in the proof of [8, VIII, Proposition 2.14 (a)] taking account moreover Theorem 2.6.

Parts (b) and (c) follow from (a) and Theorem 2.7. □

PROPOSITION 3.12. *Let $\mathscr{X}$ be an $\mathscr{F}$-Fitting class, let G be a group and let $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ be a chain of subgroups such that $G_i^{\mathscr{F}} \leq G_{i-1}$, for every $i = 1, \ldots, n$.*

For a subgroup V of G, the following statements are equivalent:

(i) $V \in \mathrm{Inj}_{\mathscr{X}}(G) = \mathrm{Inj}_{(\mathscr{X}, \mathscr{F})}(G)$;

(ii) *$V \cap G_i$ is an $\mathscr{X}$-maximal subgroup of G_i, for $i = 0, \ldots, n$.*

PROOF. If $V \in \mathrm{Inj}_{\mathscr{X}}(G)$, then statement (ii) is clear because every G_i is $\mathscr{F}$-subnormal in G.

For the converse, argue as in the proof of [8, VIII, Proposition 2.12] taking Corollary 3.10 into account. □

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